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ON A NEW ONE PARAMETER GENERALIZATION OF PELL NUMBERS

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Abstract. In this paper we present a new one parameter generalization of the classical Pell numbers. We investigate the generalized Binet's formula, the generating function and some identities for r-Pell numbers. Moreover, we give a graph interpretation of these numbers.

1. Introduction

The Pell sequence $\{P_n\}$ is one of the special cases of sequences $\{a_n\}$ which are defined recurrently as a linear combination of the preceding k terms

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_k a_{n-k} \quad \text{for } n \geq k, \tag{1.1}$$

where $k \geq 2$, b_i are integers, $i = 1, 2, \ldots, k$ and $a_0, a_1, \ldots, a_{k-1}$ are given numbers.

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By recurrence (1.1) for $k = 2$ we get (among others) the well-known recurrences:

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1 \quad \text{(Fibonacci numbers)},$$
$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \quad L_1 = 1 \quad \text{(Lucas numbers)},$$
$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \quad J_1 = 1 \quad \text{(Jacobsthal numbers)},$$
$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \quad P_1 = 1 \quad \text{(Pell numbers)}.$$

The first ten terms of the Pell sequence are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985$. The n-th Pell number is explicitly given by the Binet-type formula

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \quad \text{for } n \geq 0.$$

Moreover, the Pell numbers are defined by the following formula

$$P_n = \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} 2^k.$$

The matrix generator of the sequence $\{P_n\}$ is $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. It is known that

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Hence we get the well-known formula (Cassini's identity) $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$. Another interesting properties of the Pell numbers are given in [4].

In the literature there are some generalizations of the Pell numbers. We recall some of them. In [5] the authors introduced p-Pell numbers $P_p(n)$ defined by the following relation: $P_p(n) = 2P_p(n-1) + P_p(n-p-1)$ for $p = 0, 1, 2\ldots$ and $n \geq p+2$ with $P_p(1) = a_1$, $P_p(2) = a_2$, $\ldots$, $P_p(p+1) = a_{p+1}$, where $a_1, a_2, \ldots, a_{p+1}$ are integers, real or complex numbers. Another generalization of the Pell numbers is given in [1], [2]: the k-Pell numbers $\{P_{k,n}\}$ are defined recurrently by $P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}$ for $k \geq 1$ and $n \geq 1$ with $P_{k,0} = 0$, $P_{k,1} = 1$.

In [6] there was presented k-distance Pell sequence defined as follows: $P_k(n) = 2P_k(n-1) + P_k(n-k)$ for $n \geq k$ with $P_k(0) = 0, P_k(n) = 2^{n-1}$ for $n = 1, 2, \ldots, k-1$. Another interesting generalizations of the Pell numbers can be found in [9].

In this paper we introduce a new one parameter generalization of Pell numbers.

2. The r-Pell numbers and some basic properties

Let $n \geq 0$, $r \geq 1$ be integers. Define *r-Pell sequence* $\{P(r,n)\}$ by the following recurrence relation

$$P(r,n) = 2^r P(r,n-1) + 2^{r-1} P(r,n-2) \quad \text{for } n \geq 2 \tag{2.1}$$

with initial conditions $P(r,0) = 2$, $P(r,1) = 1 + 2^{r+1}$.

It is easily seen that $P(1,n) = P_{n+2}$. By (2.1) we obtain

$$\begin{aligned}
P(r,0) &= 2,\\
P(r,1) &= 1 + 2^{r+1},\\
P(r,2) &= 2^{r+1} + 2 \cdot 4^r,\\
P(r,3) &= 2^{r-1} + 3 \cdot 4^r + 2 \cdot 8^r,\\
P(r,4) &= \tfrac{3}{2} \cdot 4^r + 4 \cdot 8^r + 2 \cdot 16^r.
\end{aligned}$$

Now we present the Binet's formula, which allows us to express the r-Pell numbers in function of the roots r_1 and r_2 of the following characteristic equation, associated with the recurrence relation (2.1)

$$x^2 - 2^r x - 2^{r-1} = 0. \tag{2.2}$$

Then

$$r_1 = \frac{2^r + \sqrt{4^r + 2^{r+1}}}{2}, \quad r_2 = \frac{2^r - \sqrt{4^r + 2^{r+1}}}{2}. \tag{2.3}$$

Proposition 2.1 (Binet's formula). *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$P(r,n) = C_1 r_1^n + C_2 r_2^n, \tag{2.4}$$

where r_1, r_2 are given by (2.3) *and*

$$C_1 = 1 + \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}, \quad C_2 = 1 - \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}.$$

PROOF. The general term of the sequence $\{P(r,n)\}$ may be expressed in the following form

$$P(r,n) = C_1 r_1^n + C_2 r_2^n$$

for some coefficients C_1 and C_2. Using initial conditions of the recurrence (2.1), we obtain the following system of two linear equations

$$\begin{cases} C_1 + C_2 = 2, \\ C_1 r_1 + C_2 r_2 = 1 + 2^{r+1}. \end{cases}$$

Hence

$$C_1 = 1 + \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}} \quad \text{and} \quad C_2 = 1 - \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}},$$

which ends the proof. □

Since r_1 and r_2 are the roots of equation (2.2), we have

$$r_1 + r_2 = 2^r, \tag{2.5}$$

$$r_1 - r_2 = \sqrt{4^r + 2^{r+1}}, \tag{2.6}$$

$$r_1 r_2 = -2^{r-1}. \tag{2.7}$$

Moreover, by simple calculations, we get

$$C_1 C_2 = -\frac{1}{4^r + 2^{r+1}}, \tag{2.8}$$

$$C_1 r_2 + C_2 r_1 = -1. \tag{2.9}$$

3. Some identities for the sequence $\{P(r,n)\}$

In this section we present some properties and identities for the r-Pell numbers. They generalize known results for classical Pell numbers.

THEOREM 3.1. *Let r be a positive integer. Then*

$$\lim_{n\to\infty}\frac{P(r,n+1)}{P(r,n)}=\frac{2^r+\sqrt{4^r+2^{r+1}}}{2}.$$

PROOF. Using Proposition 2.1, we have

$$\lim_{n\to\infty}\frac{P(r,n+1)}{P(r,n)}=\lim_{n\to\infty}\frac{C_1{r_1}^{n+1}+C_2r_2^{n+1}}{C_1{r_1}^n+C_2r_2^n}=\lim_{n\to\infty}\frac{C_1r_1+C_2r_2\left(\frac{r_2}{r_1}\right)^n}{C_1+C_2(\frac{r_2}{r_1})^n}.$$

Since $\lim\limits_{n\to\infty}(\frac{r_2}{r_1})^n=0$, we get

$$\lim_{n\to\infty}\frac{P(r,n+1)}{P(r,n)}=r_1=\frac{2^r+\sqrt{4^r+2^{r+1}}}{2}. \qquad \square$$

THEOREM 3.2 (Cassini's identity). *Let n,r be positive integers. Then*

$$P(r,n+1)P(r,n-1)-P^2(r,n)=(-1)^n2^{(r-1)(n-1)}. \tag{3.1}$$

PROOF. By Binet's formula (2.4) we obtain

$$\begin{aligned}&P(r,n+1)P(r,n-1)-P^2(r,n)\\&\quad=(C_1r_1^{n+1}+C_2r_2^{n+1})(C_1r_1^{n-1}+C_2r_2^{n-1})-(C_1r_1^n+C_2r_2^n)^2\\&\qquad=C_1C_2(r_1r_2)^n(\frac{r_1}{r_2}+\frac{r_2}{r_1}-2)=C_1C_2(r_1r_2)^{n-1}(r_1-r_2)^2,\end{aligned}$$

where r_1, r_2 are given by (2.3).

Using formulas (2.8), (2.7) and (2.6), we have

$$P(r,n+1)P(r,n-1)-P^2(r,n)=-(-2^{r-1})^{n-1}=(-1)^n2^{(r-1)(n-1)}. \qquad \square$$

By formula (3.1), considering $r=1$ and taking into account that $P(1,n)=P_{n+2}$, we obtain Cassini's identity for the classical Pell numbers.

COROLLARY 3.3. *For $n\geq 1$, $P_{n+1}P_{n-1}-P_n^2=(-1)^n$.*

The next theorem presents a summation formula for the r-Pell numbers.

THEOREM 3.4. *Let n, r be positive integers. Then*

$$\sum_{i=0}^{n-1} P(r,i) = \frac{P(r,n) + 2^{r-1}P(r,n-1) - 3}{3 \cdot 2^{r-1} - 1}.$$

PROOF. Using formula (2.4), we have

$$\sum_{i=0}^{n-1} P(r,i) = \sum_{i=0}^{n-1}(C_1 r_1^i + C_2 r_2^i) = C_1 \frac{1 - r_1^n}{1 - r_1} + C_2 \frac{1 - r_2^n}{1 - r_2}$$
$$= \frac{C_1 + C_2 - (C_1 r_2 + C_2 r_1) - (C_1 r_1^n + C_2 r_2^n) + r_1 r_2 (C_1 r_1^{n-1} + C_2 r_2^{n-1})}{1 - (r_1 + r_2) + r_1 r_2}.$$

By Binet's formula we get

$$\sum_{i=0}^{n-1} P(r,i) = \frac{C_1 + C_2 - (C_1 r_2 + C_2 r_1) - P(r,n) + r_1 r_2 P(r,n-1)}{1 - (r_1 + r_2) + r_1 r_2}.$$

By (2.9), (2.7) and (2.5) we obtain

$$\sum_{i=0}^{n-1} P(r,i) = \frac{P(r,n) + 2^{r-1}P(r,n-1) - 3}{3 \cdot 2^{r-1} - 1}. \qquad \square$$

Using twice the recurrence (2.1), we obtain the following result.

PROPOSITION 3.5. *Let n, r be integers such that $n \geq 4$, $r \geq 1$. Then*

$$P(r,n) = (8^r + 4^r)P(r,n-3) + (2^{3r-1} + 2^{2r-2})P(r,n-4).$$

THEOREM 3.6. *The generating function of the sequence $\{P(r,n)\}$ has the following form*

$$f(x) = \frac{2 + x}{1 - 2^r x - 2^{r-1} x^2}.$$

Proof. Assuming that the generating function of the sequence $\{P(r,n)\}$ has the form $f(x)=\sum_{n=0}^{\infty} P(r,n)x^n$, we get

$$\begin{aligned}(1-2^r x-2^{r-1}x^2)f(x) &= (1-2^r x-2^{r-1}x^2)\sum_{n=0}^{\infty} P(r,n)x^n\\ &= \sum_{n=0}^{\infty} P(r,n)x^n - 2^r\sum_{n=0}^{\infty} P(r,n)x^{n+1} - 2^{r-1}\sum_{n=0}^{\infty} P(r,n)x^{n+2}\\ &= \sum_{n=2}^{\infty}(P(r,n)-2^rP(r,n-1)-2^{r-1}P(r,n-2))x^n\\ &\qquad + (P(r,0)+P(r,1)x)-2^rP(r,0)x\end{aligned}$$

By recurrence (2.1) we have

$$(1-2^r x-2^{r-1}x^2)f(x)=2+(1+2^{r+1}-2^{r+1})x.$$

Hence

$$(1-2^r x-2^{r-1}x^2)f(x)=2+x.$$

Thus

$$f(x)=\frac{2+x}{1-2^r x-2^{r-1}x^2},$$

which ends the proof. □

4. A graph interpretation of the r-Pell numbers

In general we use the standard terminology and notation of graph theory, see [3]. Let G be a simple, undirected, finite graph with vertex set $V(G)$ and edge set $E(G)$. By P_n, C_m, $n\geq 1$, $m\geq 3$, we mean n-vertex path, m-vertex cycle, respectively. A set $S\subseteq V(G)$ is independent if no edge of G has both its endpoints in S. Moreover, a subset of $V(G)$ containing only one vertex and the empty set are independent sets of G. The total number of independent sets of a graph G, including the empty set, is known as the Merrifield-Simmons index. It is denoted by $i(G)$ or $NI(G)$. For a graph G with $V(G)=\emptyset$ we put

$i(G) = 1$. The Merrifield-Simmons index is an example of topological index, which is of interest in combinatorial chemistry. This parameter was introduced in 1982 by Prodinger and Tichy in [7]. It was called the Fibonacci number of a graph. It has been proved that $i(P_n) = F_{n+1}$, $i(C_n) = L_n$. In recent years, many researches have investigated this index, see for example [8]. We will show that the r-Pell numbers can be used for counting independent sets in special classes of graphs.

Let $x \in V(G)$. By $i_x(G)$ ($i_{-x}(G)$, respectively) we denote the number of independent sets S of G such that $x \in S$ ($x \notin S$, respectively). Hence we get the basic rule for counting of independent sets of a graph G

$$i(G) = i_x(G) + i_{-x}(G). \tag{4.1}$$

Consider a graph $H_{n,r}$ (Figure 1), where $n \geq 1$, $r \geq 1$, $H_{1,r} = K_{1,r+1}$.

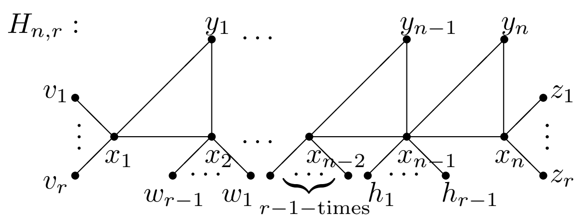

Figure 1. A graph $H_{n,r}$

THEOREM 4.1. *Let n, r be integers such that $n \geq 1$, $r \geq 1$. Then*

$$i(H_{n,r}) = P(r, n).$$

PROOF. Let $n \geq 3$. Assume that vertices of $H_{n,r}$ are numbered as in Figure 1. Using formula (4.1), we have

$$i(H_{n,r}) = i_{x_n}(H_{n,r}) + i_{-x_n}(H_{n,r}).$$

Let S be any independent set of $H_{n,r}$. Consider two cases.

Case 1. $x_n \in S$. Then $x_{n-1}, y_n, z_1, \ldots, z_r \notin S$. Hence $S = S' \cup \{x_n\} \cup Z$, where S' is any independent set of the graph

$$H_{n,r} \setminus \{x_{n-1}, y_n, z_1, \ldots, z_r, h_1, \ldots, h_r\},$$

which is isomorphic to $H_{n-2,r}$, and Z is any subset of the set $\{h_1, h_2, \ldots, h_{r-1}\}$. Hence we get

$$i_{x_n}(H_{n,r}) = 2^{r-1} i(H_{n-2,r}).$$

Case 2. $x_n \notin S$. Proving analogously as in Case 1, we have

$$i_{-x_n}(H_{n,r}) = 2^r i(H_{n-2,r}).$$

Consequently, for $n \geq 3$ we get

$$i(H_{n,r}) = 2^{r-1} i(H_{n-1,r}) + 2^r i(H_{n-2,r}).$$

Now we consider graphs $H_{1,r}$ and $H_{2,r}$. It is easy to check that $i(H_{1,r}) = 1 + 2^{r+1} = P(r, 1)$. Using the same method for the graph $H_{2,r}$ as in Case 1, we have

$$\begin{aligned} i(H_{2,r}) &= i_{x_2}(H_{2,r}) + i_{-x_2}(H_{2,r}) \\ &= 2^r + 2^r(1 + 2^{r+1}) = 2(4^r + 2^r) = P(r, 2). \quad \square \end{aligned}$$

Corollary 4.2. *For* $n \geq 1$

$$i(H_{n,1}) = P(1, n) = P_{n+2}.$$

The graph interpretation of r-Pell numbers can be used for proving some identities.

Theorem 4.3. (*Convolution identity*) *Let* n, m, r *be integers such that* $m \geq 2, n \geq 1,\ r \geq 1$. *Then*

$$P(r, m + n) = 2^{r-1} P(r, m - 1) P(r, n) + 2^{2r-2} P(r, m - 2) P(r, n - 1).$$

Proof. It is easy to check that the theorem is true for $m = 2$ and $n = 1$, we have namely

$$P(r, 3) = 2^{r-1}(1 + 2^{r+1})^2 + 4 \cdot 2^{2r-2} = 2^{r-1} + 3 \cdot 4^r + 2 \cdot 8^r.$$

Moreover, for $m = 2$ and $n = 2$ we obtain

$$\begin{aligned} P(r, 4) &= 2^{r-1}(1 + 2^{r+1})(2^{r+1} + 2 \cdot 4^r) + 2^{2r-2}(2 + 2^{r+2}) \\ &= 2 \cdot 16^r + 4 \cdot 8^r + \frac{3}{2} \cdot 4^r. \end{aligned}$$

Assume now that $m \geq 3$, $n \geq 2$. Consider the graph $H_{m+n,r}$. Assume that vertices of the graph are numbered analogously as in Figure 1. By Theorem 4.1 we have $i(H_{m+n,r}) = P(r, m + n)$. Assume that x_m is any vertex of the graph $H_{m+n,r}$, such that $\deg x_m = r + 3$. Let S be any independent set of the graph $H_{m+n,r}$. Denote by $L(x_i)$ the set of pendant vertices attached to the vertex x_i, $i = 1, 2, 3, \ldots, m+n$. Consider two cases.

Case 1. $x_m \in S$. Then $x_{m-1}, x_{m+1}, y_m, y_{m-1} \notin S$. Moreover, $L(x_m) \not\subset S$. Then $S = S^* \cup S^{**} \cup Z_1 \cup Z_2 \cup \{x_m\}$, where S^* is an independent set of the graph $H_{m+n,r} \setminus \bigcup_{i=0}^{n+1} \{x_{m+n-i}\} \setminus \bigcup_{j=0}^{n+2} \{y_{m+n-j}\} \setminus L(x_i)$, which is isomorphic to the graph $H_{m-2,r}$, Z_1, Z_2 is any subset of the set $L(x_{m-1})$, $L(x_{m+1})$, resp. Moreover, S^{**} is an independent set of the graph $H_{m+n,r} \setminus \bigcup_{i=1}^{m+1} \{x_i, y_i\} \setminus L(x_i)$, which is isomorphic to the graph $H_{n-1,r}$. Thus we obtain

$$i_{x_m}(H_{m+n,r}) = (2^{r-1})^2 P(r, m-2) P(r, n-1).$$

Case 2. $x_m \notin S$. Using the same method as in Case 1, we have

$$i_{-x_m}(H_{m+n,r}) = 2^{r-1} P(r, m-1) P(r, n).$$

Consequently,

$$\begin{aligned} i(H_{m+n,r}) &= P(r, m+n) \\ &= 2^{r-1} P(r, m-1) P(r, n) + 2^{2r-2} P(r, m-2) P(r, n-1). \qquad \square \end{aligned}$$

Using the fact that $P(0, n) = P_{n+2}$, we get known identity for classical Pell numbers.

Corollary 4.4. $P_{m+n} = P_m P_{n+1} + P_{m-1} P_n$.

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