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m-CONVEX FUNCTIONS OF HIGHER ORDER

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Abstract. In this research we introduce the concept of m-convex function of higher order by means of the so called m-divided difference; elementary properties of this type of functions are exhibited and some examples are provided.

1. Introduction

The concept of m-convex function, $0 \leq m \leq 1$, was introduced in [2, 13] and since then many properties, especially inequalities and algebraic properties have been obtained for them ([8]). This concept has evolved and nowadays there are many generalizations of it, examples of both, analytic and numerics, are also available, among these new stuff we ought to mention, strongly m-convex functions, approximate m-convex functions and Jensen m-convex functions; interested readers may consult for instance [7, 8, 9]. In this work we introduce the concepts of m-difference operator and m-divided difference in a similar manner to difference operator and divided difference respectively ([6]), and from here the concept of m-convexity of higher order is set for functions $f\colon [0, b] \to \mathbb{R}$. Our research is based and motivated basically by the works of Popoviciu ([12]) and more recently in the works of [3, 6, 11] and references therein.

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We need to set a couple of known definitions and remarks before go over the matter of our investigation. Along this work and unless otherwise is said, the real number m will be in $[0,1]$.

Definition 1.1. Let D be any nonempty set of $\mathbb{R}$. D is said to be *m-convex* if, for all x and y in D and all t in the interval $[0,1]$, the point $tx+(1-t)my$ also belongs to D.

In the following, D always will be the interval $[0,b]$ which, of course, is m-convex.

Definition 1.2 ([13]). A function $f\colon [0,b] \to \mathbb{R}$ is called *m-convex*, if for any $x,\, y \in [0,b]$ and $t \in [0,1]$ we have

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y).$$

Remark 1.3. It is important to point out that the above definition is equivalent to $f(mtx+(1-t)y) \leq mtf(x)+(1-t)f(y)$, with x, y and t as before.

The incoming result is very similar to the one given in [10, Proposition 1.1.1] (see also the references therein); the proof also goes in a similar fashion.

Proposition 1.4. *Let $f\colon [0,b] \to \mathbb{R}$. The following statements are equivalent:*

(1) *f is m-convex.*
(2) $f(msx+ty) \leq msf(x)+tf(y)$, $x,\, y \in [0,b]$; $s,t \in (0,1)$ *and* $s+t=1$.
(3) *If* $x,y,z \in [0,b]$, $x<z<y$,

$$(y-z)mf(x)+(z-mx)f(y)+(mx-y)f(z) \geq 0.$$

Following ideas given in [6] and [11] we set the following

Definition 1.5. Consider the function $r\colon \mathbb{N} \to \mathbb{N} \cup \{0\}$ given by

$$r(x) = \begin{cases} x-2 & \text{if } x \neq 1, \\ 0 & \text{if } x = 1, \end{cases}$$

and, for a fixed $k \in \mathbb{N}$, let $x_k, x_{k+1}, \ldots, x_{k+n-1}$ be n points in $[0,b]$. If $m \in [0,1]$, we say that these points are *m-ordered* if they verify

$$m^{r(j)}x_j < m^{r(j+1)}x_{j+1} \quad \text{with } j = k, \ldots, k+n-2.$$

The *m-divided difference*, $[x_k, x_{k+1}, \dots, x_{k+n-1}; f]_m$, of order n of a real valued function f defined on $[0, b]$ at the m-ordered points $x_k, x_{k+1}, \dots, x_{k+n-1}$, is given by

$$[x_k; f]_m = m^{r(k)} f(x_k),$$

and for $n \geq 2$

$$[x_k, \dots, x_{k+n-1}; f]_m = \frac{[x_{k+1}, \dots, x_{k+n-1}; f]_m - [x_k, \dots, x_{k+n-2}; f]_m}{m^{r(k+n-1)} x_{k+n-1} - m^{r(k)} x_k}.$$

The case $m = 1$ corresponds to the classical definition of divided difference ([4, 5, 6]).

It is clear that known properties of divided differences of functions hold true for m-divided difference of functions defined here; in the incoming section we list some of them.

2. Properties of m-divided differences

Here we show a couple of results involving the foregoing concept of m-divided difference; basically a way of writing it, as a sum and also in terms of some determinants.

THEOREM 2.1. *For any $n \geq 2$, the following equality holds*

$$[x_1, \dots, x_n; f]_m = \sum_{i=1}^{n} \frac{m^{r(i)} f(x_i)}{\prod_{j=1, j\neq i}^{n} \left(m^{r(i)} x_i - m^{r(j)} x_j\right)}.$$

PROOF. The proof runs by induction; for $n = 2$,

$$[x_1, x_2; f]_m = \frac{[x_2; f]_m - [x_1; f]_m}{m^{r(2)} x_2 - m^{r(1)} x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

while

$$\sum_{i=1}^{2} \frac{m^{r(i)} f(x_i)}{\prod_{j=1, j\neq i}^{2} \left(m^{r(i)} x_i - m^{r(j)} x_j\right)} = \frac{f(x_1)}{\prod_{j=1, j\neq 1}^{2} \left(m^{r(1)} x_1 - m^{r(j)} x_j\right)} + \frac{f(x_2)}{\prod_{j=1, j\neq 2}^{2} \left(m^{r(2)} x_2 - m^{r(j)} x_j\right)}$$

$$= \frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Thus, the result holds for $n = 2$.

Assume now that it is true for n. Then,

$$[x_1, \ldots, x_{n+1}; f]_m = \frac{[x_2, \ldots, x_{n+1}; f]_m - [x_1, \ldots, x_n; f]_m}{m^{r(n+1)}x_{n+1} - m^{r(1)}x_1}$$

$$= \frac{\displaystyle\sum_{i=2}^{n+1} \frac{m^{r(i)} f(x_i)}{\prod_{j=2, j\neq i}^{n+1}(m^{r(i)}x_i - m^{r(j)}x_j)} - \sum_{i=1}^{n} \frac{m^{r(i)} f(x_i)}{\prod_{j=1, j\neq i}^{n}(m^{r(i)}x_i - m^{r(j)}x_j)}}{m^{r(n+1)}x_{n+1} - x_1}.$$

So,

$$\left(m^{r(n+1)}x_{n+1} - x_1\right)[x_1, \ldots, x_{n+1}; f]_m$$

$$= \sum_{i=2}^{n} \frac{m^{r(i)} f(x_i)}{\prod_{j=2, j\neq i}^{n+1}(m^{r(i)}x_i - m^{r(j)}x_j)} + \frac{m^{r(n+1)} f(x_{n+1})}{\prod_{j=2, j\neq n+1}^{n+1}(m^{r(n+1)}x_{n+1} - m^{r(j)}x_j)}$$

$$- \frac{m^{r(1)} f(x_1)}{\prod_{j=1, j\neq 1}^{n}(m^{r(1)}x_1 - m^{r(j)}x_j)} - \sum_{i=2}^{n} \frac{m^{r(i)} f(x_i)}{\prod_{j=1, j\neq i}^{n}(m^{r(i)}x_i - m^{r(j)}x_j)}$$

$$= \sum_{i=2}^{n} \left[\frac{1}{\prod_{j=2, j\neq i}^{n+1}(m^{r(i)}x_i - m^{r(j)}x_j)} - \frac{1}{\prod_{j=1, j\neq i}^{n}(m^{r(i)}x_i - m^{r(j)}x_j)}\right] m^{r(i)} f(x_i)$$

$$+ \frac{m^{r(n+1)} f(x_{n+1})}{\prod_{j=2, j\neq n+1}^{n+1}(m^{r(n+1)}x_{n+1} - m^{r(j)}x_j)} - \frac{f(x_1)}{\prod_{j=1, j\neq 1}^{n}(x_1 - m^{r(j)}x_j)}$$

$$= \sum_{i=2}^{n} \frac{\left(m^{r(n+1)}x_{n+1} - x_1\right) m^{r(i)} f(x_i)}{\prod_{j=1, j\neq i}^{n+1}(m^{r(i)}x_i - m^{r(j)}x_j)}$$

$$+ \frac{(m^{r(n+1)}x_{n+1} - x_1) m^{r(n+1)} f(x_{n+1})}{\prod_{j=2, j\neq n+1}^{n+1}(m^{r(n+1)}x_{n+1} - m^{r(j)}x_j)(m^{r(n+1)}x_{n+1} - x_1)}$$

$$- \frac{(x_1 - m^{r(n+1)}x_{n+1}) f(x_1)}{\prod_{j=1, j\neq 1}^{n}(x_1 - m^{r(j)}x_j)(x_1 - m^{r(n+1)}x_{n+1})}.$$

Therefore,

$$
\begin{aligned}
&[x_1,\dots,x_{n+1};f]_m\\
&\quad=\frac{f(x_1)}{\prod_{j=1,j\neq 1}^{n+1}(x_1-m^{r(j)}x_j)}+\sum_{i=2}^{n}\frac{m^{r(i)}f(x_i)}{\prod_{j=1,j\neq i}^{n+1}(m^{r(i)}x_i-m^{r(j)}x_j)}\\
&\qquad+\frac{m^{r(n+1)}f(x_{n+1})}{\prod_{j=1,j\neq n+1}^{n+1}(m^{r(n+1)}x_{n+1}-m^{r(j)}x_j)}\\
&\quad=\sum_{i=1}^{n+1}\frac{m^{r(i)}f(x_i)}{\prod_{j=1,j\neq i}^{n+1}(m^{r(i)}x_i-m^{r(j)}x_j)}.
\end{aligned}
$$

And the result takes place for all natural $n\geq 2$. □

Again, by following ideas from [6], for a function $f\colon[0,b]\to\mathbb{R}$ and m-ordered points $x_1,\dots,x_n\in[0,b]$, we set $U(x_1,\dots,m^{r(n)}x_n;f)$ to be the determinant

$$
\begin{aligned}
&U(x_1,\dots,m^{r(n)}x_n;f)\\
&\quad=\begin{vmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & f(x_1)\\
1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & f(x_2)\\
1 & mx_3 & m^2x_3^2 & \cdots & m^{n-2}x_3^{n-2} & mf(x_3)\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & m^{r(n)}x_n & m^{2r(n)}x_n^2 & \dots & m^{(n-2)r(n)}x_n^{n-2} & m^{r(n)}f(x_n)
\end{vmatrix},
\end{aligned}
$$

and $V(x_1,\dots,m^{r(n)}x_n)$ the classical Vandermonde determinant

$$
\begin{aligned}
V(x_1,\dots,m^{r(n)}x_n)&=\begin{vmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-1}\\
1 & x_2 & x_2^2 & \cdots & x_2^{n-1}\\
1 & mx_3 & m^2x_3^2 & \cdots & m^{n-1}x_3^{n-1}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & m^{r(n)}x_n & m^{2r(n)}x_n^2 & \dots & m^{(n-1)r(n)}x_n^{n-1}
\end{vmatrix}\\
&=\prod_{\substack{i,j=1\\ i>j}}^{n}(m^{r(i)}x_i-m^{r(j)}x_j).
\end{aligned}
$$

Note that $V(x_1, \ldots, m^{r(n)}x_n) > 0$ because of the manner in picking up the points. We have the following

THEOREM 2.2. *Let $f\colon [0,b] \to \mathbb{R}$ be an arbitrary function. Then, for any m-ordered points $x_1, \ldots, x_n \in [0,b]$ $(n \geq 2)$ we have*

$$[x_1, \ldots, x_n; f]_m = \frac{U(x_1, \ldots, m^{r(n)}x_n; f)}{V(x_1, \ldots, m^{r(n)}x_n)}.$$

PROOF. It runs by induction on n; for $n = 2$, the result is clear since

$$U(x_1, x_2; f) = \begin{vmatrix} 1 & f(x_1) \\ 1 & f(x_2) \end{vmatrix} = f(x_2) - f(x_1)$$

and

$$V(x_1, x_2) = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1.$$

Suppose it holds for n. Then

$$[x_1, \ldots, x_{n+1}; f]_m = \frac{[x_2, \ldots, x_{n+1}; f]_m - [x_1, \ldots, x_n; f]_m}{m^{r(n+1)}x_{n+1} - x_1}$$
$$= \frac{1}{m^{r(n+1)}x_{n+1} - x_1}\left[\frac{U(x_2, \ldots, m^{r(n+1)}x_{n+1}; f)}{V(x_2, \ldots, m^{r(n+1)}x_{n+1})} - \frac{U(x_1, \ldots, m^{r(n)}x_n; f)}{V(x_1, \ldots, m^{r(n)}x_n)}\right].$$

Now, developing the determinants U according to the last column, we get

$$U(x_2, \ldots, m^{r(n+1)}x_{n+1}; f)$$
$$= \sum_{i=2}^{n+1}(-1)^{n+i+1}m^{r(i)}f(x_i)V(x_2, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1})$$

and

$$U(x_1, \ldots, m^{r(n)}x_n; f)$$
$$= \sum_{i=1}^{n}(-1)^{n+i}m^{r(i)}f(x_i)V(x_1, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n)}x_n).$$

Moreover,

$$V(x_1,\ldots,m^{r(n+1)}x_{n+1}) = V(x_2,\ldots,m^{r(n+1)}x_{n+1})\prod_{j=2}^{n+1}\left(m^{r(j)}x_j - x_1\right)$$
$$= \left(m^{r(n+1)}x_{n+1} - x_1\right)V(x_2,\ldots,m^{r(n+1)}x_{n+1})\prod_{j=2}^{n}\left(m^{r(j)}x_j - x_1\right);$$

and

$$V(x_1,\ldots,m^{r(n+1)}x_{n+1}) = V(x_1,\ldots,m^{r(n)}x_n)\prod_{j=1}^{n}\left(m^{r(n+1)}x_{n+1} - m^{r(j)}x_j\right)$$
$$= \left(m^{r(n+1)}x_{n+1} - x_1\right)V(x_1,\ldots,m^{r(n)}x_n)\prod_{j=2}^{n}\left(m^{r(n+1)}x_{n+1} - m^{r(j)}x_j\right).$$

Therefore,

$$[x_1,\ldots,x_{n+1};f]_m = \frac{1}{m^{r(n+1)}x_{n+1} - x_1}$$
$$\times\left[\frac{\sum_{i=2}^{n+1}(-1)^{n+i+1}m^{r(i)}f(x_i)V(x_2,\ldots,m^{r(i-1)}x_{i-1},m^{r(i+1)}x_{i+1},\ldots,m^{r(n+1)}x_{n+1})}{\dfrac{V(x_1,\ldots,m^{r(n+1)}x_{n+1})}{(m^{r(n+1)}x_{n+1} - x_1)\prod_{j=2}^{n}(m^{r(j)}x_j - x_1)}}\right.$$
$$\left.+\frac{\sum_{i=1}^{n}(-1)^{n+i+1}m^{r(i)}f(x_i)V(x_1,\ldots,m^{r(i-1)}x_{i-1},m^{r(i+1)}x_{i+1},\ldots,m^{r(n)}x_n)}{\dfrac{V(x_1,\ldots,m^{r(n+1)}x_{n+1})}{(m^{r(n+1)}x_{n+1} - x_1)\prod_{j=2}^{n}(m^{r(n+1)}x_{n+1} - m^{r(j)}x_j)}}\right]$$
$$= \frac{1}{V(x_1,\ldots,m^{r(n+1)}x_{n+1})}\left[\sum_{i=2}^{n+1}(-1)^{n+i+1}m^{r(i)}f(x_i)\right.$$
$$\times V(x_2,\ldots,m^{r(i-1)}x_{i-1},m^{r(i+1)}x_{i+1},\ldots,m^{r(n+1)}x_{n+1})\prod_{j=2}^{n}(m^{r(j)}x_j - x_1)$$
$$+\sum_{i=1}^{n}(-1)^{n+i+1}m^{r(i)}f(x_i)V(x_1,\ldots,m^{r(i-1)}x_{i-1},m^{r(i+1)}x_{i+1},\ldots,m^{r(n)}x_n)$$
$$\left.\times\prod_{j=2}^{n}(m^{r(n+1)}x_{n+1} - m^{r(j)}x_j)\right].$$

Actually, since for $i = 2, \ldots, n$

$$V(x_1, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1})(m^{r(n+1)}x_{n+1} - m^{r(i)}x_i)$$
$$= V(x_1, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n)}x_n)$$
$$\times \prod_{j=2}^{n}(m^{r(n+1)}x_{n+1} - m^{r(j)}x_j)(m^{r(n+1)}x_{n+1} - x_1),$$

and

$$V(x_1, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1})(m^{r(i)}x_i - x_1)$$
$$= V(x_2, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1})$$
$$\times \prod_{j=2}^{n}(m^{r(j)}x_j - x_1)(m^{r(n+1)}x_{n+1} - x_1),$$

we can write

$$[x_1, \ldots, x_{n+1}; f]_m = \frac{1}{V(x_1, \ldots, m^{r(n+1)}x_{n+1})}\Bigg[\sum_{i=2}^{n+1}(-1)^{n+i+1}m^{r(i)}f(x_i)$$
$$\times V(x_1, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1})\frac{(m^{r(i)}x_i - x_1)}{m^{r(n+1)}x_{n+1} - x_1}$$
$$+\sum_{i=1}^{n}(-1)^{n+i+1}m^{r(i)}f(x_i)V(x_1, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1})$$
$$\times \frac{(m^{r(n+1)}x_{n+1} - m^{r(i)}x_i)}{m^{r(n+1)}x_{n+1} - x_1}\Bigg]$$
$$= \frac{1}{V(x_1, \ldots, m^{r(n+1)}x_{n+1})}\Big[(-1)^{n+2}f(x_1)V(x_2, \ldots, m^{r(n+1)}x_{n+1})$$
$$+\sum_{i=2}^{n}(-1)^{n+i+1}!m^{r(i)}f(x_i)V(x_1, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1})$$
$$+ m^{r(n+1)}f(x_{n+1})V(x_1, \ldots, m^{r(n)}x_n)\Big]$$
$$= \frac{\sum_{i=1}^{n+1}(-1)^{n+i+1}m^{r(i)}f(x_i)V(x_1, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1})}{V(x_1, \ldots, m^{r(n+1)}x_{n+1})}.$$

Thus,

$$[x_1, \ldots, x_{n+1}; f]_m = \frac{U(x_1, \ldots, m^{r(n+1)} x_{n+1}; f)}{V(x_1, \ldots, m^{r(n+1)} x_{n+1})}.$$

Hence, the result is true for all $n \geq 2$. □

3. m-convexity of higher order

In the last section, we use the divided differences of functions to define the concept of m-convex function of higher order and show some of their properties.

Definition 3.1. Let $m \in [0, 1]$ and $n \in \mathbb{N}$ be fixed numbers. A function $f\colon [0, b] \to \mathbb{R}$ is called *m-convex of order n* if

$$[x_1, \ldots, x_{n+1}; f]_m \geq 0 \tag{3.1}$$

for all m-ordered points $x_1, \ldots, x_{n+1} \in [0, b]$.

Remark 3.2. Note that if $n = 2$ Theorem 2.1 implies that condition (3.1) is equivalent to

$$\frac{f(x_1)}{(x_1 - x_2)(x_1 - mx_3)} + \frac{f(x_2)}{(x_2 - x_1)(x_2 - mx_3)} + \frac{mf(x_3)}{(mx_3 - x_1)(mx_3 - x_2)} \geq 0,$$

or

$$f(x_2) \leq \frac{mx_3 - x_2}{mx_3 - x_1} f(x_1) + m \frac{x_2 - x_1}{mx_3 - x_1} f(x_3).$$

By putting $t = \frac{mx_3 - x_2}{mx_3 - x_1}$ it follows that

$$1 - t = \frac{x_2 - x_1}{mx_3 - x_1} \quad \text{and} \quad x_2 = tx_1 + m(1 - t)x_3;$$

thus,

$$f(tx_1 + m(1 - t)x_3) \leq tf(x_1) + m(1 - t)f(x_3),$$

implying that f is an m-convex function. In other words, the m-convexity of order 2, is precisely the usual m-convexity.

REMARK 3.3. If $n=2$, Theorem 2.2 implies

$$[x_1,x_2,x_3;f]_m = \frac{U(x_1,x_2,mx_3;f)}{V(x_1,x_2,mx_3)} = \frac{\begin{vmatrix} 1 & x_1 & f(x_1) \\ 1 & x_2 & f(x_2) \\ 1 & mx_3 & mf(x_3) \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & mx_3 & m^2x_3^2 \end{vmatrix}}.$$

That is, the m-convexity of f is determined for the nonnegativity of the above quotient. This fact is known from [1, Theorem 1]. Even more, Theorem 2.2 generalizes the above-cited result to m-convexity of higher order.

REMARK 3.4. Constant functions are 1-convex of any order n.

In the case of m-convexity it is known ([1, 9]) that if a function f is m-convex then it is also n-convex for any $0<n<m$. For a similar result in higher order it is necessary to consider some additional hypothesis.

PROPOSITION 3.5. *Let $m_1,m_2\in(0,1]$ with $m_1\le m_2$, and $f\colon[0,b]\to\mathbb{R}$ be a function such that*

$$f(\lambda x)=\lambda f(x) \text{ for all } \lambda\in[0,1]. \tag{3.2}$$

If f is m_2-convex of order n, then f is m_1-convex of order n as well.

PROOF. Let $x_1,\dots,x_{n+1}$ be m_1-ordered points in $[0,b]$. By Theorem 2.2,

$$[x_1,\dots,x_{n+1};f]_{m_1} = \frac{U(x_1,\dots,m_1^{r(n+1)}x_{n+1};f)}{V(x_1,\dots,m_1^{r(n+1)}x_{n+1})}.$$

But

$$U(x_1,\dots,m_1^{r(n+1)}x_{n+1};f)$$
$$=\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & f(x_1) \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & f(x_2) \\ 1 & m_1x_3 & m_1^2x_3^2 & \cdots & m_1^{n-1}x_3^{n-1} & m_1f(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & m_1^{r(n+1)}x_{n+1} & m_1^{2r(n+1)}x_{n+1}^2 & \dots & m_1^{(n-1)r(n+1)}x_{n+1}^{n-1} & m_1^{r(n+1)}f(x_{n+1}) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} & f(y_1) \\ 1 & y_2 & y_2^2 & \cdots & y_2^{n-1} & f(y_2) \\ 1 & m_2 y_3 & m_2^2 y_3^2 & \cdots & m_2^{n-1} y_3^{n-1} & m_2 f(y_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & m_2^{r(n+1)} y_{n+1} & m_2^{2r(n+1)} y_{n+1}^2 & \cdots & m_2^{(n-1)r(n+1)} y_{n+1}^{n-1} & m_2^{r(n+1)} f(y_{n+1}) \end{vmatrix},$$

where $y_j = (m_1/m_2)^{r(j)} x_j$, $j = 1, \ldots, n+1$; and the last column is obtained by using the additional hypothesis on f.

Now, since $x_1, \ldots, x_{n+1}$ are m_1-ordered, $m_1^{r(j)} x_j < m_1^{r(j+1)} x_{j+1}$ with $j = 1, \ldots, n$, which in turn implies $m_2^{r(j)} y_j < m_2^{r(j+1)} y_{j+1}$, also points $y_1, \ldots, y_{n+1} \in [0, b]$ are m_2-ordered, $U(x_1, \ldots, m_1^{r(n+1)} x_{n+1}; f) = U(y_1, \ldots, m_2^{r(n+1)} y_{n+1}; f)$ and $V(x_1, \ldots, m_1^{r(n+1)} x_{n+1}) = V(y_1, \ldots, m_2^{r(n+1)} y_{n+1})$. Therefore,

$$[x_1, \ldots, x_{n+1}; f]_{m_1} = [y_1, \ldots, y_{n+1}; f]_{m_2} \geq 0,$$

and the last inequality is a consequence of the m_2-convexity of order n of f. □

Remark 3.6. The condition (3.2) can not be omitted in Proposition 3.5. Note that, according to Remark 3.4, the function $f\colon [0, 8] \to \mathbb{R}$ given by $f(x) = -1$ is 1-convex of order 3. Nonetheless, f is not $\frac{1}{2}$-convex of order 3; indeed, if we consider the $\frac{1}{2}$-ordered points of the interval $[0, 8]$ as $x_1 = 0$, $x_2 = 1, x_3 = 3, x_4 = 7$, then

$$[0, 1, 3, 7; f]_{\frac{1}{2}} = \frac{U\left(0, 1, \frac{3}{2}, \frac{7}{4}; f\right)}{V\left(0, 1, \frac{3}{2}, \frac{7}{4}\right)} = -\frac{8}{21} < 0.$$

Proposition 3.7. *If $f, g\colon [0, b] \to \mathbb{R}$ are m-convex functions of order n, then $f + g$ and $\alpha f, \alpha > 0$ are m-convex functions of order n as well.*

Proof. Let $x_1, \ldots, x_{n+1}$ be m-ordered points in $[0, b]$, by Theorem 2.2

$$\begin{aligned} [x_1, \ldots, x_{n+1}; f+g]_m &= \frac{U(x_1, \ldots, m^{r(n+1)} x_{n+1}; f+g)}{V(x_1, \ldots, m^{r(n+1)} x_{n+1})} \\ &= \frac{U(x_1, \ldots, m^{r(n+1)} x_{n+1}; f)}{V(x_1, \ldots, m^{r(n+1)} x_{n+1})} + \frac{U(x_1, \ldots, m^{r(n+1)} x_{n+1}; g)}{V(x_1, \ldots, m^{r(n+1)} x_{n+1})} \\ &= [x_1, \ldots, x_{n+1}; f]_m + [x_1, \ldots, x_{n+1}; g]_m \\ &\geq 0. \end{aligned}$$

Also,

$$\begin{aligned}
[x_1, \ldots, x_{n+1}; \alpha f]_m &= \frac{U(x_1, \ldots, m^{r(n+1)} x_{n+1}; \alpha f)}{V(x_1, \ldots, m^{r(n+1)} x_{n+1})} \\
&= \frac{\alpha U(x_1, \ldots, m^{r(n+1)} x_{n+1}; f)}{V(x_1, \ldots, m^{r(n+1)} x_{n+1})} \\
&= \alpha [x_1, \ldots, x_{n+1}; f]_m \\
&\geq 0.
\end{aligned}$$

□

In [9] it was proved that if f and g are both nonnegative, increasing and m-convex functions (m-convex of order 2), then the product function, fg, is m-convex as well. Nevertheless, in case of higher order, this is not necessarily true.

Example 1. The function $f\colon [0, b] \to \mathbb{R}$, $b > 3$, given by $f(x) = ax, a > 0$ is clearly m-convex of any order n (and any $m \in [0, 1]$), since

$$[x_1, \ldots, x_{n+1}; f]_m = \frac{U(x_1, \ldots, m^{r(n+1)} x_{n+1}; f)}{V(x_1, \ldots, m^{r(n+1)} x_{n+1})} = 0;$$

actually, f is $\frac{1}{2}$-convex of order 3. However, the function $g = f^2$ is not $\frac{1}{2}$-convex of order 3. Indeed, if we consider the four $\frac{1}{2}$-ordered points $x_1 = 0, x_2 = \frac{1}{3}, x_3 = \frac{7}{10}, x_4 = 3$ in $[0, b]$,

$$\left[0, \frac{1}{3}, \frac{7}{10}, 3; g\right]_{\frac{1}{2}} = \frac{U(0, \frac{1}{3}, \frac{7}{20}, \frac{3}{4}; g)}{V(0, \frac{1}{3}, \frac{7}{20}, \frac{3}{4})} = \frac{-\frac{91}{9600} a^2}{\frac{7}{28800}} = -39a^2 < 0.$$

Theorem 3.8. *Let $f\colon (0, +\infty) \to \mathbb{R}$ be an m-convex function of order n. If f is bounded in a neighborhood of one point $p \in (0, +\infty)$, and the above-mentioned neighborhood contains some m-ordered collection $x_1, \ldots, x_{n+1}$ of points of $(0, +\infty)$, then f is locally bounded.*

Proof. Consider $A \subset D \subset (0, +\infty)$, where A is an m-ordered collection of points $x_1, \ldots, x_{n+1}$, and D is a neighborhood of p with radius r, such that $|f(u)| \leq M$ for all $u \in D$, with $M \in \mathbb{R}^+$. We must prove that for any $z \in (0, +\infty)$ there exists a neighborhood K on which f is bounded. If $z \in D$, we pick K as the neighborhood centered at z and radius $r - |p - z|$, in this case $K \subset D$ and f is bounded on K.

If $z \notin D$, we can choose $y \in (0, +\infty)$ with $z = \lambda p + (1 - \lambda) y$ for some $\lambda \in (0, 1)$, hence $K = \lambda D + (1 - \lambda) y$ is a neighborhood of z with radius λr.

Now, let $v \in K$. It is clear that if $v \in D$, then $|f(v)| \leq M$; so, we can assume $v \in K \backslash D$. Hence, two possibilities may occur:

(1) $y < v < w$ for all $w \in A$, this happens if $z < x$ for all $x \in D$;

(2) $y > v > w$ for all $w \in A$, which occurs only if $z > x$ for all $x \in D$.

If (1) happens, we consider the points $y, v, x_3, \ldots, x_{n+1}$ and because $v < x_2$, these points become m-ordered. Consequently, by the m-convexity of order n of f and Theorem 2.2,

$$\frac{U(y, v, mx_3, \ldots, m^{r(n+1)}x_{n+1}; f)}{V(y, v, mx_3, \ldots, m^{r(n+1)}x_{n+1})} \geq 0.$$

Even more, $V(y, v, mx_3, \ldots, m^{r(n+1)}x_{n+1}) > 0$ therefore,

$$U(y, v, mx_3, \ldots, m^{r(n+1)}x_{n+1}; f) \geq 0.$$

By developing this determinant U according to the last column,

$$\begin{aligned} &(-1)^{n+2}V(v, mx_3, \ldots, m^{r(n+1)}x_{n+1})f(y) \\ &\qquad + (-1)^{n+3}V(y, mx_3, \ldots, m^{r(n+1)}x_{n+1})f(v) \\ &+ \sum_{i=3}^{n+1}(-1)^{n+i+1}V(y, v, mx_3, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1}) \\ &\qquad \times m^{r(i)}f(x_i) \geq 0. \end{aligned}$$

Hence, if n is even,

$$\begin{aligned} (3.3)\quad f(v) &\leq \frac{V(v, mx_3, \ldots, m^{r(n+1)}x_{n+1})}{V(y, mx_3, \ldots, m^{r(n+1)}x_{n+1})}f(y) \\ &+ \sum_{i=3}^{n+1}(-1)^{i+1}\frac{V(y, v, mx_3, \ldots, m^{r(i-1)}x_{i-1}, m^{r(i+1)}x_{i+1}, \ldots, m^{r(n+1)}x_{n+1})}{V(y, mx_3, \ldots, m^{r(n+1)}x_{n+1})} \\ &\qquad \times m^{r(i)}f(x_i). \end{aligned}$$

The Vandermonde determinants involved in (3.3) are bounded, since all the differences between each pair of points corresponding may be estimated by the radii of D and K, and the distances between y and these neighborhoods. Moreover, since $x_i \in D$ for all $i \in \{3, \ldots, n+1\}$, $|f(x_i)| \leq M$.

If n is odd, we obtain the opposite inequality in (3.3); and thus, f is bounded from below on K.

For the boundedness of f on K in the opposite direction (from below if n is even, and from above if n is odd), we may choose the m-ordered points $v, x_2, x_3, \ldots, x_{n+1}$, and by reasoning in a similar manner to the above arguments, $U(v, x_2, mx_3, \ldots, m^{r(n+1)}x_{n+1}; f) \geq 0$. Whereupon, the desired boundedness is shown.

For the case (2), if necessary, we can decrease K so that $v < my$, in this case $m^{r(n)}v < m^{r(n+1)}y$ and because $x_n < v$, points $x_1, \ldots, x_{n-1}, v, y$ become m-ordered, as well as $x_1, \ldots, x_n, v$. Now from the m-convexity of order n of f, and arguing as before, we obtain

$$U(x_1, \ldots, m^{r(n-1)}x_{n-1}, m^{r(n)}v, m^{r(n+1)}y; f) \geq 0$$

and

$$U(x_1, \ldots, m^{r(n)}x_n, m^{r(n+1)}v; f) \geq 0.$$

The rest of the proof goes in a similar way to the previous case. □

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