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# JACOBSTHAL REPRESENTATION HYBRINOMIALS

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**Abstract.** Jacobsthal numbers are a special case of numbers defined recursively by the second order linear relation and for these reasons they are also named as numbers of the Fibonacci type. They have many interpretations, representations and applications in distinct areas of mathematics. In this paper we present the Jacobsthal representation hybrinomials, i.e. polynomials, which are a generalization of Jacobsthal hybrid numbers.

## 1. Introduction

Let $n \geq 0$ be an integer. Numbers defined recursively by the second order linear recurrence relation of the form

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} \quad \text{for } n \geq 2,$$

where $b_1 \geq 0$ and $b_2 \geq 0$ are integers with given non negative integers $a_0, a_1$ are named as numbers of the Fibonacci type.

For special values of $b_1$, $b_2$, $a_0$ and $a_1$ we obtain well-known recurrences which define numbers of the Fibonacci type. We list some of them

(1) Fibonacci numbers $F_n$,
$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0$, $F_1 = 1$,

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(2) Lucas numbers $L_n$,
$L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$,
(3) Jacobsthal numbers $J_n$,
$J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 2$ with $J_0 = 0$, $J_1 = 1$,
(4) Jacobsthal–Lucas numbers $j_n$,
$j_n = j_{n-1} + 2j_{n-2}$ for $n \geq 2$ with $j_0 = 2$, $j_1 = 1$.

Jacobsthal numbers and Jacobsthal–Lucas numbers were introduced in [3] and [4], respectively. A natural extension of Jacobsthal numbers is given by Jacobsthal polynomials, which were introduced by Horadam in [5] and defined as follows.

For any variable quantity $x$, the Jacobsthal polynomial $J_n(x)$ is defined as $J_n(x) = J_{n-1}(x) + 2x \cdot J_{n-2}(x)$ for $n \geq 2$ with $J_0(x) = 0$, $J_1(x) = 1$.

The Jacobsthal–Lucas polynomial $j_n(x)$ is defined as $j_n(x) = j_{n-1}(x) + 2x \cdot j_{n-2}(x)$ for $n \geq 2$ with initial terms $j_0(x) = 2$, $j_1(x) = 1$.

For $x = 1$ we obtain Jacobsthal numbers and Jacobsthal–Lucas numbers, respectively. Moreover, observe that $J_n(\frac{1}{2}) = F_n$ and $j_n(\frac{1}{2}) = L_n$.

Since $J_n(x)$ and $j_n(x)$ are defined by the second-order linear recurrence relation, so we can solve it and then we obtain direct formulas of the form

$$J_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \tag{1.1}$$

$$j_n(x) = \alpha^n(x) + \beta^n(x), \tag{1.2}$$

where $\alpha(x) = \frac{1}{2}\left(1 + \sqrt{8x+1}\right)$ and $\beta(x) = \frac{1}{2}\left(1 - \sqrt{8x+1}\right)$.

These equations are named as Binet formulas for Jacobsthal and Jacobsthal–Lucas polynomials.

Jacobsthal numbers and Jacobsthal–Lucas numbers belong to the family of numbers of the Fibonacci type which have many interesting applications not only in number theory and combinatorics also in the theory of hypercomplex numbers, see for details [11]. Jacobsthal polynomials and Jacobsthal–Lucas polynomials can be applied to different problems related to combinatorics, graph theory, algebra, see e.g. [3, 10, 12]. In the literature we can find generalized Jacobsthal sequences which were used in studing hypercomplex numbers, see for example [1].

In this paper we use following results.

Theorem 1.1 ([5]). *Let $n$ be an integer. Then*

$$j_n(x) = J_{n+1}(x) + 2x \cdot J_{n-1}(x) \quad \text{for } n \geq 1, \tag{1.3}$$

$$J_n(x) + j_n(x) = 2J_{n+1}(x) \quad \text{for } n \geq 0, \tag{1.4}$$

$$\sum_{l=0}^{n} J_l(x) = \frac{J_{n+2}(x) - 1}{2x} \quad \textit{for } n \geq 0, \tag{1.5}$$

$$\sum_{l=0}^{n} j_l(x) = \frac{j_{n+2}(x) - 1}{2x} \quad \textit{for } n \geq 0. \tag{1.6}$$

Properties of some generalizations of Jacobsthal polynomials can be found in [2, 6]. In this paper we use Jacobsthal and Jacobsthal–Lucas polynomials in the theory of hybrid numbers.

Hybrid numbers were introduced by Özdemir in [8] as a new generalization of complex, hyperbolic and dual numbers.

Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\boldsymbol{\epsilon} + d\mathbf{h},$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, $\boldsymbol{\epsilon}$, $\mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1, \quad \boldsymbol{\epsilon}^2 = 0, \quad \mathbf{h}^2 = 1, \tag{1.7}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\epsilon} + \mathbf{i}. \tag{1.8}$$

If $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\epsilon} + d_1\mathbf{h}$, and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\boldsymbol{\epsilon} + d_2\mathbf{h}$, are any two hybrid numbers then equality, addition, subtraction and multiplication by scalar are defined as follows:
equality: $\mathbf{Z}_1 = \mathbf{Z}_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\boldsymbol{\epsilon} + (d_1 + d_2)\mathbf{h}$,
subtraction: $\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\boldsymbol{\epsilon} + (d_1 - d_2)\mathbf{h}$,
multiplication by scalar $s \in \mathbb{R}$: $s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\boldsymbol{\epsilon} + sd_1\mathbf{h}$.

The hybrid numbers multiplication is defined using (1.7) and (1.8). Note that using formulas (1.7) and (1.8) we can find the product of any two hybrid units. Table 1 presents products of $\mathbf{i}$, $\boldsymbol{\epsilon}$, and $\mathbf{h}$.

Table 1. The hybrid number multiplication

| $\cdot$ | $\mathbf{i}$ | $\boldsymbol{\epsilon}$ | $\mathbf{h}$ |
|---|---|---|---|
| $\mathbf{i}$ | $-1$ | $1 - \mathbf{h}$ | $\boldsymbol{\epsilon} + \mathbf{i}$ |
| $\boldsymbol{\epsilon}$ | $\mathbf{h} + 1$ | $0$ | $-\boldsymbol{\epsilon}$ |
| $\mathbf{h}$ | $-\boldsymbol{\epsilon} - \mathbf{i}$ | $\boldsymbol{\epsilon}$ | $1$ |

Using rules given in Table 1 the multiplication of hybrid numbers can be made analogously as multiplications of algebraic expressions. For hybrid numbers details, see [8].

A special kind of hybrid numbers, namely Jacobsthal hybrid numbers and Jacobsthal–Lucas hybrid numbers, were introduced as a sequel of Fibonacci type hybrid numbers in [9] as follows.

The $n$th Jacobsthal hybrid number $JH_n$ and the $n$th Jacobsthal–Lucas hybrid number $jH_n$ are defined as

$$JH_n = J_n + \mathbf{i}J_{n+1} + \boldsymbol{\epsilon}J_{n+2} + \mathbf{h}J_{n+3}, \tag{1.9}$$

$$jH_n = j_n + \mathbf{i}j_{n+1} + \boldsymbol{\epsilon}j_{n+2} + \mathbf{h}j_{n+3}, \tag{1.10}$$

respectively.

Interesting results of Jacobsthal and Jacobsthal–Lucas hybrid numbers obtained recently can be found in [12].

The concept of hybrinomials first appears in [13] with respect to Fibonacci and Lucas hybrid numbers and next applied for Pell and Pell–Lucas hybrid numbers, see [7]. In the book [11] we defined Jacobsthal and Jacobsthal–Lucas hybrinomials and we presented some results for them, however no proofs were given. This article is a complementary of our results mentioned in [11].

For $n \geq 0$ Jacobsthal and Jacobsthal–Lucas hybrinomials are defined by

$$JH_n(x) = J_n(x) + \mathbf{i}J_{n+1}(x) + \boldsymbol{\epsilon}J_{n+2}(x) + \mathbf{h}J_{n+3}(x) \tag{1.11}$$

and

$$jH_n(x) = j_n(x) + \mathbf{i}j_{n+1}(x) + \boldsymbol{\epsilon}j_{n+2}(x) + \mathbf{h}j_{n+3}(x), \tag{1.12}$$

where $J_n(x)$ is the $n$th Jacobsthal polynomial, $j_n(x)$ is the $n$-th Jacobsthal–Lucas polynomial and $\mathbf{i}$, $\boldsymbol{\epsilon}$, $\mathbf{h}$ are hybrid units satisfying (1.7) and (1.8).

For $x = 1$ we obtain Jacobsthal hybrid numbers and Jacobsthal–Lucas hybrid numbers, respectively.

## 2. Properties of Jacobsthal hybrinomials

THEOREM 2.1. *For any variable quantity $x$, we have*

$$JH_n(x) = JH_{n-1}(x) + 2x \cdot JH_{n-2}(x) \quad \textit{for } n \geq 2 \tag{2.1}$$

*with* $JH_0(x) = \mathbf{i} + \boldsymbol{\epsilon} + \mathbf{h} \cdot (2x+1)$ *and* $JH_1(x) = 1 + \mathbf{i} + \boldsymbol{\epsilon} \cdot (2x+1) + \mathbf{h} \cdot (4x+1)$.

PROOF. If $n=2$ we have

$$\begin{aligned}
JH_2(x) &= JH_1(x) + 2x\cdot FH_0(x)\\
&= 1+\mathbf{i}+\boldsymbol{\epsilon}\cdot(2x+1)+\mathbf{h}\cdot(4x+1)\\
&\quad + 2x\cdot(\mathbf{i}+\boldsymbol{\epsilon}+\mathbf{h}\cdot(2x+1))\\
&= 1+\mathbf{i}\cdot(2x+1)+\boldsymbol{\epsilon}\cdot(4x+1)+\mathbf{h}\cdot(4x^2+6x+1)\\
&= J_2(x)+\mathbf{i}J_3(x)+\boldsymbol{\epsilon}J_4(x)+\mathbf{h}J_5(x).
\end{aligned}$$

If $n \geq 3$ then using the definition of Jacobsthal polynomials we have

$$\begin{aligned}
JH_n(x) &= J_n(x)+\mathbf{i}J_{n+1}(x)+\boldsymbol{\epsilon}J_{n+2}(x)+\mathbf{h}J_{n+3}(x)\\
&= (J_{n-1}(x)+2x\cdot J_{n-2}(x))+\mathbf{i}(J_n(x)+2x\cdot J_{n-1}(x))\\
&\quad+\boldsymbol{\epsilon}(J_{n+1}(x)+2x\cdot J_n(x))+\mathbf{h}(J_{n+2}(x)+2x\cdot J_{n+1}(x))\\
&= J_{n-1}(x)+\mathbf{i}\cdot J_n(x)+\boldsymbol{\epsilon}\cdot J_{n+1}(x)+\mathbf{h}\cdot J_{n+2}(x)\\
&\quad+2x\cdot(J_{n-2}(x)+\mathbf{i}\cdot J_{n-1}(x)+\boldsymbol{\epsilon}\cdot J_n(x)+\mathbf{h}\cdot J_{n+1}(x))\\
&= JH_{n-1}(x)+2x\cdot JH_{n-2}(x),
\end{aligned}$$

which ends the proof. □

In the same way one can easily prove the next theorem.

THEOREM 2.2. *For any variable quantity $x$, we have*

$$jH_n(x) = jH_{n-1}(x) + 2x\cdot jH_{n-2}(x) \quad \textit{for } n \geq 2$$

*with* $jH_0(x) = 2+\mathbf{i}+\boldsymbol{\epsilon}\cdot(4x+1)+\mathbf{h}\cdot(6x+1)$ *and* $jH_1(x) = 1+\mathbf{i}\cdot(4x+1) +\boldsymbol{\epsilon}\cdot(6x+1)+\mathbf{h}\cdot(8x^2+8x+1)$.

Now we give identities for Jacobsthal and Jacobsthal–Lucas hybrinomials which relate to Theorem 1.1.

THEOREM 2.3. *Let $n \geq 1$ be an integer. Then*

$$jH_n(x) = JH_{n+1}(x) + 2x\cdot JH_{n-1}(x).$$

PROOF. Using (1.3) we have

$$\begin{aligned}
&JH_{n+1}(x) + 2x \cdot JH_{n-1}(x) \\
&\quad = J_{n+1}(x) + \mathbf{i}J_{n+2}(x) + \boldsymbol{\epsilon}J_{n+3}(x) + \mathbf{h}J_{n+4}(x) \\
&\qquad + 2x \cdot (J_{n-1}(x) + \mathbf{i}J_n(x) + \boldsymbol{\epsilon}J_{n+1}(x) + \mathbf{h}J_{n+2}(x)) \\
&\quad = (J_{n+1}(x) + 2x \cdot J_{n-1}(x)) + \mathbf{i}(J_{n+2}(x) + 2x \cdot J_n(x)) \\
&\qquad + \boldsymbol{\epsilon}(J_{n+3}(x) + 2x \cdot J_{n+1}(x)) + \mathbf{h}(J_{n+4}(x) + 2x \cdot J_{n+2}(x)) \\
&\quad = j_n(x) + \mathbf{i}j_{n+1}(x) + \boldsymbol{\epsilon}j_{n+2}(x) + \mathbf{h}j_{n+3}(x) = jH_n(x).
\end{aligned}$$

□

THEOREM 2.4. *Let $n \geq 0$ be an integer. Then*

$$JH_n(x) + jH_n(x) = 2JH_{n+1}(x).$$

PROOF. Using (1.4) and proceeding in the same way as in Theorem 2.3 the result follows. □

THEOREM 2.5. *Let $n \geq 0$ be an integer. Then*

$$\sum_{l=0}^{n} JH_l(x) = \frac{JH_{n+2}(x) - JH_1(x)}{2x}.$$

PROOF. For an integer $n \geq 0$ we have

$$\begin{aligned}
\sum_{l=0}^{n} JH_l(x) &= JH_0(x) + JH_1(x) + \ldots + JH_n(x) \\
&= J_0(x) + \mathbf{i}J_1(x) + \boldsymbol{\epsilon}J_2(x) + \mathbf{h}J_3(x) \\
&\quad + J_1(x) + \mathbf{i}J_2(x) + \boldsymbol{\epsilon}J_3(x) + \mathbf{h}J_4(x) + \cdots \\
&\quad + J_n(x) + \mathbf{i}J_{n+1}(x) + \boldsymbol{\epsilon}J_{n+2}(x) + \mathbf{h}J_{n+3}(x) \\
&= J_0(x) + J_1(x) + \cdots + J_n(x) \\
&\quad + \mathbf{i}(J_1(x) + J_2(x) + \cdots + J_{n+1}(x) + J_0(x) - J_0(x)) \\
&\quad + \boldsymbol{\epsilon}(J_2(x) + J_3(x) + \cdots + J_{n+2}(x) + J_0(x) + J_1(x) \\
&\quad - J_0(x) - J_1(x)) \\
&\quad + \mathbf{h}(J_3(x) + J_4(x) + \cdots + J_{n+3}(x) + J_0(x) + J_1(x) + J_2(x) \\
&\quad - J_0(x) - J_1(x) - J_2(x))
\end{aligned}$$

and using (1.5) we have

$$\begin{aligned}\sum_{l=0}^{n} JH_l(x) &= \frac{J_{n+2}(x)-1}{2x} + \mathbf{i}\left(\frac{J_{n+3}(x)-1}{2x} - J_0(x)\right)\\ &\quad + \boldsymbol{\epsilon}\left(\frac{J_{n+4}(x)-1}{2x} - J_0(x) - J_1(x)\right)\\ &\quad + \mathbf{h}\left(\frac{J_{n+5}(x)-1}{2x} - J_0(x) - J_1(x) - J_2(x)\right)\\ &= \frac{J_{n+2}(x)-1}{2x} + \mathbf{i}\left(\frac{J_{n+3}(x)-1}{2x}\right)\\ &\quad + \boldsymbol{\epsilon}\left(\frac{J_{n+4}(x)-(1+2x)}{2x}\right) + \mathbf{h}\left(\frac{J_{n+5}(x)-(1+4x)}{2x}\right),\end{aligned}$$

which completes the proof. □

THEOREM 2.6. *Let $n \geq 0$ be an integer. Then*

$$\sum_{l=0}^{n} jH_l(x) = \frac{jH_{n+2}(x) - jH_1(x)}{2x}.$$

PROOF. Using (1.6) and proceeding in the same way as in Theorem 2.5 the result follows. □

Next we shall give the generating function for Jacobsthal hybrinomials.

THEOREM 2.7. *The generating function for the Jacobsthal hybrinomial sequence $\{JH_n(x)\}$ is*

$$G(t) = \frac{\mathbf{i} + \boldsymbol{\epsilon} + \mathbf{h}\cdot(2x+1) + (1 + \boldsymbol{\epsilon}\cdot(2x) + \mathbf{h}\cdot(2x))t}{1 - t - 2xt^2}.$$

PROOF. Assume that the generating function of the Jacobsthal hybrinomial sequence $\{JH_n(x)\}$ has the form $G(t) = \sum_{n=0}^{\infty} JH_n(x)t^n$. Then

$$G(t) = JH_0(x) + JH_1(x)t + JH_2(x)t^2 + \dots$$

Multiplying the above equality on both sides by $-t$ and then by $-2xt^2$ we obtain

$$-G(t)t = -JH_0(x)t - JH_1(x)t^2 - JH_2(x)t^3 - \dots$$

$$-G(t)\cdot(2x)t^2 = -JH_0(x)\cdot(2x)t^2 - JH_1(x)\cdot(2x)t^3 - JH_2(x)\cdot(2x)t^4 - \dots$$

By adding these three equalities above, we will get

$$G(t)(1-t-2xt^2) = JH_0(x) + (JH_1(x) - JH_0(x))t$$

since $JH_n(x) = JH_{n-1}(x) + 2x\cdot JH_{n-2}(x)$ (see (2.1)) and the coefficients of $t^n$ for $n \geq 2$ are equal to zero. Moreover, $JH_0(x) = \mathbf{i} + \boldsymbol{\epsilon} + \mathbf{h}\cdot(2x+1)$, $JH_1(x) - JH_0(x) = 1 + \boldsymbol{\epsilon}\cdot(2x) + \mathbf{h}\cdot(2x)$. □

In the same way we obtain the generating function $g(t)$ for Jacobsthal–Lucas hybrinomials.

THEOREM 2.8. *The generating function for the Jacobsthal–Lucas hybrinomial sequence* $\{jH_n(x)\}$ *is*

$$g(t) = \frac{jH_0(x) + (jH_1(x) - jH_0(x))t}{1-t-2xt^2},$$

*where* $jH_0(x) = 2 + \mathbf{i} + \boldsymbol{\epsilon}\cdot(4x+1) + \mathbf{h}\cdot(6x+1)$ *and* $jH_1(x) - jH_0(x) = -1 + \mathbf{i}\cdot(4x) + \boldsymbol{\epsilon}\cdot(2x) + \mathbf{h}\cdot(8x^2+2x)$.

Now we give so called Binet formulas for Jacobsthal and Jacobsthal–Lucas hybrinomials being their direct formulas.

THEOREM 2.9. *Let* $n \geq 0$ *be an integer. Then*

$$\begin{aligned} JH_n(x) = {} & \frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\left(1 + \mathbf{i}\alpha(x) + \boldsymbol{\epsilon}\alpha^2(x) + \mathbf{h}\alpha^3(x)\right) \\ & - \frac{\beta^n(x)}{\alpha(x)-\beta(x)}\left(1 + \mathbf{i}\beta(x) + \boldsymbol{\epsilon}\beta^2(x) + \mathbf{h}\beta^3(x)\right), \end{aligned} \tag{2.2}$$

*where* $\alpha(x) = \frac{1}{2}\left(1+\sqrt{8x+1}\right)$ *and* $\beta(x) = \frac{1}{2}\left(1-\sqrt{8x+1}\right)$.

PROOF. Using (1.1), (1.9) and (1.11) we have

$$\begin{aligned} JH_n(x) &= J_n(x) + \mathbf{i}J_{n+1}(x) + \boldsymbol{\epsilon}J_{n+2}(x) + \mathbf{h}J_{n+3}(x) \\ &= \frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)} + \mathbf{i}\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)} \\ &\quad + \boldsymbol{\epsilon}\frac{\alpha^{n+2}(x)-\beta^{n+2}(x)}{\alpha(x)-\beta(x)} + \mathbf{h}\frac{\alpha^{n+3}(x)-\beta^{n+3}(x)}{\alpha(x)-\beta(x)} \end{aligned}$$

and after calculations the result follows. □

In the same way, using (1.2), (1.10) and (1.12), we obtain the Binet formula for Jacobsthal–Lucas hybrinomials.

THEOREM 2.10. *Let $n \geq 0$ be an integer. Then*

$$jH_n(x) = \alpha^n(x)\left(1 + \mathbf{i}\alpha(x) + \boldsymbol{\epsilon}\alpha^2(x) + \mathbf{h}\alpha^3(x)\right) + \beta^n(x)\left(1 + \mathbf{i}\beta(x) + \boldsymbol{\epsilon}\beta^2(x) + \mathbf{h}\beta^3(x)\right), \tag{2.3}$$

*where* $\alpha(x) = \frac{1}{2}\left(1 + \sqrt{8x+1}\right)$ *and* $\beta(x) = \frac{1}{2}\left(1 - \sqrt{8x+1}\right)$.

Now we will give some identities which will be named as Catalan, Cassini and d'Ocagne identities for the Jacobsthal and Jacobsthal–Lucas hybrinomials since they are analogous to Catalan, Cassini and d'Ocagne identities for the classical Fibonacci numbers. These identities can be proved using Binet formulas.

For simplicity of notation let

$$\Delta(x) = \alpha(x) - \beta(x),$$

$$\hat{\alpha}(x) = 1 + \mathbf{i}\alpha(x) + \boldsymbol{\epsilon}\alpha^2(x) + \mathbf{h}\alpha^3(x),$$

$$\hat{\beta}(x) = 1 + \mathbf{i}\beta(x) + \boldsymbol{\epsilon}\beta^2(x) + \mathbf{h}\beta^3(x).$$

Then we can write (2.2) and (2.3) as

$$JH_n(x) = \frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x)$$

and

$$jH_n(x) = \alpha^n(x)\hat{\alpha}(x) + \beta^n(x)\hat{\beta}(x),$$

respectively. Moreover, $\alpha(x)\cdot\beta(x) = -2x$ and $\Delta^2(x) = 8x+1$.

THEOREM 2.11 (Catalan identity for Jacobsthal hybrinomials). *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$JH_{n-r}(x)\cdot JH_{n+r}(x) - \left(JH_n(x)\right)^2$$
$$= \frac{(-2x)^n}{8x+1}\hat{\alpha}(x)\hat{\beta}(x)\left(1 - \frac{\beta^r(x)}{\alpha^r(x)}\right) + \frac{(-2x)^n}{8x+1}\hat{\beta}(x)\hat{\alpha}(x)\left(1 - \frac{\alpha^r(x)}{\beta^r(x)}\right).$$

Proof. Let $n, r$ be as in the statement of the theorem. Then

$$\begin{aligned}
&JH_{n-r}(x) \cdot JH_{n+r}(x) - (JH_n(x))^2 \\
&= \left(\frac{\alpha^{n-r}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n-r}(x)}{\Delta(x)}\hat{\beta}(x)\right) \cdot \left(\frac{\alpha^{n+r}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n+r}(x)}{\Delta(x)}\hat{\beta}(x)\right) \\
&\quad - \left(\frac{\alpha^{n}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n}(x)}{\Delta(x)}\hat{\beta}(x)\right) \cdot \left(\frac{\alpha^{n}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n}(x)}{\Delta(x)}\hat{\beta}(x)\right) \\
&= -\frac{\alpha^{n-r}(x)}{\Delta(x)}\hat{\alpha}(x)\frac{\beta^{n+r}(x)}{\Delta(x)}\hat{\beta}(x) - \frac{\beta^{n-r}(x)}{\Delta(x)}\hat{\beta}(x)\frac{\alpha^{n+r}(x)}{\Delta(x)}\hat{\alpha}(x) \\
&\quad + \frac{\alpha^{n}(x)}{\Delta(x)}\hat{\alpha}(x)\frac{\beta^{n}(x)}{\Delta(x)}\hat{\beta}(x) + \frac{\beta^{n}(x)}{\Delta(x)}\hat{\beta}(x)\frac{\alpha^{n}(x)}{\Delta(x)}\hat{\alpha}(x) \\
&= -\frac{\alpha^{n-r}(x)\beta^{n+r}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{\beta^{n-r}(x)\alpha^{n+r}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x) \\
&\quad + \frac{\alpha^{n}(x)\beta^{n}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x) + \frac{\beta^{n}(x)\alpha^{n}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x) \\
&= \frac{\alpha^{n}(x)\beta^{n}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)\left(1 - \frac{\beta^r(x)}{\alpha^r(x)}\right) \\
&\quad + \frac{\alpha^{n}(x)\beta^{n}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)\left(1 - \frac{\alpha^r(x)}{\beta^r(x)}\right) \\
&= \frac{(-2x)^n}{8x+1}\hat{\alpha}(x)\hat{\beta}(x)\left(1 - \frac{\beta^r(x)}{\alpha^r(x)}\right)\frac{(-2x)^n}{8x+1}\hat{\beta}(x)\hat{\alpha}(x)\left(1 - \frac{\alpha^r(x)}{\beta^r(x)}\right),
\end{aligned}$$

which completes the proof. □

In the same way one can easily prove the next theorem, which gives Catalan identity for Jacobsthal–Lucas hybrinomials.

Theorem 2.12 (Catalan identity for Jacobsthal–Lucas hybrinomials). *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$\begin{aligned}
&jH_{n-r}(x) \cdot jH_{n+r}(x) - (jH_n(x))^2 \\
&\qquad = (-2x)^n\hat{\alpha}(x)\hat{\beta}(x)\left(\frac{\beta^r(x)}{\alpha^r(x)} - 1\right) + (-2x)^n\hat{\beta}(x)\hat{\alpha}(x)\left(\frac{\alpha^r(x)}{\beta^r(x)} - 1\right).
\end{aligned}$$

Note that for $r = 1$ we get Cassini identities for Jacobsthal and Jacobsthal–Lucas hybrinomials. Moreover, for $r = 1$ we have

$$1 - \frac{\beta(x)}{\alpha(x)} = \frac{\alpha(x) - \beta(x)}{\alpha(x)} = \frac{\Delta(x)}{\alpha(x)} \quad \text{and} \quad 1 - \frac{\alpha(x)}{\beta(x)} = \frac{\beta(x) - \alpha(x)}{\beta(x)} = -\frac{\Delta(x)}{\beta(x)}.$$

Corollary 2.13 (Cassini identities for Jacobsthal and Jacobsthal–Lucas hybrinomials). *Let $n \geq 1$ be an integer. Then*

$$\begin{aligned} &JH_{n-1}(x) \cdot JH_{n+1}(x) - (JH_n(x))^2 \\ &\qquad = \frac{(-2x)^{n-1}\beta(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{(-2x)^{n-1}\alpha(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x), \\ &jH_{n-1}(x) \cdot jH_{n+1}(x) - (jH_n(x))^2 \\ &\qquad = (-2x)^n \hat{\alpha}(x)\hat{\beta}(x)\left(\frac{\beta(x)}{\alpha(x)} - 1\right) + (-2x)^n \hat{\beta}(x)\hat{\alpha}(x)\left(\frac{\alpha(x)}{\beta(x)} - 1\right). \end{aligned}$$

Theorem 2.14 (d'Ocagne identity for Jacobsthal hybrinomials). *Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then*

$$\begin{aligned} &JH_m(x) \cdot JH_{n+1}(x) - JH_{m+1}(x) \cdot JH_n(x) \\ &\qquad = \frac{(-2x)^n \alpha^{m-n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{(-2x)^n \beta^{m-n}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x). \end{aligned}$$

Proof. For integers $m \geq 0$, $n \geq 0$ and $m \geq n$ we have

$$\begin{aligned} &JH_m(x) \cdot JH_{n+1}(x) - JH_{m+1}(x) \cdot JH_n(x) \\ &= \left(\frac{\alpha^m(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^m(x)}{\Delta(x)}\hat{\beta}(x)\right) \cdot \left(\frac{\alpha^{n+1}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n+1}(x)}{\Delta(x)}\hat{\beta}(x)\right) \\ &\quad - \left(\frac{\alpha^{m+1}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{m+1}(x)}{\Delta(x)}\hat{\beta}(x)\right) \cdot \left(\frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x)\right) \\ &= \frac{\alpha^{m+n+1}(x)}{\Delta^2(x)}\hat{\alpha}^2(x) - \frac{\alpha^m(x)\beta^{n+1}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{\alpha^{n+1}(x)\beta^m(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x) \\ &\quad + \frac{\beta^{m+n+1}(x)}{\Delta^2(x)}\hat{\beta}^2(x) - \frac{\alpha^{m+1+n}(x)}{\Delta^2(x)}\hat{\alpha}^2(x) + \frac{\alpha^{m+1}(x)\beta^n(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x) \\ &\quad + \frac{\alpha^n(x)\beta^{m+1}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x) - \frac{\beta^{m+1+n}(x)}{\Delta^2(x)}\hat{\beta}^2(x) \end{aligned}$$

$$= \frac{\alpha^{m+1}(x)\beta^n(x) - \alpha^m(x)\beta^{n+1}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)$$

$$+ \frac{\alpha^n(x)\beta^{m+1}(x) - \alpha^{n+1}(x)\beta^m(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)$$

$$= \frac{\alpha^m(x)\beta^n(x)(\alpha(x) - \beta(x))}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)$$

$$+ \frac{\alpha^n(x)\beta^m(x)(\beta(x) - \alpha(x))}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)$$

$$= \frac{\alpha^m(x)\beta^n(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{\alpha^n(x)\beta^m(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x)$$

$$= \frac{(-2x)^n\alpha^{m-n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{(-2x)^n\beta^{m-n}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).$$

Thus the theorem is proved. □

In the same way we can prove next theorems.

THEOREM 2.15 (d'Ocagne identity for Jacobsthal–Lucas hybrinomials). *Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then*

$$jH_m(x) \cdot jH_{n+1}(x) - jH_{m+1}(x) \cdot jH_n(x)$$

$$= (-2x)^n\beta^{m-n}(x)\Delta(x)\hat{\beta}(x)\hat{\alpha}(x) - (-2x)^n\alpha^{m-n}(x)\Delta(x)\hat{\alpha}(x)\hat{\beta}(x).$$

THEOREM 2.16. *Let $m \geq 0$, $n \geq 0$ be integers. Then*

$$JH_m(x) \cdot jH_n(x) - jH_m(x) \cdot JH_n(x)$$

$$= \frac{2(-2x)^n\alpha^{m-n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{2(-2x)^n\beta^{m-n}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).$$

We will give the matrix representation of Jacobsthal hybrinomials.

THEOREM 2.17. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} JH_{n+2}(x) & JH_{n+1}(x) \\ JH_{n+1}(x) & JH_n(x) \end{bmatrix} = \begin{bmatrix} JH_2(x) & JH_1(x) \\ JH_1(x) & JH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2x & 0 \end{bmatrix}^n.$$

Proof. (by induction on $n$)
If $n = 0$ then assuming that the matrix to the power 0 is the identity matrix the result is obvious. Now assume that for any $n \geq 0$ holds

$$\begin{bmatrix} JH_{n+2}(x) & JH_{n+1}(x) \\ JH_{n+1}(x) & JH_n(x) \end{bmatrix} = \begin{bmatrix} JH_2(x) & JH_1(x) \\ JH_1(x) & JH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2x & 0 \end{bmatrix}^n .$$

We shall show that

$$\begin{bmatrix} JH_{n+3}(x) & JH_{n+2}(x) \\ JH_{n+2}(x) & JH_{n+1}(x) \end{bmatrix} = \begin{bmatrix} JH_2(x) & JH_1(x) \\ JH_1(x) & JH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2x & 0 \end{bmatrix}^{n+1} .$$

By simple calculation using induction's hypothesis we have

$$\begin{bmatrix} JH_2(x) & JH_1(x) \\ JH_1(x) & JH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2x & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 & 1 \\ 2x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} JH_{n+2}(x) & JH_{n+1}(x) \\ JH_{n+1}(x) & JH_n(x) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} JH_{n+2}(x) + 2x \cdot JH_{n+1}(x) & JH_{n+2}(x) \\ JH_{n+1}(x) + 2x \cdot JH_n(x) & JH_{n+1}(x) \end{bmatrix} = \begin{bmatrix} JH_{n+3}(x) & JH_{n+2}(x) \\ JH_{n+2}(x) & JH_{n+1}(x) \end{bmatrix},$$

which ends the proof. □

In the same way we obtain the matrix representation for Jacobsthal–Lucas hybrinomials.

Theorem 2.18. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} jH_{n+2}(x) & jH_{n+1}(x) \\ jH_{n+1}(x) & jH_n(x) \end{bmatrix} = \begin{bmatrix} jH_2(x) & jH_1(x) \\ jH_1(x) & jH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2x & 0 \end{bmatrix}^n .$$

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