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HYPO-q-NORMS ON A CARTESIAN PRODUCT OF ALGEBRAS OF OPERATORS ON BANACH SPACES

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Abstract. In this paper we consider the hypo-q-operator norm and hypo-q-numerical radius on a Cartesian product of algebras of bounded linear operators on Banach spaces. A representation of these norms in terms of semi-inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy-Buniakowski-Schwarz inequality are also given.

1. Introduction

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. On $\mathbb{K}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit ball

$$\mathbb{B}\left(\|\cdot\|_n\right) := \left\{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n \middle|\, \|\lambda\|_n \leq 1\right\}.$$

As an example of such norms we should mention the usual *p-norms*

$$\|\lambda\|_{n,p} := \begin{cases} \max\left\{|\lambda_1|, \ldots, |\lambda_n|\right\} & \text{if } \ p = \infty, \\ \left(\sum_{k=1}^{n} |\lambda_k|^p\right)^{\frac{1}{p}} & \text{if } \ p \in [1, \infty). \end{cases}$$

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The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\lambda\|_{n,2} = \left(\sum_{k=1}^{n} |\lambda_k|^2 \right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following *p-norms*:

$$\|\mathbf{x}\|_{n,p} := \begin{cases} \max\{\|x_1\|, \dots, \|x_n\|\} & \text{if } p = \infty, \\ \left(\sum_{k=1}^{n} \|x_k\|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \end{cases}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

Following [6], for a given norm $\|\cdot\|_n$ on $\mathbb{K}^n$, we define the functional $\|\cdot\|_{h,n} : E^n \to [0, \infty)$ given by

$$\|\mathbf{x}\|_{h,n} := \sup_{\lambda \in B(\|\cdot\|_n)} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.1}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

It is easy to see, by the properties of the norm $\|\cdot\|$, that:

(i) $\|\mathbf{x}\|_{h,n} \geq 0$ for any $\mathbf{x} \in E^n$,
(ii) $\|\mathbf{x} + \mathbf{y}\|_{h,n} \leq \|\mathbf{x}\|_{h,n} + \|\mathbf{y}\|_{h,n}$ for any $\mathbf{x}$, $\mathbf{y} \in E^n$,
(iii) $\|\alpha\mathbf{x}\|_{h,n} = |\alpha| \|\mathbf{x}\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $\mathbf{x} \in E^n$,

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n.

We observe that $\|\mathbf{x}\|_{h,n} = 0$ if and only if $\sum_{j=1}^{n} \lambda_j x_j = 0$ for any $(\lambda_1, \dots, \lambda_n) \in B(\|\cdot\|_n)$. Since $(0, \dots, 1, ..., 0) \in B(\|\cdot\|_n)$ then the semi-norm $\|\cdot\|_{h,n}$ generated by $\|\cdot\|_n$ is a *norm* on E^n.

If by $\mathbb{B}_{n,p}$ with $p \in [1, \infty]$ we denote the balls generated by the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{K}^n$, then we can obtain the following *hypo-q-norms* on E^n :

$$\|\mathbf{x}\|_{h,n,q} := \sup_{\lambda \in \mathbb{B}_{n,p}} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.2}$$

with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

For $p = 2$, we have the Euclidean ball in $\mathbb{K}^n$, which we denote by $\mathbb{B}_n$, $\mathbb{B}_n = \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n \,\middle|\, \sum_{i=1}^n |\lambda_i|^2 \le 1 \right\}$ that generates the *hypo-Euclidean norm* on E^n, i.e.,

$$\|\mathbf{x}\|_{h,e} := \sup_{\lambda \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|.$$

Moreover, if $E = H$, where H is an inner product space over $\mathbb{K}$, then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$\|\mathbf{x}\|_e := \sup_{\lambda \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|.$$

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and $n \in \mathbb{N}$, $n \ge 1$. In the Cartesian product $H^n := H \times \cdots \times H$, for the n-tuples of vectors $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in H^n$, we can define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^n \langle x_j, y_j \rangle, \quad \mathbf{x},\ \mathbf{y} \in H^n,$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n, i.e.,

$$\|\mathbf{x}\|_2 := \left(\sum_{j=1}^n \|x_j\|^2 \right)^{\frac{1}{2}}, \quad \mathbf{x} \in H^n.$$

The following result established in [6] connects the usual Euclidean norm $\|\cdot\|_2$ with the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 1.1 (Dragomir, 2007, [6]). *For any $\mathbf{x} \in H^n$ we have the inequalities*

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_e \le \|\mathbf{x}\|_2,$$

i.e., $\|\cdot\|_2$ and $\|\cdot\|_e$ are equivalent norms on H^n.

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 1.2 (Dragomir, 2007, [6]). *For any* $\mathbf{x} \in H^n$ *with* $\mathbf{x} = (x_1, \ldots, x_n)$*, we have*

$$\|\mathbf{x}\|_e = \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle x, x_j \rangle|^2 \right)^{\frac{1}{2}}.$$

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. We denote by E^* its dual space endowed with the norm $\|\cdot\|$ defined by

$$\|f\| := \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|u\|=1} |f(u)| < \infty, \text{ where } f \in E^*.$$

The following representation result for the *hypo-q-norms* on E^n plays a key role in obtaining different bounds for these norms (see [7]):

Theorem 1.3 (Dragomir, 2017, [7]). *Let* $(E, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$*. For any* $\mathbf{x} \in E^n$ *with* $\mathbf{x} = (x_1, \ldots, x_n)$*, we have*

$$\|\mathbf{x}\|_{h,n,q} = \sup_{\|f\|=1} \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^q \right)^{1/q} \right\}$$

where $q \geq 1$*, and*

$$\|\mathbf{x}\|_{h,n,\infty} = \|\mathbf{x}\|_{n,\infty} = \max_{j \in \{1,\ldots,n\}} \|x_j\|.$$

We have the following inequalities of interest:

Corollary 1.4. *With the assumptions of Theorem* 1.3 *we have for* $q \geq 1$ *that*

$$\frac{1}{n^{1/q}} \|\mathbf{x}\|_{n,q} \leq \|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{n,q}$$

for any any $\mathbf{x} \in E^n$*.*

We have for $r \geq q \geq 1$ *that*

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|\mathbf{x}\|_{h,n,r}$$

for any $\mathbf{x} \in E^n$*.*

In this paper we introduce the hypo-q-operator norms and hypo-q-numerical radius on a Cartesian product of algebras of bounded linear operators on Banach spaces. A representation of these norms in terms of semi-inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy–Buniakowski–Schwarz inequality are also given.

2. Semi-inner products and preliminary results

In what follows, we assume that E is a linear space over the real or complex number field $\mathbb{K}$.

The following concept was introduced in 1961 by G. Lumer [11] but the main properties of it were discovered by J. R. Giles [9], P. L. Papini [17], P. M. Miličić [12]–[14], I. Roşca [18], B. Nath [16] and others (see also [3]).

In this section we give the definition of this concept and point out the main facts which are derived directly from the definition.

Definition 2.1. The mapping $[\cdot,\cdot] : E \times E \to \mathbb{K}$ will be called the *semi-inner product in the sense of Lumer-Giles* or L-G-*s.i.p.*, for short, if the following properties are satisfied:

(i) $[x+y,z] = [x,z] + [y,z]$ for all x, y, $z \in E$,
(ii) $[\lambda x, y] = \lambda [x,y]$ for all x, $y \in E$ and λ a scalar in $\mathbb{K}$,
(iii) $[x,x] \geq 0$ for all $x \in E$ and $[x,x] = 0$ implies that $x = 0$,
(iv) $|[x,y]|^2 \leq [x,x][y,y]$ (Schwarz's inequality) for all x, $y \in E$,
(v) $[x,\lambda y] = \bar{\lambda}[x,y]$ for all x, $y \in E$ and λ a scalar in $\mathbb{K}$.

The following result collects some fundamental facts concerning the connection between the semi-inner products and norms.

Proposition 2.2. *Let E be a linear space and $[\cdot,\cdot]$ a L-G-s.i.p on E. Then the following statements are true:*

(i) *The mapping $E \ni x \xrightarrow{\|\cdot\|} [x,x]^{\frac{1}{2}} \in \mathbb{R}_+$ is a norm on E.*
(ii) *For every $y \in E$ the functional $E \ni x \xrightarrow{f_y} [x,y] \in \mathbb{K}$ is a continuous linear functional on E endowed with the norm generated by the L-G-s.i.p. Moreover, one has the equality $\|f_y\| = \|y\|$.*

Definition 2.3. The mapping $J : E \to 2^{E^*}$, where E^* is the dual space of E, given by:

$$J(x) := \{x^* \in E^* | \langle x^*, x\rangle = \|x^*\| \|x\|, \ \ \|x^*\| = \|x\|\}, \ \ x \in E,$$

will be called the *normalised duality mapping* of normed linear space $(E, \|\cdot\|)$.

Definition 2.4. A mapping $\tilde{J} : E \to E^*$ will be called *a section* of normalised duality mapping if $\tilde{J}(x) \in J(x)$ for all x in E.

The following theorem due to I. Roşca ([18]) establishes a natural connection between the normalised duality mapping and the semi-inner products in the sense of Lumer-Giles.

Theorem 2.5. *Let $(E, \|\cdot\|)$ be a normed space. Then every L-G-s.i.p. which generates the norm $\|\cdot\|$ is of the form*

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \quad \textit{for all } x, y \textit{ in } E,$$

where $\tilde{J}$ is a section of the normalised duality mapping.

The following proposition is a natural consequence of Roşca's result.

Proposition 2.6. *Let $(E, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) *E is smooth.*
(ii) *There exists a unique L-G-s.i.p. which generates the norm $\|\cdot\|$.*

We need the following lemma holding for n-tuples of complex numbers:

Lemma 2.7. *Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{C}^n$. If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, or $p = 1$, $q = \infty$ or $p = \infty, q = 1$, then*

$$\sup_{\|\alpha\|_{n,p} \le 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| = \|\beta\|_{n,q} . \tag{2.1}$$

The proof follows by using Hölder's discrete inequality and its sharpness for the three cases under consideration and we omit the details.

THEOREM 2.8. *Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$ and $[\cdot,\cdot]$ a L-G-s.i.p on E that generates the norm $\|\cdot\|$, i.e. $[x,x]^{1/2} = \|x\|$. For any $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$, we have*

$$\|\mathbf{x}\|_{h,n,q} = \sup_{\|u\|=1} \left\{ \left(\sum_{j=1}^{n} |[x_j, u]|^q \right)^{1/q} \right\}, \tag{2.2}$$

where $q \geq 1$.

PROOF. If $[\cdot,\cdot]$ is a L-G-s.i.p. that generates the norm $\|\cdot\|$, then

$$\sup_{\|u\|=1} |[x,u]| = \|x\| \text{ for any } x \in X. \tag{2.3}$$

Indeed, if $x = 0$ the equality is obvious. If $x \neq 0$, then by Schwarz's inequality we have

$$|[x,u]| \leq \|x\| \|u\| \text{ for any } u \in X.$$

By taking the supremum in this inequality we have

$$\sup_{\|u\|=1} |[x,u]| \leq \|x\|.$$

On the other hand by taking $u_0 := \frac{x}{\|x\|}$ we have that $\|u_0\| = 1$ and since

$$\sup_{\|u\|=1} |[x,u]| \geq |[x,u_0]| = \left| \left[x, \frac{x}{\|x\|} \right] \right| = \frac{\|x\|^2}{\|x\|} = \|x\|,$$

then we get the desired equality (2.3).

Assume that $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by the definition (1.2) and representation (2.3) we have

$$\begin{aligned} \|\mathbf{x}\|_{h,n,q} &:= \sup_{|\alpha|_p \leq 1} \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| = \sup_{|\alpha|_p \leq 1} \left(\sup_{\|u\|=1} \left| \left[\left(\sum_{j=1}^{n} \alpha_j x_j \right), u \right] \right| \right) \\ &= \sup_{\|u\|=1} \left(\sup_{|\alpha|_p \leq 1} \left| \sum_{j=1}^{n} \alpha_j [x_j, u] \right| \right) = \sup_{\|u\|=1} \left(\sum_{j=1}^{n} |[x_j, u]|^q \right)^{1/q}, \end{aligned} \tag{2.4}$$

where the last equality in (2.4) follows by the representation (2.1) for $\beta_j = [x_j, u]$, $j \in \{1, ..., n\}$.

For $q=1$, $p=\infty$ the representation (2.2) follows in a similar way by utilising the equality (2.1). We omit the details. $\square$

REMARK 2.9. If $(E,\|\cdot\|)$ is an inner product space with $\langle\cdot,\cdot\rangle$ generating the norm, then we recapture the representation result obtained in the recent paper [8].

REMARK 2.10. We observe that the representation (2.2) provides a stronger result than the one from Theorem 1.3 since it makes use of a smaller class of bounded linear functionals, namely the ones generated by a given L-G-*s.i.p* on E that generates the norm $\|\cdot\|$.

3. The case of operators on Banach spaces

A fundamental result due to Lumer ([11]), in the theory of operators on complex Banach spaces X, is that if $T\in\mathcal{B}(X)$, then

$$w(T)\le\|T\|\le 4w(T), \tag{3.1}$$

where $w(T):=\sup_{\|x\|=1}|[Tx,x]|$ is the numerical radius of the operator T and $[\cdot,\cdot]$ is a s-L-G-s.i.p. that generates the norm $\|\cdot\|$. The numerical radius is independent of the choice of $[\cdot,\cdot]$ (see [11], Theorem 14). Also, the numerical radius is a norm.

As shown by Glickfeld ([10]), the second inequality in (3.1) holds with $e=\exp(1)$ instead of 4 and e is the best possible constant. Therefore we have the sharp inequalities

$$\frac{1}{e}\|T\|\le w(T)\le\|T\| \tag{3.2}$$

for any $T\in\mathcal{B}(X)$.

On the Cartesian product $B^{(n)}(X):=\mathcal{B}(X)\times\ldots\times\mathcal{B}(X)$ we can define the *hypo-q-operator norms* of $(T_1,\ldots,T_n)\in B^{(n)}(X)$ by

$$\|(T_1,\ldots,T_n)\|_{h,n,q}:=\sup_{\|\lambda\|_{n,p}\le 1}\left\|\sum_{j=1}^{n}\lambda_jT_j\right\| \quad \text{where } p,\ q\in[1,\infty], \tag{3.3}$$

with the convention that if $p=1$, $q=\infty$; if $p=\infty$, $q=1$ and if $p>1$, then $\frac{1}{p}+\frac{1}{q}=1$.

If $[\cdot,\cdot]$ is a s-L-G-s.i.p. that generates the norm $\|\cdot\|$ of X and $w(T) := \sup_{\|x\|=1} |[Tx,x]|$ is the numerical radius of the operator T we can also define the *hypo-q-numerical radius* of $(T_1,\dots,T_n) \in B^{(n)}(X)$ by

$$w_{h,n,q}(T_1,\dots,T_n) := \sup_{\|\lambda\|_{n,p}\le 1} w\left(\sum_{j=1}^{n} \lambda_j T_j\right) \quad \text{with } p,\ q \in [1,\infty], \tag{3.4}$$

with the convention that if $p=1$, $q=\infty$; if $p=\infty$, $q=1$ and if $p>1$, then $\frac{1}{p}+\frac{1}{q}=1$.

We observe that (3.3) and (3.4) are special cases of (1.1), for two different norms on $E=B(X)$.

Using (3.2) we have

$$\frac{1}{e}\left\|\sum_{j=1}^{n} \lambda_j T_j\right\| \le w\left(\sum_{j=1}^{n} \lambda_j T_j\right) \le \left\|\sum_{j=1}^{n} \lambda_j T_j\right\|$$

and by taking the supremum over $\|\lambda\|_{n,p} \le 1$ in this inequality, we get the following fundamental result

$$\frac{1}{e}\|(T_1,\dots,T_n)\|_{h,n,q} \le w_{h,n,q}(T_1,\dots,T_n) \le \|(T_1,\dots,T_n)\|_{h,n,q} \tag{3.5}$$

for any $(T_1,\dots,T_n) \in B^{(n)}(X)$ and $q \ge 1$. The inequalities (3.5) are sharp, which follow by the unidimensional case.

THEOREM 3.1. *Let $(X,\|\cdot\|)$ be a Banach space and $[\cdot,\cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$ of X. Let $(T_1,\dots,T_n) \in B^{(n)}(X)$ and $x,\ y \in X$, then for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ or $p=1$, $q=\infty$ or $p=\infty, q=1$, we have*

$$\sup_{\|\alpha\|_{n,p}\le 1}\left|\left[\left(\sum_{j=1}^{n} \alpha_j T_j\right)x, y\right]\right| = \left(\sum_{j=1}^{n} |[T_j x, y]|^q\right)^{1/q}. \tag{3.6}$$

PROOF. If we take $\beta = ([T_1x,y],\dots,[T_nx,y]) \in \mathbb{C}^n$ in (2.1), then we get

$$\begin{aligned}\left(\sum_{j=1}^{n} |[T_j x, y]|^q\right)^{1/q} &= \|\beta\|_{n,q} = \sup_{\|\alpha\|_p\le 1}\left|\sum_{j=1}^{n} \alpha_j\beta_j\right| \\ &= \sup_{\|\alpha\|_{n,p}\le 1}\left|\sum_{j=1}^{n} \alpha_j [T_j x, y]\right| = \sup_{\|\alpha\|_{n,p}\le 1}\left|\left[\sum_{j=1}^{n} \alpha_j T_j x, y\right]\right|,\end{aligned}$$

which proves (3.6). □

Corollary 3.2. *With the assumptions of Theorem 3.1, if* $(T_1,\dots,T_n)\in B^{(n)}(X)$ *and* $x\in X$*, then for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *or* $p=1,\ q=\infty$ *or* $p=\infty, q=1$*, we have*

$$\sup_{\|\alpha\|_{n,p}\le 1}\left\|\sum_{j=1}^{n}\alpha_j T_j x\right\|=\sup_{\|y\|=1}\left(\sum_{j=1}^{n}\left|[T_jx,y]\right|^q\right)^{1/q}. \tag{3.7}$$

Proof. By the properties of semi-inner product, we have for any $u\in X$, $u\neq 0$ (see also (2.3)) that

$$\|u\|=\sup_{\|y\|=1}\left|[u,y]\right|. \tag{3.8}$$

Let $x\in X$, then by taking the supremum over $\|y\|=1$ in (3.6) we get for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned}\sup_{\|y\|=1}\left(\sum_{j=1}^{n}\left|[T_jx,y]\right|^q\right)^{1/q}&=\sup_{\|y\|=1}\left(\sup_{\|\alpha\|_{n,p}\le 1}\left|\left[\left(\sum_{j=1}^{n}\alpha_jT_j\right)x,y\right]\right|\right)\\&=\sup_{\|\alpha\|_{n,p}\le 1}\left(\sup_{\|y\|=1}\left|\left[\left(\sum_{j=1}^{n}\alpha_jT_j\right)x,y\right]\right|\right)\\&=\sup_{\|\alpha\|_{n,p}\le 1}\left\|\left(\sum_{j=1}^{n}\alpha_jT_j\right)x\right\|,\end{aligned}$$

which proves the equality (3.7). We used for the last equality the property (3.8). □

We can state and prove our main representation result.

Theorem 3.3. *Let* $(X,\|\cdot\|)$ *be a Banach space,* $[\cdot,\cdot]$ *a s-L-G-s.i.p. that generates the norm* $\|\cdot\|$ *of* X *and* $(T_1,\dots,T_n)\in B^{(n)}(X)$.

(i) *For* $q\ge 1$ *we have the representation for the hypo-q-operator norm*

$$\|(T_1,\dots,T_n)\|_{h,n,q}=\sup_{\|x\|=\|y\|=1}\left(\sum_{j=1}^{n}\left|[T_jx,y]\right|^q\right)^{1/q} \tag{3.9}$$

and

$$\|(T_1,\dots,T_n)\|_{h,n,\infty}=\max_{j\in\{1,\dots,n\}}\|T_j\|.$$

(ii) *For $q \geq 1$ we have the representation for the hypo-q-numerical radius*

$$w_{h,n,q}(T_1,\dots,T_n) = \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |[T_j x, x]|^q \right)^{1/q} \tag{3.10}$$

and

$$w_{h,n,\infty}(T_1,\dots,T_n) = \max_{j\in\{1,\dots,n\}} w(T_j).$$

Proof. (i) By using the equality (3.7) we have for $(T_1,\dots,T_n) \in B^{(n)}(X)$ that

$$\begin{aligned}
\sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^{n} |[T_j x, y]|^q \right)^{1/q} &= \sup_{\|x\|=1} \left(\sup_{\|y\|=1} \left(\sum_{j=1}^{n} |[T_j x, y]|^q \right)^{1/q} \right) \\
&= \sup_{\|x\|=1} \left(\sup_{\|\alpha\|_{n,p}\leq 1} \left\| \sum_{j=1}^{n} \alpha_j T_j x \right\| \right) \\
&= \sup_{\|\alpha\|_{n,p}\leq 1} \left(\sup_{\|x\|=1} \left\| \sum_{j=1}^{n} \alpha_j T_j x \right\| \right) \\
&= \sup_{\|\alpha\|_{n,p}\leq 1} \left\| \sum_{j=1}^{n} \alpha_j T_j \right\| = \|(T_1,\dots,T_n)\|_{h,n,q},
\end{aligned}$$

which proves (3.9). The rest is obvious.

(ii) By using the equality (3.6) we have for $(T_1,\dots,T_n) \in B^{(n)}(X)$ that

$$\begin{aligned}
\sup_{\|x\|=1} \left(\sum_{j=1}^{n} |[T_j x, x]|^q \right)^{1/q} &= \sup_{\|x\|=1} \left(\sup_{\|\alpha\|_{n,p}\leq 1} \left| \left[\left(\sum_{j=1}^{n} \alpha_j T_j \right) x, x \right] \right| \right) \\
&= \sup_{\|\alpha\|_{n,p}\leq 1} \left(\sup_{\|x\|=1} \left| \left[\left(\sum_{j=1}^{n} \alpha_j T_j \right) x, x \right] \right| \right) \\
&= \sup_{\|\alpha\|_{n,p}\leq 1} w\left(\sum_{j=1}^{n} \alpha_j T_j \right) = w_{h,n,q}(T_1,\dots,T_n),
\end{aligned}$$

which proves (3.10). The rest is obvious. □

We can consider on $B^{(n)}(X)$ the following usual operator and numerical radius q-norms, for $q \geq 1$

$$\|(T_1, \ldots, T_n)\|_{n,q} := \left(\sum_{j=1}^{n} \|T_j\|^q \right)^{1/q}$$

and

$$w_{n,q}(T_1, \ldots, T_n) := \left(\sum_{j=1}^{n} w^q(T_j) \right)^{1/q}$$

where $(T_1, \ldots, T_n) \in B^{(n)}(X)$. For $q = \infty$ we put

$$\|(T_1, \ldots, T_n)\|_{n,\infty} := \max_{j \in \{1,\ldots,n\}} \|T_j\|$$

and

$$w_{n,\infty}(T_1, \ldots, T_n) := \max_{j \in \{1,\ldots,n\}} w(T_j).$$

Corollary 3.4. *With the assumptions of Theorem 3.3 we have for $q \geq 1$ that*

$$\frac{1}{n^{1/q}} \|(T_1, \ldots, T_n)\|_{n,q} \leq \|(T_1, \ldots, T_n)\|_{h,n,q} \leq \|(T_1, \ldots, T_n)\|_{n,q}$$

and

$$\frac{1}{n^{1/q}} w_{n,q}(T_1, \ldots, T_n) \leq w_{h,n,q}(T_1, \ldots, T_n) \leq w_{n,q}(T_1, \ldots, T_n)$$

for any $(T_1, \ldots, T_n) \in B^{(n)}(X)$.

The proof follows from Corollary 3.2 for $E = B(X)$ and we omit the details.

Corollary 3.5. *With the assumptions of Theorem 3.3 we have for $r \geq q \geq 1$ that*

$$\|(T_1, \ldots, T_n)\|_{h,n,r} \leq \|(T_1, \ldots, T_n)\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|(T_1, \ldots, T_n)\|_{h,n,r} \tag{3.11}$$

and

$$w_{h,n,r}(T_1, \ldots, T_n) \leq w_{h,n,q}(T_1, \ldots, T_n) \leq n^{\frac{r-q}{rq}} w_{h,n,r}(T_1, \ldots, T_n) \tag{3.12}$$

for any $(T_1, \ldots, T_n) \in B^{(n)}(X)$.

PROOF. We use the following elementary inequalities for the nonnegative numbers a_j, $j = 1, ..., n$ and $r \geq q > 0$ (see for instance [19] and [15])

$$\left(\sum_{j=1}^{n} a_j^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} a_j^q\right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} a_j^r\right)^{1/r}. \tag{3.13}$$

Let $(T_1, \ldots, T_n) \in B^{(n)}(X)$ and $x, y \in X$ with $\|x\| = \|y\| = 1$. Then by (3.13) we get

$$\left(\sum_{j=1}^{n} |[T_j x, y]|^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} |[T_j x, y]|^q\right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} |[T_j x, y]|^r\right)^{1/r}.$$

By taking the supremum over $\|x\| = \|y\| = 1$ we get (3.11).

The inequality (3.12) follows in a similar way and we omit the details. $\square$

For $q = 2$, we put

$$\|(T_1, \ldots, T_n)\|_{h,n,e} := \|(T_1, \ldots, T_n)\|_{h,n,2}$$

and

$$w_{h,n,e}(T_1, \ldots, T_n) := w_{h,n,2}(T_1, \ldots, T_n).$$

REMARK 3.6. We draw the readers' particular attention to special cases of Corollary 3.5: $r = 2$, $q = 2$, $q = 1$.

We have:

PROPOSITION 3.7. *For any $(T_1, \ldots, T_n) \in B^{(n)}(X)$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\|(T_1, \ldots, T_n)\|_{h,n,q} \geq \frac{1}{n^{1/p}} \left\| \sum_{j=1}^{n} T_j \right\|$$

and

$$w_{h,n,q}(T_1, \ldots, T_n) \geq \frac{1}{n^{1/p}} w\left(\sum_{j=1}^{n} T_j\right). \tag{3.14}$$

Proof. Let $\lambda_j = \frac{1}{n^{1/p}}$ for $j \in \{1, ..., n\}$, then $\sum_{j=1}^n |\lambda_j|^p = 1$. Therefore by (3.3) we get

$$\|(T_1, \dots, T_n)\|_{h,n,q} = \sup_{\|\lambda\|_{n,p} \le 1} \left\| \sum_{j=1}^n \lambda_j T_j \right\| \ge \left\| \sum_{j=1}^n \frac{1}{n^{1/p}} T_j \right\| = \frac{1}{n^{1/p}} \left\| \sum_{j=1}^n T_j \right\|.$$

The inequality (3.14) follows in a similar way. □

We can also introduce the following norms for $(T_1, \dots, T_n) \in B^{(n)}(X)$,

$$\|(T_1, \dots, T_n)\|_{s,n,p} := \sup_{\|x\|=1} \left(\sum_{j=1}^n \|T_j x\|^p \right)^{1/p},$$

where $p \ge 1$ and

$$\|(T_1, \dots, T_n)\|_{s,n,\infty} := \sup_{\|x\|=1} \left(\max_{j \in \{1,\dots,n\}} \|T_j x\| \right) = \max_{j \in \{1,\dots,n\}} \|T_j\|.$$

The triangle inequality for $\|\cdot\|_{s,n,q}$ follows from Minkowski inequality, while the other properties of the norm are obvious.

Proposition 3.8. *Let $(T_1, \dots, T_n) \in B^{(n)}(X)$. We have for $p \ge 1$, that*

$$\|(T_1, \dots, T_n)\|_{h,n,p} \le \|(T_1, \dots, T_n)\|_{s,n,p} \le \|(T_1, \dots, T_n)\|_{n,p}. \tag{3.15}$$

Proof. We have for $p \ge 2$ and $x, y \in X$ with $\|x\| = \|y\| = 1$, that

$$|[T_j x, y]|^p \le \|T_j x\|^p \|y\|^p = \|T_j x\|^p \le \|T_j\|^p \|x\|^p = \|T_j\|^p$$

for $j \in \{1, ..., n\}$.

This implies

$$\sum_{j=1}^n |[T_j x, y]|^p \le \sum_{j=1}^n \|T_j x\|^p \le \sum_{j=1}^n \|T_j\|^p,$$

so

$$\left(\sum_{j=1}^n |[T_j x, y]|^p \right)^{1/p} \le \left(\sum_{j=1}^n \|T_j x\|^p \right)^{1/p} \le \left(\sum_{j=1}^n \|T_j\|^p \right)^{1/p}, \tag{3.16}$$

for any $x, y \in X$ with $\|x\| = \|y\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.16), we get the desired result (3.15). □

4. Reverse inequalities

Recall the following reverse of Cauchy-Buniakowski-Schwarz inequality ([2], see also [1, Theorem 5.14]):

Lemma 4.1. *Let $a,\ A \in \mathbb{R}$ and $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ be two sequences of real numbers with the property that:*

$$ay_j \le z_j \le Ay_j \quad \text{for each} \;\; j \in \{1, \ldots, n\}.$$

Then for any $\mathbf{w} = (w_1, \ldots, w_n)$ a sequence of positive real numbers, one has the inequality

$$(4.1) \quad 0 \le \sum_{j=1}^{n} w_j z_j^2 \sum_{j=1}^{n} w_j y_j^2 - \left(\sum_{j=1}^{n} w_j z_j y_j \right)^2 \le \frac{1}{4} (A-a)^2 \left(\sum_{j=1}^{n} w_j y_j^2 \right)^2.$$

The constant $\frac{1}{4}$ is sharp in (4.1).

O. Shisha and B. Mond obtained in 1967 (see [19]) the following counterparts of (CBS)-inequality (see also [1, Theorem 5.20 & 5.21]):

Lemma 4.2. *Assume that $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ are such that there exist a, A, b, B with the property that:*

$$0 \le a \le a_j \le A \quad \text{and} \quad 0 < b \le b_j \le B \quad \text{for any } j \in \{1, \ldots, n\}.$$

Then we have the inequality

$$(4.2) \qquad \sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j \right)^2 \le \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2 \sum_{j=1}^{n} a_j b_j \sum_{j=1}^{n} b_j^2.$$

and

Lemma 4.3. *Assume that* $\mathbf{a}$, $\mathbf{b}$ *are nonnegative sequences and there exist* γ, Γ *with the property that:*

$$0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \quad \text{for any} \quad j \in \{1, \dots, n\}.$$

Then we have the inequality

$$0 \leq \left(\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2. \tag{4.3}$$

We have:

Theorem 4.4. *Let* $(X, \|\cdot\|)$ *be a Banach space,* $[\cdot, \cdot]$ *a s-L-G-s.i.p. that generates the norm* $\|\cdot\|$ *of* X *and* $(T_1, \dots, T_n) \in B^{(n)}(X)$.

(i) *We have*

$$0 \leq \|(T_1, \dots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \dots, T_n)\|_{h,n,1}^2 \leq \frac{1}{4} n \|(T_1, \dots, T_n)\|_{n,\infty}^2 \tag{4.4}$$

and

$$0 \leq w_{n,e}^2 (T_1, \dots, T_n) - \frac{1}{n} w_{h,n,1}^2 (T_1, \dots, T_n) \leq \frac{1}{4} n \|(T_1, \dots, T_n)\|_{n,\infty}^2. \tag{4.5}$$

(ii) *We have*

$$\begin{aligned} 0 &\leq \|(T_1, \dots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \dots, T_n)\|_{h,n,1}^2 \\ &\leq \|(T_1, \dots, T_n)\|_{n,\infty} \|(T_1, \dots, T_n)\|_{h,n,1} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} 0 &\leq w_{n,e}^2 (T_1, \dots, T_n) - \frac{1}{n} w_{h,n,1}^2 (T_1, \dots, T_n) \\ &\leq \|(T_1, \dots, T_n)\|_{n,\infty} w_{h,n,1} (T_1, \dots, T_n). \end{aligned} \tag{4.7}$$

(iii) *We have*

$$0 \le \|(T_1, \dots, T_n)\|_{h,n,e} - \frac{1}{\sqrt{n}} \|(T_1, \dots, T_n)\|_{h,n,1} \tag{4.8}$$
$$\le \frac{1}{4}\sqrt{n}\, \|(T_1, \dots, T_n)\|_{n,\infty}$$

and

$$0 \le w_{n,e}(T_1, \dots, T_n) - \frac{1}{\sqrt{n}} w_{h,n,1}(T_1, \dots, T_n) \tag{4.9}$$
$$\le \frac{1}{4}\sqrt{n}\, \|(T_1, \dots, T_n)\|_{n,\infty}.$$

Proof. (i). Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and put

$$R = \max_{j \in \{1,\dots,n\}} \|T_j\| = \|(T_1, \dots, T_n)\|_{n,\infty}.$$

If x, $y \in H$ with $\|x\| = \|y\| = 1$ then

$$|[T_j x, y]| \le \|T_j x\| \le \|T_j\| \le R$$

for any $j \in \{1, ..., n\}$.

If we write the inequality (4.1) for $z_j = |[T_j x, y]|$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \le n \sum_{j=1}^{n} |[T_j x, y]|^2 - \left(\sum_{j=1}^{n} |[T_j x, y]| \right)^2 \le \frac{1}{4} n^2 R^2$$

for any x, $y \in H$ with $\|x\| = \|y\| = 1$.

This implies that

$$\sum_{j=1}^{n} |[T_j x, y]|^2 \le \frac{1}{n} \left(\sum_{j=1}^{n} |[T_j x, y]| \right)^2 + \frac{1}{4} n R^2 \tag{4.10}$$

for any x, $y \in H$ with $\|x\| = \|y\| = 1$ and, in particular

$$\sum_{j=1}^{n} |[T_j x, x]|^2 \le \frac{1}{n} \left(\sum_{j=1}^{n} |[T_j x, x]| \right)^2 + \frac{1}{4} n R^2 \tag{4.11}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (4.10) and over $\|x\| = 1$ in (4.11), we get (4.4) and (4.5).

(ii). Let $(T_1, \ldots, T_n) \in B^{(n)}(H)$. If we write the inequality (4.2) for $a_j = |[T_j x, y]|$, $b_j = 1$, $b = B = 1$, $a = 0$ and $A = R$, then we get

$$0 \leq n \sum_{j=1}^{n} |[T_j x, y]|^2 - \left(\sum_{j=1}^{n} |[T_j x, y]| \right)^2 \leq nR \sum_{j=1}^{n} |[T_j x, y]|,$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

This implies that

$$\sum_{j=1}^{n} |[T_j x, y]|^2 \leq \frac{1}{n} \left(\sum_{j=1}^{n} |[T_j x, y]| \right)^2 + R \sum_{j=1}^{n} |[T_j x, y]|, \tag{4.12}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and, in particular

$$\sum_{j=1}^{n} |[T_j x, x]|^2 \leq \frac{1}{n} \left(\sum_{j=1}^{n} |[T_j x, x]| \right)^2 + R \sum_{j=1}^{n} |[T_j x, x]|, \tag{4.13}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (4.12) and over $\|x\| = 1$ in (4.13), we get (4.6) and (4.7).

(iii). If we write the inequality (4.3) for $a_j = |[T_j x, y]|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$ we have

$$0 \leq \left(n \sum_{j=1}^{n} |[T_j x, y]|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} |[T_j x, y]| \leq \frac{1}{4} nR,$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

This implies that

$$\left(\sum_{j=1}^{n} |[T_j x, y]|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |[T_j x, y]| + \frac{1}{4} \sqrt{n} R, \tag{4.14}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and, in particular

$$\left(\sum_{j=1}^{n} |[T_j x, x]|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |[T_j x, x]| + \frac{1}{4} \sqrt{n} R, \tag{4.15}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (4.14) and over $\|x\| = 1$ in (4.15), we get (4.8) and (4.9). $\square$

Before we proceed with establishing some reverse inequalities for the hypo-Euclidean numerical radius, we recall some reverse results of the Cauchy-Bunyakovsky-Schwarz inequality for complex numbers as follows:

If γ, $\Gamma \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$ with the property that

$$
\begin{aligned}
(4.16) \qquad 0 &\le \operatorname{Re}\left[(\Gamma - \alpha_j)\left(\overline{\alpha_j} - \bar{\gamma}\right)\right] \\
&= (\operatorname{Re}\Gamma - \operatorname{Re}\alpha_j)(\operatorname{Re}\alpha_j - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im}\alpha_j)(\operatorname{Im}\alpha_j - \operatorname{Im}\gamma)
\end{aligned}
$$

or, equivalently,

$$\left|\alpha_j - \frac{\gamma + \Gamma}{2}\right| \le \frac{1}{2}\left|\Gamma - \gamma\right|$$

for each $j \in \{1, \dots, n\}$, then (see for instance [4, p. 9])

$$
n \sum_{j=1}^{n} |\alpha_j|^2 - \left|\sum_{j=1}^{n} \alpha_j\right|^2 \le \frac{1}{4} n^2 \left|\Gamma - \gamma\right|^2. \tag{4.17}
$$

In addition, if $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then (see for example [4, p. 26]):

$$
\begin{aligned}
(4.18) \qquad n \sum_{j=1}^{n} |\alpha_j|^2 &\le \frac{1}{4} \frac{\left\{\operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma}\right)\sum_{j=1}^{n} \alpha_j\right]\right\}^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \\
&\le \frac{1}{4} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left|\sum_{j=1}^{n} \alpha_j\right|^2.
\end{aligned}
$$

Also, if $\Gamma \ne -\gamma$, then (see for instance [4, p. 32]):

$$
\left(n \sum_{j=1}^{n} |\alpha_j|^2\right)^{\frac{1}{2}} - \left|\sum_{j=1}^{n} \alpha_j\right| \le \frac{1}{4} n \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}. \tag{4.19}
$$

Finally, from [5] we can also state that

$$
n \sum_{j=1}^{n} |\alpha_j|^2 - \left|\sum_{j=1}^{n} \alpha_j\right|^2 \le n\left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\left|\sum_{j=1}^{n} \alpha_j\right|, \tag{4.20}
$$

provided $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$.

We notice that a simple sufficient condition for (4.16) to hold is that

$$\operatorname{Re}\Gamma \geq \operatorname{Re}\alpha_j \geq \operatorname{Re}\gamma \quad \text{and} \quad \operatorname{Im}\Gamma \geq \operatorname{Im}\alpha_j \geq \operatorname{Im}\gamma$$

for each $j \in \{1, \ldots, n\}$.

THEOREM 4.5. *Let $(X, \|\cdot\|)$ be a Banach space, $[\cdot,\cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$ of X and $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq \gamma$. Assume that*

$$w\left(T_j - \frac{\gamma+\Gamma}{2} I\right) \leq \frac{1}{2}|\Gamma - \gamma| \text{ for any } j \in \{1, \ldots, n\}. \tag{4.21}$$

(i) *We have*

$$w_{h,n,e}^2(T_1, \ldots, T_n) \leq \frac{1}{n} w^2\left(\sum_{j=1}^n T_j\right) + \frac{1}{4} n |\Gamma - \gamma|^2. \tag{4.22}$$

(ii) *If* $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, *then*

$$w_{h,n,e}(T_1, \ldots, T_n) \leq \frac{1}{2\sqrt{n}} \frac{|\Gamma+\gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} w\left(\sum_{j=1}^n T_j\right) \tag{4.23}$$

and

$$w_{h,n,e}^2(T_1, \ldots, T_n) \leq \left[\frac{1}{n} w^2\left(\sum_{j=1}^n T_j\right) + \left[|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\right] \times w\left(\sum_{j=1}^n T_j\right). \tag{4.24}$$

(iii) *If* $\Gamma \neq -\gamma$, *then*

$$w_{h,n,e}(T_1, \ldots, T_n) \leq \frac{1}{\sqrt{n}}\left(w\left(\sum_{j=1}^n T_j\right) + \frac{1}{4} n \frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\right). \tag{4.25}$$

PROOF. Let $x \in H$ with $\|x\| = 1$ and $(T_1, \ldots, T_n) \in B^{(n)}(H)$ with the property (4.21). By taking $\alpha_j = [T_j x, x]$ we have

$$\begin{aligned}\left|\alpha_j - \frac{\gamma+\Gamma}{2}\right| &= \left|[T_j x, x] - \frac{\gamma+\Gamma}{2}[x,x]\right| = \left|\left[\left(T_j - \frac{\gamma+\Gamma}{2}I\right)x, x\right]\right| \\ &\leq \sup_{\|x\|=1}\left|\left[\left(T_j - \frac{\gamma+\Gamma}{2}I\right)x, x\right]\right| = w\left(T_j - \frac{\gamma+\Gamma}{2}I\right) \\ &\leq \frac{1}{2}|\Gamma - \gamma|\end{aligned}$$

for any $j \in \{1, \ldots, n\}$.

(i) By using the inequality (4.17), we have

$$\begin{aligned}\sum_{j=1}^{n}|[T_j x, x]|^2 &\leq \frac{1}{n}\left|\sum_{j=1}^{n}[T_j x, x]\right|^2 + \frac{1}{4}n|\Gamma-\gamma|^2 \\ &= \frac{1}{n}\left|\left[\sum_{j=1}^{n}T_j x, x\right]\right|^2 + \frac{1}{4}n|\Gamma-\gamma|^2\end{aligned} \tag{4.26}$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in (4.26) we get

$$\begin{aligned}\sup_{\|x\|=1}\left(\sum_{j=1}^{n}|[T_j x, x]|^2\right) &\leq \frac{1}{n}\sup_{\|x\|=1}\left|\left[\sum_{j=1}^{n}T_j x, x\right]\right|^2 + \frac{1}{4}n|\Gamma-\gamma|^2 \\ &= \frac{1}{n}w^2\left(\sum_{j=1}^{n}T_j\right) + \frac{1}{4}n|\Gamma-\gamma|^2,\end{aligned}$$

which proves (4.22).

(ii) If $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then by (4.18) we have for $\alpha_j = [T_j x, x]$, $j \in \{1, \ldots, n\}$ that

$$\begin{aligned}\sum_{j=1}^{n}|[T_j x, x]|^2 &\leq \frac{1}{4n}\frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\sum_{j=1}^{n}[T_j x, x]\right|^2 \\ &= \frac{1}{4n}\frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\left[\sum_{j=1}^{n}T_j x, x\right]\right|^2\end{aligned} \tag{4.27}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = 1$ in (4.27) we get (4.23).

Also, by (4.20) we get

$$\sum_{j=1}^{n} |[T_j x, x]|^2 \le \frac{1}{n}\left|\left[\sum_{j=1}^{n} T_j x, x\right]\right|^2 + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\left|\left[\sum_{j=1}^{n} T_j x, x\right]\right|$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in this inequality, we have

$$\begin{aligned}
&\sup_{\|x\|=1} \sum_{j=1}^{n} |[T_j x, x]|^2 \\
&\le \sup_{\|x\|=1}\left[\frac{1}{n}\left|\left[\sum_{j=1}^{n} T_j x, x\right]\right|^2 + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\left|\left[\sum_{j=1}^{n} T_j x, x\right]\right|\right] \\
&\le \frac{1}{n}\sup_{\|x\|=1}\left|\left[\sum_{j=1}^{n} T_j x, x\right]\right|^2 + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\sup_{\|x\|=1}\left|\left[\sum_{j=1}^{n} T_j x, x\right]\right| \\
&= \frac{1}{n} w^2\left(\sum_{j=1}^{n} T_j\right) + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right] w\left(\sum_{j=1}^{n} T_j\right),
\end{aligned}$$

which proves (4.24).

(iii) By the inequality (4.19) we have

$$\begin{aligned}
\left(\sum_{j=1}^{n} |[T_j x, x]|^2\right)^{\frac{1}{2}} &\le \frac{1}{\sqrt{n}}\left(\left|\sum_{j=1}^{n} [T_j x, x]\right| + \frac{1}{4} n \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}\right) \\
&= \frac{1}{\sqrt{n}}\left(\left|\left[\sum_{j=1}^{n} T_j x, x\right]\right| + \frac{1}{4} n \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}\right)
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in this inequality, we get (4.25). $\square$

Remark 4.6. By the use of the elementary inequality $w(T) \le \|T\|$ that holds for any $T \in B(X)$, a sufficient condition for (4.21) to hold is that

$$\left\|T_j - \frac{\gamma + \Gamma}{2}\right\| \le \frac{1}{2}|\Gamma - \gamma| \quad \text{for any } j \in \{1, \ldots, n\}.$$

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