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OPERATOR SUBADDITIVITY OF THE $\mathcal{D}$-LOGARITHMIC INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $[0, \infty)$ we consider the following *$\mathcal{D}$-logarithmic integral transform*

$$\mathcal{DL}og\,(w, \mu)\,(T) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda + T}{\lambda}\right) d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if A, $B > 0$ with $BA + AB \geq 0$, then

$$\mathcal{DL}og\,(w, \mu)\,(A) + \mathcal{DL}og\,(w, \mu)\,(B) \geq \mathcal{DL}og\,(w, \mu)\,(A + B).$$

In particular we have

$$\frac{1}{6}\pi^2 + \operatorname{dilog}(A + B) \geq \operatorname{dilog}(A) + \operatorname{dilog}(B),$$

where the *dilogarithmic function* $\operatorname{dilog} : [0, \infty) \to \mathbb{R}$ is defined by

$$\operatorname{dilog}(t) := \int_1^t \frac{\ln s}{1 - s} ds, \quad t \geq 0.$$

Some examples for integral transform $\mathcal{DL}og\,(\cdot, \cdot)$ related to the operator monotone functions are also provided.

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1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions ([7], [6]), see for instance [1, p. 144–145]:

THEOREM 1. *A function $f\colon [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.1}$$

where $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty.$$

For some examples of operator monotone functions see [3]-[5], [8]-[9] and the references therein.

We have the following integral representation for the power function when $s > 0$, $r \in (0, 1]$, see for instance [1, p. 145]:

$$s^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + s} d\lambda. \tag{1.2}$$

Observe that for $s > 0$, $s \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + s)(\lambda + 1)} = \frac{\ln s}{s - 1} + \frac{1}{1 - s} \ln\left(\frac{u + s}{u + 1}\right)$$

for all $u > 0$. By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln s}{s - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + s)(\lambda + 1)}, \tag{1.3}$$

which gives the representation for the logarithm

$$\ln s = (s - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + s)}. \tag{1.4}$$

If we integrate (1.2) over s from 0 to $t>0$, we get by Fubini's theorem

$$\begin{aligned}\frac{t^r}{r} &= \frac{\sin(r\pi)}{\pi}\int_0^\infty \left(\int_0^t \left(\frac{1}{\lambda+s}\right)ds\right)\lambda^{r-1}d\lambda\\ &= \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda\end{aligned}$$

giving the identity of interest

$$t^r = \frac{r\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda, \quad t>0 \text{ and } r\in(0,1].$$

Recall the *dilogarithmic function* dilog: $[0,\infty)\to\mathbb{R}$ defined by

$$\operatorname{dilog}(t) := \int_1^t \frac{\ln s}{1-s}ds, \quad t\geq 0.$$

Some particular values of interest are

$$\operatorname{dilog}(1)=0,\ \operatorname{dilog}(0)=\int_1^0 \frac{\ln s}{1-s}ds = \int_0^1 \frac{\ln s}{s-1}ds = \frac{1}{6}\pi^2,$$

and

$$\operatorname{dilog}\left(\frac{1}{2}\right) = \frac{1}{12}\pi^2 - \frac{1}{2}(\ln 2)^2.$$

If we integrate the identity (1.3) over s from 0 to $t>0$, we get by Fubini's theorem

$$\int_0^t \frac{\ln s}{s-1}ds = \int_0^\infty \left(\int_0^t \frac{1}{\lambda+s}ds\right)\frac{1}{(\lambda+1)}d\lambda = \int_0^\infty \frac{1}{(\lambda+1)}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda$$

and since

$$\begin{aligned}\int_0^t \frac{\ln s}{s-1}ds &= \int_0^1 \frac{\ln s}{s-1}ds + \int_1^t \frac{\ln s}{s-1}ds = \frac{1}{6}\pi^2 - \int_1^t \frac{\ln s}{1-s}ds\\ &= \frac{1}{6}\pi^2 - \operatorname{dilog}(t)\end{aligned}$$

then we get the identity of interest

$$\frac{1}{6}\pi^2 - \operatorname{dilog}(t) = \int_0^\infty \frac{1}{(\lambda+1)}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda, \quad t>0.$$

Motivated by the above representations, we define the $\mathcal{D}$-*logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda > 0$ by

$$\mathcal{DL}og\,(w,\mu)(t) := \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\mu(\lambda), \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.5) exists for all $t > 0$. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{DL}og\,(w)(t) := \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\lambda.$$

Obviously,

$$\begin{aligned}\mathcal{DL}og\,(w,\mu)(t) &= \int_0^\infty w(\lambda)\ln\left(1+\frac{t}{\lambda}\right)d\mu(\lambda)\\ &= \int_0^\infty w(\lambda)\left[\ln(\lambda+t)-\ln(\lambda)\right]d\mu(\lambda)\end{aligned}$$

and one can use either of these representations when is needed.

By utilising the continuous functional calculus for selfadjoint operators, we can define the operator $\mathcal{D}$-*logarithmic transform* by

$$\mathcal{DL}og\,(w,\mu)(T) = \int_0^\infty w(\lambda)\ln\left(1+\frac{1}{\lambda}T\right)d\mu(\lambda)$$

for $T > 0$.

If we use the $\mathcal{D}$-logarithmic transform for the kernel $w_{\ell^{r-1}}(\lambda) := \frac{r\sin(r\pi)}{\pi}\lambda^{r-1}$, $r \in (0,1]$ we have

$$\mathcal{DL}og\,(w_{\ell^{r-1}})(T) = T^r, \quad T \geq 0$$

while for the kernel $w_{(\ell+1)^{-1}}(\lambda) := \frac{1}{\lambda+1}$ we have

$$\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)(T) = \frac{1}{6}\pi^2 - \operatorname{dilog}(T), \quad T \geq 0.$$

In the recent paper [2] we introduced the following *integral transform*

$$\mathcal{D}(w,\mu)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s}d\mu(\lambda), \quad s > 0, \tag{1.6}$$

for a continuous and positive function $w(\lambda)$, $\lambda > 0$,where μ is a positive measure on $(0,\infty)$ and the integral (1.6) exists for all $s > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda + s} d\lambda, \quad s > 0.$$

Several examples of integral transforms $\mathcal{D}(w,\mu)$ have also been given in [2].

If we integrate the identity (1.3) over s from 0 to $t > 0$, we get by Fubini's theorem

$$\begin{aligned}\int_0^t \mathcal{D}(w,\mu)(s)\, ds &:= \int_0^\infty \left(\int_0^t \frac{1}{\lambda + s} ds\right) w(\lambda)\, d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \ln\left(\frac{\lambda + t}{\lambda}\right) d\mu(\lambda)\end{aligned}$$

for $t > 0$, which provides the equality of interest

$$\mathcal{D}\mathcal{L}og(w,\mu)(t) = \int_0^t \mathcal{D}(w,\mu)(s)\, ds, \quad t > 0,$$

provided that the integral on the right side exists for all $t > 0$.

2. Main results

We have the following identity of interest:

LEMMA 1. *For all A, $B > 0$ and $\lambda > 0$ we have*

$$\begin{aligned}(2.1)\quad &\ln\left(\frac{A+\lambda}{\lambda}\right) + \ln\left(\frac{B+\lambda}{\lambda}\right) - \ln\left(\frac{A+B+\lambda}{\lambda}\right) \\ &= \ln(A+\lambda) + \ln(B+\lambda) - \ln(A+B+\lambda) - \ln\lambda \\ &= \int_0^\infty \frac{1}{s+\lambda} (A+B+s+\lambda)^{-1} \\ &\quad \times [A(B+s+\lambda)^{-1}BA + B(A+s+\lambda)^{-1}AB](A+B+s+\lambda)^{-1} ds \\ &\quad + \int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}(BA+AB)(A+B+s+\lambda)^{-1} ds.\end{aligned}$$

PROOF. We have, by (1.4), that

$$\begin{aligned}\ln(T+\lambda) &= \int_0^\infty \frac{1}{(s+1)}(T+\lambda-1)(s+T+\lambda)^{-1}\,ds\\ &= \int_0^\infty \frac{1}{(s+1)}(T+\lambda+s-1-s)(s+T+\lambda)^{-1}\,ds\\ &= \int_0^\infty \frac{1}{(s+1)}\left[1-(1+s)(s+T+\lambda)^{-1}\right]ds\\ &= \int_0^\infty \left[\frac{1}{s+1}-(s+T+\lambda)^{-1}\right]ds.\end{aligned}$$

For A, $B>0$ and $u\geq 0$, define

$$K_u := (A+u)^{-1}+(B+u)^{-1}-(A+B+u)^{-1}$$

and $W_u := 1-uK_u$.

Therefore

$$\begin{aligned}&\ln(A+\lambda)+\ln(B+\lambda)-\ln(A+B+\lambda)-\ln\lambda \qquad (2.2)\\ &= \int_0^\infty \left[\frac{1}{(s+1)}-(s+A+\lambda)^{-1}\right]ds\\ &\quad+\int_0^\infty \left[\frac{1}{(s+1)}-(s+B+\lambda)^{-1}\right]ds\\ &\quad-\int_0^\infty \left[\frac{1}{(s+1)}-(s+A+B+\lambda)^{-1}\right]ds\\ &\quad-\int_0^\infty \left[\frac{1}{(s+1)}-(s+\lambda)^{-1}\right]ds\\ &= \int_0^\infty \Big[(s+A+B+\lambda)^{-1}+(s+\lambda)^{-1}\\ &\quad-(s+A+\lambda)^{-1}-(s+B+\lambda)^{-1}\Big]ds\\ &= \int_0^\infty \left[(s+\lambda)^{-1}-K_{s+\lambda}\right]ds = \int_0^\infty \left(\frac{1}{s+\lambda}-K_{s+\lambda}\right)ds\\ &= \int_0^\infty \frac{1}{s+\lambda}\left[1-(s+\lambda)K_{s+\lambda}\right]ds = \int_0^\infty \frac{1}{s+\lambda}W_{s+\lambda}ds.\end{aligned}$$

We have successively

$$
\begin{aligned}
&(A+B+\lambda)\,W_\lambda\,(A+B+\lambda)\\
&=(A+B+\lambda)\,(1-\lambda K_\lambda)\,(A+B+\lambda)\\
&=(A+B+\lambda)^2-\lambda\,(A+B+\lambda)\,K_\lambda\,(A+B+\lambda)\\
&=(A+B+\lambda)\,(A+B+\lambda)\\
&\quad-\lambda\left[B\,(A+\lambda)^{-1}\,B+A\,(B+\lambda)^{-1}\,A+2\,(A+B)+\lambda\right]\\
&=A^2+BA+\lambda A+AB+B^2+\lambda B+\lambda A+\lambda B+\lambda^2\\
&\quad-\lambda B\,(A+\lambda)^{-1}\,B-\lambda A\,(B+\lambda)^{-1}\,A-2\lambda\,(A+B)-\lambda^2\\
&=A^2+B^2+BA+AB-\lambda B\,(A+\lambda)^{-1}\,B-\lambda A\,(B+\lambda)^{-1}\,A\\
&=A\,(B+\lambda)^{-1}\,(B+\lambda)\,A-\lambda A\,(B+\lambda)^{-1}\,A\\
&\quad+B\,(A+\lambda)^{-1}\,(A+\lambda)\,B-\lambda B\,(A+\lambda)^{-1}\,B+BA+AB\\
&=A\,(B+\lambda)^{-1}\,BA+B\,(A+\lambda)^{-1}\,AB+BA+AB,
\end{aligned}
$$

therefore

$$
\begin{aligned}
W_\lambda&=(A+B+\lambda)^{-1}\left[A\,(B+\lambda)^{-1}\,BA\right.\\
&\qquad\left.+B\,(A+\lambda)^{-1}\,AB+BA+AB\right](A+B+\lambda)^{-1}\,.
\end{aligned}
\tag{2.3}
$$

From (2.3) we obtain

$$
\begin{aligned}
W_{s+\lambda}&=(A+B+s+\lambda)^{-1}\left[A(B+s+\lambda)^{-1}BA\right.\\
&\qquad\left.+B(A+s+\lambda)^{-1}AB+BA+AB\right](A+B+s+\lambda)^{-1}\\
&=(A+B+s+\lambda)^{-1}\left[A(B+s+\lambda)^{-1}BA\right.\\
&\qquad\left.+B(A+s+\lambda)^{-1}AB\right](A+B+s+\lambda)^{-1}\\
&\qquad+(A+B+s+\lambda)^{-1}\,(BA+AB)\,(A+B+s+\lambda)^{-1}\,.
\end{aligned}
\tag{2.4}
$$

On making use of (2.2) and (2.4) we obtain the desired result (2.1). □

Theorem 2. *For all A, $B > 0$ we have*

$$\begin{aligned}(2.5)\qquad & \mathcal{DL}og\,(w,\mu)\,(A) + \mathcal{DL}og\,(w,\mu)\,(B) - \mathcal{DL}og\,(w,\mu)\,(A+B) \\ &= \int_0^\infty w\,(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda} \,(A+B+s+\lambda)^{-1} \right. \\ &\quad \times \left[A\,(B+s+\lambda)^{-1}\,BA + B\,(A+s+\lambda)^{-1}\,AB \right] \\ &\quad \left. \times\,(A+B+s+\lambda)^{-1}\,ds \right) d\mu\,(\lambda) \\ &\quad + \int_0^\infty w\,(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda}\,(A+B+s+\lambda)^{-1} \right. \\ &\quad \left. \times\,(BA+AB)\,(A+B+s+\lambda)^{-1}\,ds \right) d\mu\,(\lambda) \\ &\geq \int_0^\infty w\,(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda}\,(A+B+s+\lambda)^{-1} \right. \\ &\quad \left. \times\,(BA+AB)\,(A+B+s+\lambda)^{-1}\,ds \right) d\mu\,(\lambda)\,.\end{aligned}$$

If $BA + AB \geq 0$, then

$$\mathcal{DL}og\,(w,\mu)\,(A) + \mathcal{DL}og\,(w,\mu)\,(B) \geq \mathcal{DL}og\,(w,\mu)\,(A+B)\,. \tag{2.6}$$

Proof. The identity (2.5) follows by multiplying the equality (2.1) with $w\,(\lambda)$ and integrating on $[0,\infty)$ over the measure $d\mu\,(\lambda)$.

Let s, $\lambda \geq 0$. Since $(B+s+\lambda)^{-1}\,B > 0$ and $(A+s+\lambda)^{-1}\,A > 0$ hence $A\,(B+s+\lambda)^{-1}\,BA > 0$ and $B\,(A+s+\lambda)^{-1}\,AB > 0$. Therefore

$$A\,(B+s+\lambda)^{-1}\,BA + B\,(A+s+\lambda)^{-1}\,AB > 0$$

and by multiplying both sides by $(A+B+s+\lambda)^{-1}$ we get

$$\begin{aligned}&(A+B+s+\lambda)^{-1} \left[A\,(B+s+\lambda)^{-1}\,BA + B\,(A+s+\lambda)^{-1}\,AB \right] \\ &\times\,(A+B+s+\lambda)^{-1} > 0.\end{aligned}$$

By multiplying with $\frac{1}{s+\lambda}$ and $w(\lambda)$ and integrating twice, we obtain

$$\begin{aligned}&\int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}\right.\\&\times\left[A(B+s+\lambda)^{-1}BA+B(A+s+\lambda)^{-1}AB\right]\\&\left.\times(A+B+s+\lambda)^{-1}ds\right)d\mu(\lambda)\geq 0,\end{aligned}$$

which proves the inequality in (2.5).

If $BA+AB\geq 0$, then by multiplying both sides by $(A+B+s+\lambda)^{-1}$ we get

$$(A+B+s+\lambda)^{-1}(BA+AB)(A+B+s+\lambda)^{-1}\geq 0$$

for s, $\lambda\geq 0$ and by integration twice, we derive

$$\begin{aligned}&\int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}\right.\\&\left.\times(BA+AB)(A+B+s+\lambda)^{-1}ds\right)d\mu(\lambda)\geq 0\end{aligned}$$

and the subadditivity property (2.6) is proved. □

Remark 1. If we write the inequality (2.6) for the transform $\mathcal{DL}og(w_{\ell^{r-1}})$ we get

$$A^r+B^r\geq(A+B)^r,\ r\in(0,1] \tag{2.7}$$

provided A, $B>0$ with $BA+AB\geq 0$.

If we write the inequality (2.6) for the transform $\mathcal{DL}og(w_{(\ell+1)^{-1}})$ we get

$$\frac{1}{6}\pi^2+\operatorname{dilog}(A+B)\geq\operatorname{dilog}(A)+\operatorname{dilog}(B)$$

provided A, $B>0$ with $BA+AB\geq 0$.

We define the function

$$G_{w,\mu}(t) := \int_0^\infty w(\lambda)\left(\int_0^\infty \frac{ds}{(s+\lambda)(s+t+\lambda)^2}\right) d\mu(\lambda), \quad t>0. \tag{2.8}$$

Observe that for a, $b>0$ we have

$$\int_0^\infty \frac{ds}{(s+a)(s+b)^2} = \frac{\ln b - \ln a}{(b-a)^2} - \frac{1}{b(b-a)}.$$

This gives that

$$\int_0^\infty \frac{ds}{(s+\lambda)(s+t+\lambda)^2} = \frac{\ln(t+\lambda)-\ln\lambda}{t^2} - \frac{1}{t(t+\lambda)}, \quad t>0.$$

Therefore

$$\begin{aligned}
G_{w,\mu}(t) &= \int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{(s+\lambda)(s+t+\lambda)^2} ds\right) d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left(\frac{\ln(t+\lambda)-\ln\lambda}{t^2} - \frac{1}{t(t+\lambda)}\right) d\mu(\lambda) \\
&= \frac{1}{t^2}\int_0^\infty w(\lambda)\ln\left(\frac{t+\lambda}{\lambda}\right) d\mu(\lambda) - \frac{1}{t}\int_0^\infty \frac{w(\lambda)}{t+\lambda} d\mu(\lambda) \\
&= \frac{1}{t^2}\mathcal{D}\mathcal{L}og(w,\mu)(t) - \frac{1}{t}\mathcal{D}(w)(t) \geq 0,
\end{aligned} \tag{2.9}$$

for all $t>0$.

Corollary 1. *If A, $B>0$ with $BA+AB \geq k$, where k is a real number, then*

$$\begin{aligned}
&\mathcal{D}\mathcal{L}og(w,\mu)(A) + \mathcal{D}\mathcal{L}og(w,\mu)(B) - \mathcal{D}\mathcal{L}og(w,\mu)(A+B) \\
&\quad \geq k\left(\mathcal{D}\mathcal{L}og(w,\mu)(A+B) - (A+B)\mathcal{D}(w)(t)\right)(A+B)^{-2}.
\end{aligned} \tag{2.10}$$

Proof. If $BA+AB \geq k$, then by multiplying both sides by $(A+B+s+\lambda)^{-1}$, we get

$$\begin{aligned}
&(A+B+s+\lambda)^{-1}(BA+AB)(A+B+s+\lambda)^{-1} \\
&\geq k(A+B+s+\lambda)^{-2},
\end{aligned}$$

for s, $\lambda \geq 0$, which by integration gives that

$$\int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda} (A+B+s+\lambda)^{-1} \right.$$
$$\left. \times (BA+AB)(A+B+s+\lambda)^{-1} ds \right) d\mu(\lambda)$$
$$\geq k \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda} (A+B+s+\lambda)^{-2} \right) d\mu(\lambda).$$

Observe that, by continuous functional calculus and by (2.8) and (2.9), we get

$$\int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda} (A+B+s+\lambda)^{-2} \right) d\mu(\lambda)$$
$$= G_{w,\mu}(A+B)$$
$$= (\mathcal{DL}og(w,\mu)(A+B) - (A+B)\mathcal{D}(w)(t))(A+B)^{-2}$$

and the inequality (2.10) is proved. □

Remark 2. If A, $B > 0$ with $BA + AB \geq k \geq 0$, then we have the following refinement of (2.6)

$$\mathcal{DL}og(w,\mu)(A) + \mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A+B)$$
$$\geq k(\mathcal{DL}og(w,\mu)(A+B) - (A+B)\mathcal{D}(w)(t))(A+B)^{-2} \geq 0.$$

If we write the inequality (2.10) for the transform $\mathcal{DL}og(w_{\ell^{r-1}})$ we get for $r \in (0,1]$ that

$$A^r + B^r - (A+B)^r \geq (1-r)k(A+B)^{r-2},$$

provided A, $B > 0$ with $BA + AB \geq k$. If $k \geq 0$, then we obtain the following refinement of (2.7)

$$A^r + B^r - (A+B)^r \geq (1-r)k(A+B)^{r-2} \geq 0.$$

3. Some examples via operator monotone functions

We have the following class of examples that are of interest:

Lemma 2. *Assume that function $f\colon [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation* (1.1)*, where $b \geq 0$ and μ is a positive measure on $[0,\infty)$. Then*

$$\mathcal{D}\mathcal{L}og\,(\ell,\mu)\,(t) = F_f\,(t) - bt$$

provided the function

$$F_f\,(t) := \int_0^t \frac{f\,(s) - f\,(0)}{s} ds$$

is defined for all $t \in (0,\infty)$.

Proof. From (1.1) we have

$$\frac{f\,(s) - f\,(0)}{s} - b = \int_0^\infty \frac{\lambda}{s+\lambda} d\mu\,(\lambda) = \mathcal{D}\,(\ell,\mu)\,(s)\,,$$

where $\ell\,(\lambda) = \lambda$, $\lambda \geq 0$.

By taking the integral over s on $(0,t)$, we have

$$F_f\,(t) = \int_0^t \frac{f\,(s) - f\,(0)}{s} ds - bt = \int_0^t \mathcal{D}\,(\ell,\mu)\,(s)\,ds = \mathcal{D}\mathcal{L}og\,(\ell,\mu)\,(t)$$

for $t > 0$, and the proposition is proved. □

Remark 3. If we take $f\,(t) = \ln\,(t+a)$, for a, $t > 0$, then we have

$$F_{\ln(t+a)}\,(t) := \int_0^t \frac{\ln\,(s+a) - \ln\,(a)}{s} ds = \int_0^t \frac{1}{s} \ln\left(\frac{s}{a} + 1\right) ds.$$

If we change the variable $u = \frac{s}{a}$, then we get

$$\int_0^t \frac{1}{s} \ln\left(\frac{s}{a} + 1\right) ds = \int_0^{t/a} \frac{1}{ua} \ln\,(u+1)\,adu = \int_0^{t/a} \frac{1}{u} \ln\,(u+1)\,du$$
$$= -\operatorname{dilog}\left(\frac{t}{a} + 1\right),$$

which gives

$$F_{\ln(t+a)}(t) = -\operatorname{dilog}\left(\frac{t}{a}+1\right), \quad t>0.$$

If $f(t)=t^r$, $r\in(0,1]$, then $F_f(t) := \frac{t^r}{r}$, $t>0$.

Proposition 1. *Assume that function $f\colon[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. If $A,B>0$ with $BA+AB\geq 0$, then*

$$F_f(A)+F_f(B)\geq F_f(A+B). \tag{3.1}$$

Proof. By Lemma 2 we have for all A, $B>0$ that

$$\begin{aligned}
&\mathcal{DL}og(\ell,\mu)(A)+\mathcal{DL}og(\ell,\mu)(B)-\mathcal{DL}og(\ell,\mu)(A+B)\\
&=F_f(A)-bA+F_f(B)-bB-F_f(A+B)+b(A+B)\\
&=F_f(A)+F_f(B)-F_f(A+B).
\end{aligned}$$

By making use of (2.6) we derive the desired result (3.1). □

Proposition 2. *If A, $B>0$ with $BA+AB\geq k$, where k is a real number, then*

$$\begin{aligned}
&F_f(A)+F_f(B)-F_f(A+B)\\
&\geq k\left[F_f(A+B)-f(A+B)+f(0)\right](A+B)^{-2}.
\end{aligned}$$

If $k\geq 0$, then we have the refinement of (3.1)

$$\begin{aligned}
&F_f(A)+F_f(B)-F_f(A+B)\\
&\geq k\left[F_f(A+B)-f(A+B)+f(0)\right](A+B)^{-2}\geq 0.
\end{aligned}$$

Remark 4. If we take $f(t)=\ln(t+a)$, for a, $t>0$, then we have

$$\begin{aligned}
&\operatorname{dilog}\left(\frac{1}{a}(A+B)+1\right)-\operatorname{dilog}\left(\frac{1}{a}A+1\right)-\operatorname{dilog}\left(\frac{1}{a}B+1\right)\\
&\geq k\left[\ln a-\operatorname{dilog}\left(\frac{1}{a}(A+B)+1\right)-\ln(A+B)\right](A+B)^{-2}\geq 0
\end{aligned}$$

provided A, $B>0$ with $BA+AB\geq k\geq 0$.

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