


## ALIENATION OF DRYGAS' AND CAUCHY'S FUNCTIONAL EQUATIONS

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**Abstract.** Inspired by the papers [2, 10] we will study, on 2-divisible groups that need not be abelian, the alienation problem between Drygas' and the exponential Cauchy functional equations, which is expressed by the equation

$$f(x+y) + g(x+y) + g(x-y) = f(x)f(y) + 2g(x) + g(y) + g(-y).$$

We also consider an analogous problem for Drygas' and the additive Cauchy functional equations as well as for Drygas' and the logarithmic Cauchy functional equations. Interesting consequences of these results are presented.

### 1. Introduction

The alienation and strong alienation problems are introduced by Dhombres ([3]), who gave the following definitions.

**DEFINITION 1.1.** Let  $E_1(f) = 0$  and  $E_2(f) = 0$  be two functional equations for a function  $f: X \rightarrow Y$ , where  $X$  and  $Y$  are non-empty sets. The equations  $E_1$  and  $E_2$  are *alien* with respect to  $X$  and  $Y$ , if any solution  $f: X \rightarrow Y$  of

$$E_1(f) + E_2(f) = 0$$

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is a solution of the system

$$\begin{cases} E_1(f) = 0, \\ E_2(f) = 0. \end{cases}$$

DEFINITION 1.2. Let  $E_1(f) = 0$  and  $E_2(g) = 0$  be two functional equations for functions  $f, g: X \rightarrow Y$ , where  $X$  and  $Y$  are non-empty sets. The equations  $E_1$  and  $E_2$  are *strongly alien*, if any solution  $f, g: X \rightarrow Y$  of

$$E_1(f) + E_2(g) = 0$$

is a solution of the system

$$\begin{cases} E_1(f) = 0, \\ E_2(g) = 0. \end{cases}$$

Later on, several papers and lectures have appeared on this subject (see [4–11, 13–16, 18, 19]). For more details concerning the alienation phenomenon in the theory of functional equations we refer to the survey article [12], which was authored by Ger and Sablik.

The aim of the present paper is to study the alienation phenomenon between Drygas' and Cauchy's functional equations. Firstly, we will solve the functional equation

$$(1.1) \quad f(x+y) + g(x+y) + g(x-y) \\ = f(x)f(y) + 2g(x) + g(y) + g(-y), \quad x, y \in X,$$

which is strictly connected with the problem of alienation of Drygas' functional equation, that is

$$(1.2) \quad g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \quad x, y \in X,$$

and the exponential Cauchy functional equation

$$(1.3) \quad f(x+y) = f(x)f(y), \quad x, y \in X,$$

where  $(X, +)$  is 2-divisible group that need not be abelian, and  $f$  and  $g$  are the unknown functions which take their values in an unital commutative ring  $Y$  of characteristic different from 2.

As consequences, we will introduce and discuss the solutions of the following functional equations

$$(1.4) \quad f(x+y) + g(x+y) + g(x-y) = f(x)f(y) + 2g(x) + 2g(y), \quad x, y \in X,$$

$$f(x+y) + g(x+y) + g(x-y)$$

$$= f(x)f(y) + f(x) + f(y) + 2g(x) + g(y) + g(-y),$$

and

$$f^2\left(\frac{x+y}{2}\right) + g(x+y) + g(x-y) = f(x)f(y) + 2g(x) + g(y) + g(-y).$$

The last equation results from summing up side by side Drygas' functional equation and Lobachevsky's functional equation, that is

$$(1.5) \quad f^2\left(\frac{x+y}{2}\right) = f(x)f(y), \quad x, y \in X.$$

Secondly, we will describe the solutions of the equation

$$(1.6) \quad f(x+y) + g(x+y) + g(x-y)$$

$$= f(x) + f(y) + 2g(x) + g(y) + g(-y), \quad x, y \in X,$$

which is derived by summing up side by side the equations (1.2) and the additive Cauchy functional equation, that is

$$(1.7) \quad f(x+y) = f(x) + f(y), \quad x, y \in X.$$

We show that modulo a constant, equations (1.2) and (1.7) are strongly alien on a 2-divisible group.

As applications, we will examine, on 2-divisible abelian groups, the alienation phenomenon between the equation (1.2) and the Jensen-additive functional equation, that is

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \quad x, y \in X,$$

and we get the result [9, Theorem 2.1], which was established by Ger about the alienation phenomenon of additivity and quadraticity up to a constant.

Finally, we will study the alienation problem, on a ring, of Drygas' and the logarithmic Cauchy functional equations, which is expressed as follows

$$(1.8) \quad f(xy) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + g(y) + g(-y), \quad x, y \in X.$$

Furthermore, we will describe the solutions of the equation

$$f(xy) + g(x + y) + g(x - y) = f(x) + f(y) + 2g(x) + 2g(y), \quad x, y \in X.$$

The monographs by Aczél and Dhombres ([1]) and by Stetkær ([17]) contain many references about Drygas' and Cauchy's functional equations.

**Notation.** The following notation will be used throughout the paper unless explicitly stated otherwise. Let  $(X, +)$  and  $(Y, +)$  be groups. We deal with the very classical functional equations defining additivity and quadraticity functions, i.e.

$$A(x + y) = A(x) + A(y), \quad x, y \in X,$$

and

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y), \quad x, y \in X$$

respectively, where  $A$  and  $Q$  are functions mapping  $X$  into  $Y$ . A map  $D: X \rightarrow Y$  is called Drygas' map provided that it satisfies the following functional equation

$$D(x + y) + D(x - y) = 2D(x) + D(y) + D(-y), \quad x, y \in X.$$

## 2. Drygas' and the exponential Cauchy functional equations

Our first main result describes the solutions  $f, g: X \rightarrow Y$  of Pexider type functional equation

$$f(x + y) + g(x + y) + g(x - y) = f(x)f(y) + 2g(x) + g(y) + g(-y), \quad x, y \in X,$$

resulting from summing up Drygas' and the exponential Cauchy functional equations side by side.

**THEOREM 2.1.** *Let  $(X, +)$  be a 2-divisible group (not necessarily commutative), and  $(Y, +, \cdot, 1)$  be an unital commutative ring of characteristic different from 2. Then, functions  $f, g: X \rightarrow Y$  satisfy the functional equation (1.1) if and only if we have one of the following two cases:*

(1)  $f(0) = 1$  and

$$\begin{cases} f(x+y) = f(x)f(y), \\ g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \quad x, y \in X. \end{cases}$$

(2)  $f(0) \neq 1$  and there exists a Drygas' map  $D: X \rightarrow Y$  such that

$$\begin{cases} (1-f(0))f(x) = 2g(0), \\ g(x)(1-f(0))^2 = D(x) - 2g(0)^2 + g(0)(1-f(0)), \quad x \in X. \end{cases}$$

Moreover if  $f(0) = 0$  then  $f \equiv 0$  and  $g$  is a solution of (1.2).

PROOF. Let  $f, g: X \rightarrow Y$  be a solution of (1.1). For  $x = y = 0$  in (1.1) we have

$$(2.1) \quad f(0)(1-f(0)) = 2g(0).$$

If we put  $y = 0$  in (1.1) we get

$$(2.2) \quad f(x)(1-f(0)) = 2g(0)$$

for all  $x \in X$ . Next, we distinguish between two cases:

Case 1: Suppose that  $f(0) = 1$ , then we get from (2.1) that  $2g(0) = 0$ . Let

$$(2.3) \quad \Gamma(x, y) = f(x)f(y) - f(x+y) = g(x+y) + g(x-y) - 2g(x) - g(y) - g(-y),$$

where  $x, y \in X$ . Since  $\Gamma$  satisfies the following equation

$$\Gamma(x+y, z) + f(z)\Gamma(x, y) = \Gamma(x, y+z) + f(x)\Gamma(y, z), \quad x, y, z \in X,$$

then

$$\begin{aligned} g(x+y+z) + g(x+y-z) - 2g(x+y) - g(z) - g(-z) + f(z)\Gamma(x, y) \\ = g(x+y+z) + g(x-y-z) - 2g(x) - g(y+z) \\ - g(-y-z) + f(x)\Gamma(y, z) \end{aligned}$$

for all  $x, y, z \in X$ , i.e.,

$$(2.4) \quad \begin{aligned} g(x+y-z) - 2g(x+y) - g(z) - g(-z) + f(z)\Gamma(x, y) \\ = g(x-y-z) - 2g(x) - g(y+z) - g(-y-z) + f(x)\Gamma(y, z) \end{aligned}$$

for all  $x, y, z \in X$ . If we replace  $z$  by  $-z$  in (2.4), we obtain that

$$(2.5) \quad \begin{aligned} g(x + y + z) - 2g(x + y) - g(-z) - g(z) + f(-z)\Gamma(x, y) \\ = g(x - y + z) - 2g(x) - g(y - z) - g(-y + z) + f(x)\Gamma(y, z), \end{aligned}$$

for all  $x, y \in X$  (because  $\Gamma(x, -y) = \Gamma(x, y)$  for all  $x, y \in X$ ).

Now, subtract equalities (2.4) and (2.5), and use the identity (2.3) to get

$$(2.6) \quad \begin{aligned} g(x + y - z) + g(x - y + z) + [f(z) - f(-z)]\Gamma(x, y) \\ = g(x + y + z) + g(x - y - z) - g(y + z) - g(-y - z) \\ + g(-y + z) + g(y - z). \end{aligned}$$

Replacing  $y$  by  $-y$  in (2.6) we get

$$(2.7) \quad \begin{aligned} g(x - y - z) + g(x + y + z) + [f(z) - f(-z)]\Gamma(x, -y) \\ = g(x - y + z) + g(x + y - z) - g(-y + z) - g(y - z) \\ + g(-y - z) + g(y + z). \end{aligned}$$

Since  $\Gamma(x, -y) = \Gamma(x, y)$  and  $Y$  has a characteristic different from 2 we get by subtracting (2.6) and (2.7) that

$$\begin{aligned} g(x + y - z) + g(x - y + z) = g(x + y + z) + g(x - y - z) - g(y + z) \\ - g(-y - z) + g(y - z) + g(-y + z), \end{aligned}$$

which, after setting  $z = y$ , leads to

$$2g(x) = g(x + 2y) + g(x - 2y) - g(2y) - g(-2y), \quad x, y \in X.$$

Since  $X$  is 2-divisible we get

$$g(x + y) + g(x - y) = 2g(x) - g(y) - g(-y), \quad x, y \in X.$$

Going back to (1.1), we infer that  $f$  is solution of (1.3).

Case 2: Suppose that  $f(0) \neq 1$ . For the special subcase  $f(0) = 0$  we get, by using (2.1), that  $2g(0) = 0$ , which implies from (2.2) that  $f \equiv 0$ .

We turn now to  $f(0) \neq 1$  (zero or not). By using (2.2), the equation (1.1) multiplied by  $(1 - f(0))^2$  gives

$$\begin{aligned} & 2g(0)(1 - f(0)) + g(x + y)(1 - f(0))^2 + g(x - y)(1 - f(0))^2 \\ & = 4g(0)^2 + 2g(x)(1 - f(0))^2 + g(y)(1 - f(0))^2 + g(y)(1 - f(0))^2, \end{aligned}$$

which means that

$$(2.8) \quad \mathcal{G}(x + y) + \mathcal{G}(x - y) = 2c + 2\mathcal{G}(x) + \mathcal{G}(y) + \mathcal{G}(-y), \quad x, y \in X,$$

where  $c = 2g(0)^2 - g(0)(1 - f(0))$  and  $\mathcal{G}: X \rightarrow Y$  is defined by

$$\mathcal{G}(x) = g(x)(1 - f(0))^2, \quad x \in X.$$

If we put  $D(x) := \mathcal{G}(x) + c$ , the equation (2.8) becomes

$$D(x + y) + D(x - y) = 2D(x) + D(y) + D(-y), \quad x, y \in X.$$

Hence, there exists a Drygas' map  $D: X \rightarrow Y$  such that

$$\begin{cases} (1 - f(0))f(x) = 2g(0), \\ g(x)(1 - f(0))^2 = D(x) - c, \quad x \in X. \end{cases}$$

The converse is straightforward. □

As a consequence of Theorem 2.1, we get the following result on the alienation of Drygasity and exponentiality up to a constant.

**COROLLARY 2.2.** *Let  $(X, +)$  be a 2-divisible group (not necessarily commutative), and  $(Y, +, \cdot)$  be a field of characteristic different from 2. Then, functions  $f, g: X \rightarrow Y$  satisfy the functional equation (1.1) if and only if we have one of the following two cases:*

- (1)  $f(0) = 1$  and  $f$  satisfies (1.3) and  $g$  solves (1.2).
- (2)  $f(0) \neq 1$  and there exist a Drygas' map  $D: X \rightarrow Y$  and a constant  $c \in Y$  such that

$$\begin{cases} f(x) = 2c, \\ g(x) = D(x) - 2c^2 + c, \quad x \in X. \end{cases}$$

Moreover, if  $f(0) = 0$  then  $f \equiv 0$  and  $g$  is a solution of (1.2).

PROOF. The assertion (1) follows from Theorem 2.1.

For (2), in view of Theorem 2.1 there exists a Drygas' map  $D: X \rightarrow Y$  such that

$$\begin{cases} (1 - f(0))f(x) = 2g(0), \\ g(x)(1 - f(0))^2 = D(x) - 2g(0)^2 + g(0)(1 - f(0)), \quad x \in X. \end{cases}$$

Then

$$\begin{cases} f(x) = 2c, \\ g(x) = ((1 - f(0))^2)^{-1} D(x) - 2c^2 + c, \quad x \in X, \end{cases}$$

where  $c = g(0)(1 - f(0))^{-1}$ . So we get the claimed result because the map  $((1 - f(0))^2)^{-1} D$  is also a Drygas' map.

The other direction is easy to check.  $\square$

The following result is devoted to study the alienation problem between exponentiality and quadraticity.

**THEOREM 2.3.** *Let  $(X, +)$  be a 2-divisible group (not necessarily commutative), and  $(Y, +, \cdot, 1)$  be an unital commutative ring of characteristic different from 2. Then, functions  $f, g: X \rightarrow Y$  satisfy the equation (1.4) if and only if there exists a quadratic function  $Q: X \rightarrow Y$  such that one of the following two cases holds:*

(1)  $f(0) = 1$  and

$$\begin{cases} f(x + y) = f(x)f(y), \\ g(x + y) + g(x - y) = 2g(x) + 2g(y), \quad x, y \in X. \end{cases}$$

(2)  $f(0) \neq 1$  and

$$\begin{cases} (1 - f(0))f(x) = 2g(0), \\ g(x)(1 - f(0))^2 = Q(x) - 2g(0)^2 + g(0)(1 - f(0)), \quad x \in X. \end{cases}$$

Moreover, if  $f(0) = 0$  then  $f \equiv 0$  and  $g$  is a solution of (1.2).

PROOF. Setting  $y = 0$  in (1.4) we get

$$(2.9) \quad f(x)(1 - f(0)) = 2g(0), \quad x \in X.$$



Replacing  $y$  by  $-y$  in (1.4) we obtain

$$(2.10) \quad f(x-y)+g(x-y)+g(x+y) = f(x)f(-y)+2g(x)+2g(-y), \quad x, y \in X.$$

Subtracting (1.4) from (2.10) we get

$$(2.11) \quad f(x+y) - f(x-y) = f(x)[f(y) - f(-y)] + 2[g(y) - g(-y)], \quad x, y \in X.$$

Setting  $x = 0$  in (2.11) gives us

$$2[g(y) - g(-y)] = (1 - f(0))[f(y) - f(-y)] \quad \text{for all } y \in X,$$

which yields, by using (2.9), that

$$\begin{aligned} 2[g(y) - g(-y)] &= (1 - f(0))f(y) - (1 - f(0))f(-y) \\ &= 2g(0) - 2g(0) = 0, \end{aligned}$$

for all  $y \in X$ . This means that  $g$  is even, i.e.,  $g(-y) = g(y)$  for all  $y \in X$ , because  $Y$  has a characteristic different from 2. Thus, from Theorem 2.1 we get the proof of the first direction.

The converse statement can be trivially shown.  $\square$

A result about the alienation of quadraticity and exponentiality up to a constant will be shown in the following corollary

**COROLLARY 2.4.** *Let  $(X, +)$  be a 2-divisible group (not necessarily commutative), and  $(Y, +, \cdot)$  be a field of characteristic different from 2. Then, functions  $f, g: X \rightarrow Y$  satisfy the functional equation (1.1) if and only if there exist a quadratic function  $Q: X \rightarrow Y$  and a constant  $c \in Y$  such that one of the following two cases holds:*

- (1)  $f(0) = 1$  and  $f$  satisfies (1.3) and  $g(x) = Q(x)$  for all  $x \in X$ .
- (2)  $f(0) \neq 1$  and

$$\begin{cases} f(x) = 2c, \\ g(x) = Q(x) - 2c^2 + c, \quad x \in X. \end{cases}$$

Moreover, if  $f(0) = 0$  then  $f \equiv 0$  and  $g(x) = Q(x)$  for all  $x \in X$ .

**PROOF.** The proof follows from Theorem 2.3 by using similar arguments as in proof of Corollary 2.2.  $\square$

Now, we will study the solutions  $f, g: X \rightarrow Y$  of the equation

$$(2.12) \quad \begin{aligned} f(x+y) + g(x+y) + g(x-y) \\ = f(x)f(y) + f(x) + f(y) + 2g(x) + g(y) + g(-y), \end{aligned}$$

where  $x, y \in X$ .

**COROLLARY 2.5.** *Let  $(X, +)$  be a 2-divisible group (not necessarily commutative), and  $(Y, +, \cdot, 1)$  be an unital commutative ring of characteristic different from 2. Then, functions  $f, g: X \rightarrow Y$  satisfy the functional equation (2.12) if and only if one of the following two cases holds:*

- (1)  $f + 1$  satisfies (1.3),  $f(0) = 0$  and  $g$  is a solution of (1.2).
- (2)  $f(0) \neq 0$  and there exists a Drygas's map  $D: X \rightarrow Y$  such that

$$\begin{cases} f(0)f(x) = -2g(0) - f(0), \\ g(x)f(0)^2 = D(x) - 2g(0)^2 - g(0)f(0), \quad x \in X. \end{cases}$$

Moreover, if  $f(0) = -1$  then  $f \equiv -1$  and  $g$  is a solution of (1.2).

**PROOF.** Let  $f, g: X \rightarrow Y$  be a solution of the equation (2.12). If we add the identity element 1 in the two sides of (2.12), we get

$$F(x+y) + g(x+y) + g(x-y) = F(x)F(y) + 2g(x) + g(y) + g(-y), \quad x, y \in X,$$

where  $F(x) := f(x) + 1$  for all  $x \in X$ . So by Theorem 2.1 we get the claimed result. The other direction is easy to check.  $\square$

As another application of our first main result, we investigate the alienation phenomenon of Lobachevsky's and the exponential Cauchy functional equations, i.e., the equation

$$(2.13) \quad \begin{aligned} f^2\left(\frac{x+y}{2}\right) + g(x+y) + g(x-y) \\ = f(x)f(y) + 2g(x) + g(y) + g(-y), \quad x, y \in X, \end{aligned}$$

on 2-divisible abelian group, where the functions  $f$  and  $g$  take their values in a field  $K$  of characteristic different from 2. We will show that (1.5) and (1.2) are strongly alien in the sense of Dhombres.

COROLLARY 2.6. *Let  $(X, +)$  be a 2-divisible abelian group, and  $K$  be a field of characteristic different from 2. The pair of functions  $f, g: X \rightarrow K$  is a solution of (2.13) if and only if it is a solution of the system of two equations*

$$\begin{cases} f^2\left(\frac{x+y}{2}\right) = f(x)f(y), \\ g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \quad x, y \in X. \end{cases}$$

Moreover, if  $f(0) = 0$  then  $f \equiv 0$  and  $g$  is a solution of (1.2).

PROOF. Setting  $x = y = 0$  in (2.13), we see that  $2g(0) = 0$ . If we put  $y = 0$  in (2.13), we get

$$(2.14) \quad f^2\left(\frac{x}{2}\right) = f(x)f(0), \quad x \in X.$$

We first suppose that  $f(0) = 0$ . So, from (2.14) we infer that  $f \equiv 0$ . Going back to (2.13) we deduce that  $g$  is a solution of the equation (1.2). We now suppose that  $f(0) \neq 0$ . By using (2.14), the equation (2.13) becomes

$$\begin{aligned} f(x+y)f(0) + g(x+y) + g(x-y) \\ = f(x)f(y) + 2g(x) + g(y) + g(-y), \end{aligned}$$

for all  $x, y \in X$ . If we multiply the last equation by  $f(0)^{-2}$ , we get

$$F(x+y) + G(x+y) + G(x-y) = F(x)F(y) + 2G(x) + G(y) + G(-y),$$

for all  $x, y \in X$ , where  $F(x) := f(x)f(0)^{-1}$  and  $G(x) := g(x)f(0)^{-2}$ . Since  $F(0) = 1$ , then according to Theorem 2.1 we get that  $F$  satisfies (1.3) and  $G$  solves (1.2). These mean that

$$(2.15) \quad f(x+y)f(0)^{-1} = f(x)f(0)^{-1}f(y)f(0)^{-1}, \quad x, y \in X,$$

and

$$(2.16) \quad \begin{aligned} g(x+y)f(0)^{-2} + g(x-y)f(0)^{-2} \\ = 2g(x)f(0)^{-2} + g(y)f(0)^{-2} + g(-y)f(0)^{-2}, \quad x, y \in X. \end{aligned}$$

If we multiply the equations (2.15) and (2.16) by  $f(0)^2$  we get, by using (2.14), that  $f$  satisfies (1.5) and  $g$  solves (1.2).

Conversely, it is elementary to show the other direction.  $\square$

### 3. Drygas' and the additive Cauchy functional equations

In this section we show that modulo a constant, the equation (1.6):

$$\begin{aligned} f(x+y) + g(x+y) + g(x-y) \\ = f(x) + f(y) + 2g(x) + g(y) + g(-y), \quad x, y \in X, \end{aligned}$$

on 2-divisible non-abelian group, forces  $f$  to be an additive function and  $g$  to be a solution of Drygas' functional equation (1.2).

**THEOREM 3.1.** *Let  $(X, +)$  be a 2-divisible group (not necessarily commutative), and  $(Y, +)$  be an abelian group. The pair of functions  $f, g: X \rightarrow Y$  is a solution of (1.6) if and only if it is a solution of the system of two equations*

$$\begin{cases} f(x+y) = f(x) + f(y) + 2c, \\ g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y) - 2c, \quad x, y \in X, \end{cases}$$

where  $c \in Y$  is a constant.

**PROOF.** Putting  $x = y = 0$  in (1.6) we get

$$(3.1) \quad f(0) + 2g(0) = 0.$$

Define  $F(x) := f(x) - f(0)$  and  $G(x) := g(x) - g(0)$  for all  $x \in X$ . So by (3.1) we see easily that the pair  $(F, G)$  is a solution of (1.6) such that  $F(0) = G(0) = 0$ .

We put  $\mathcal{G} := F + G$ , then the equation (1.6) becomes

$$\mathcal{G}(x+y) + G(x-y) = \mathcal{G}(x) + \mathcal{G}(y) + G(x) + G(-y), \quad x, y \in X.$$

We define

$$\Gamma(x, y) := \mathcal{G}(x+y) - \mathcal{G}(x) - \mathcal{G}(y) = G(x) + G(-y) - G(x-y),$$

for all  $x, y \in X$ . The map  $\Gamma$ , as a Cauchy difference, satisfies the cocycle equation

$$\Gamma(x+y, z) + \Gamma(x, y) = \Gamma(x, y+z) + \Gamma(y, z), \quad x, y, z \in X.$$

Consequently,

$$\begin{aligned} & G(x+y) + G(-z) - G(x+y-z) + G(x) + G(-y) - G(x-y) \\ &= G(x) + G(-y-z) - G(x-y-z) + G(y) + G(-z) - G(y-z), \end{aligned}$$

for all  $x, y, z \in X$ , which is equivalent to

$$\begin{aligned} (3.2) \quad & G(x+y) - G(x+y-z) - G(x-y) + G(-y) \\ &= G(-y-z) - G(x-y-z) + G(y) - G(y-z), \quad x, y, z \in X. \end{aligned}$$

Replacing  $z$  by  $-z$  in (3.2), we get

$$\begin{aligned} (3.3) \quad & G(x+y) - G(x+y+z) - G(x-y) + G(-y) \\ &= G(-y+z) - G(x-y+z) + G(y) - G(y+z), \quad x, y, z \in X. \end{aligned}$$

Subtracting (3.2) from (3.3) we obtain

$$\begin{aligned} & G(x+y+z) - G(x+y-z) = G(x-y+z) - G(x-y-z) \\ & \quad + G(y+z) + G(-y-z) - G(y-z) - G(-y+z) \end{aligned}$$

i.e.,

$$\begin{aligned} & G(x+y+z) + G(x-y-z) = G(x-y+z) + G(x+y-z) \\ & \quad + G(-y-z) - G(y-z) - G(-y+z) + G(y+z), \end{aligned}$$

which, after setting  $z = y$ , leads to

$$G(x+2y) + G(x-2y) = 2G(x) + G(-2y) + G(2y), \quad x, y \in X.$$

Due to the 2-divisibility of  $X$ , we deduce that

$$G(x+y) + G(x-y) = 2G(x) + G(-y) + G(y), \quad x, y \in X.$$

Hence  $g$  is a solution of the equation

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y) - 2g(0), \quad x, y \in X.$$

Going back to the equation (1.6), we see that  $f$  is a solution of the equation

$$f(x+y) = f(x) + f(y) + 2g(0), \quad x, y \in X.$$

The converse statement can be trivially shown. □

The following corollary states that Drygas' functional equation and Jensen's functional equation are strongly alien, on 2-divisible abelian group, in the sense of Dhombres.

**COROLLARY 3.2.** *Let  $(X, +)$  be a 2-divisible abelian group, and  $(Y, +)$  be an abelian group. The pair of functions  $f, g: X \rightarrow Y$  is a solution of the equation*

$$(3.4) \quad 2f\left(\frac{x+y}{2}\right) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + g(y) + g(-y),$$

for all  $x, y \in X$  if and only if it is a solution of the system of two equations

$$\begin{cases} 2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \\ g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \quad x, y \in X. \end{cases}$$

**PROOF.** Substituting  $x = y = 0$  in (3.4), we get  $2g(0) = 0$ . For  $y = 0$  in (3.4), we obtain

$$(3.5) \quad 2f\left(\frac{x}{2}\right) = f(x) + f(0), \quad x \in X.$$

So by (3.5), the equation (3.4) becomes

$$\begin{aligned} f(x+y) + f(0) + g(x+y) + g(x-y) &= f(x) + f(y) + 2g(x) \\ &\quad + g(y) + g(-y). \end{aligned}$$

If we define  $F(\cdot) := f(\cdot) - f(0)$ , we get

$$F(x+y) + g(x+y) + g(x-y) = F(x) + F(y) + 2g(x) + g(y) + g(-y),$$

for all  $x, y \in X$ . Then, according to Theorem 3.1, we have

$$(3.6) \quad \begin{cases} f(x+y) = f(x) + f(y) - f(0), \\ g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \quad x, y \in X, \end{cases}$$

(because, using the notation in Theorem 3.1, here the constant  $2c = -F(0) = 0$ ). Setting  $x = y$  in the equation  $f(x+y) = f(x) + f(y) - f(0)$ , gives

$$f(2x) = 2f(x) - f(0), \quad x \in X.$$

Hence, (3.6) becomes

$$\begin{cases} 2f(x+y) = f(2x) + f(2y), \\ g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \quad x, y \in X, \end{cases}$$

which is equivalent to

$$\begin{cases} 2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \\ g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \quad x, y \in X, \end{cases}$$

because  $X$  is divisible by 2. The converse statement is easy to shown.  $\square$

As another consequence of Theorem 3.1, we get the following result due to Ger ([9]) about the alienation phenomenon of additivity and quadraticity up to a constant.

**COROLLARY 3.3** ([9]). *Let  $(X, +)$  be a 2-divisible group (not necessarily commutative), and  $(Y, +)$  be an abelian group. Then, functions  $f, g: X \rightarrow Y$  satisfy the functional equation*

$$(3.7) \quad f(x+y) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + 2g(y), \quad x, y \in X,$$

*if and only if there exist an additive function  $A: X \rightarrow Y$ , a quadratic function  $Q: X \rightarrow Y$ , and a constant  $c \in Y$  such that*

$$f(x) = A(x) - 2c \quad \text{and} \quad g(x) = Q(x) + c \quad \text{for all } x \in X.$$

**PROOF.** Putting  $x = y = 0$  in (3.7) we get

$$(3.8) \quad f(0) + 2g(0) = 0.$$

Replacing  $x$  by 0 in (3.7) and using (3.8) we obtain that  $g(-y) = g(y)$  for all  $y \in X$ . So by applying Theorem 3.1 we get the desired result. Conversely, it is elementary to verify that the above formulas of  $f$  and  $g$  is a solution of (3.7).  $\square$

#### 4. Drygas' and the logarithmic Cauchy functional equations

This section is devoted to study the equation (1.8), namely

$$f(xy) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + g(y) + g(-y), \quad x, y \in X,$$

which results from summing up the well known Drygas' and the logarithmic functional equations side by side.

**THEOREM 4.1.** *Let  $(X, +, \cdot)$  be a ring, and  $(Y, +)$  be a 2-torsion-free abelian group. Then the pair of functions  $f, g: X \rightarrow Y$  is a solution of the functional equation (1.8) if and only if there exist a Drygas' map  $D: X \rightarrow Y$  and a constant  $c \in Y$  such that*

$$f(x) = -2c \quad \text{and} \quad g(x) = D(x) + c, \quad x \in X.$$

**PROOF.** Putting  $x = y = 0$  in (1.8) we get

$$f(0) + 2g(0) = 0.$$

If we put  $y = 0$  in (1.8), we see that  $f(x) = f(0) = -2g(0)$  for all  $x \in X$ . So, going back to the equation (1.8), we obtain

$$g(x+y) + g(x-y) = -2g(0) + 2g(x) + g(y) + g(-y), \quad x, y \in X.$$

This is equivalent to

$$D(x+y) + D(x-y) = 2D(x) + D(y) + D(-y), \quad x, y \in X,$$

where  $D(x) := g(x) - g(0)$  for all  $x \in X$ . Thus,

$$f(x) = f(0) = -2g(0) \quad \text{and} \quad g(x) = D(x) + g(0),$$

for all  $x \in X$ , where  $D$  is a Drygas's map from  $X$  to  $Y$ . The converse statement can be trivially shown.  $\square$

As a consequence we have the following result:

**COROLLARY 4.2.** *Let  $(X, +, \cdot, 1)$  be an unital ring, and  $(Y, +)$  be a 2-torsion-free abelian group. The pair of functions  $f, g: X \rightarrow Y$  is a solution of the functional equation*

$$(4.1) \quad f(xy) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + 2g(y), \quad x, y \in X,$$



if and only if there exist a quadratic function  $Q: X \rightarrow Y$  and a constant  $c \in Y$  such that

$$f(x) = -2c \quad \text{and} \quad g(x) = Q(x) + c, \quad x \in X.$$

PROOF. Let  $f, g: X \rightarrow Y$  be a solution of (4.1). Replacing  $y$  by  $-y$  in (4.1) gives us

$$(4.2) \quad f(-xy) + g(x+y) + g(x-y) = f(x) + f(-y) + 2g(x) + 2g(-y), \quad x, y \in X.$$

If we subtract (4.1) from (4.2), we get

$$(4.3) \quad f(xy) - f(-xy) = f(y) - f(-y) + 2g(y) - 2g(-y).$$

Setting  $x = 1$  in (4.3), we obtain that  $g$  is even. So by applying Theorem 4.1 we get the desired result. Conversely, it is elementary to show that the above formulas of  $f$  and  $g$  are solutions of (4.1).  $\square$

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