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CHARACTERIZATION OF CARATHÉODORY FUNCTIONS

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Abstract. We study Carathéodory functions $f\colon D \to Y$, where $(T, \mathcal{T})$ is a measurable space, X, Y are metric spaces and $D \subset T \times X$. In the case when $\mathcal{T}$ is complete and Y is a separable Banach space, we give a characterization of such functions.

1. Preliminaries

In this section we introduce notation and definitions, and quote some auxiliary results.

Throughout the whole paper $(T, \mathcal{T})$ is a measurable space, and X, Y are metric spaces. We say that $\mathcal{T}$ is *complete*, if there exists a σ-finite measure μ such that $\mathcal{T}$ is complete with respect to μ. For $S \subset T$ by $S \cap \mathcal{T}$ we denote the trace σ-field on S, i.e., $S \cap \mathcal{T} = \{S \cap U : U \in \mathcal{T}\}$. $\mathcal{B}(X)$ stands for the Borel σ-field on X, and $\mathcal{T} \otimes \mathcal{B}(X)$ for the product σ-field on $T \times X$.

Let φ be a multifunction from T to X, i.e., $\varphi\colon T \to 2^X$ and $\varphi(t) \neq \emptyset$ for all $t \in T$. We refer to [5] for terminology and proofs of auxiliary results on multifunctions. By the *graph* of φ we mean the set $\operatorname{Gr}\varphi = \{(t, x) \in T \times X : x \in \varphi(t)\}$. We say that φ is *measurable* (*weakly measurable*) if for each closed (open) set $A \subset X$ the preimage $\varphi^-(A) = \{t \in T : \varphi(t) \cap A \neq \emptyset\}$ belongs to the σ-field $\mathcal{T}$. If φ is measurable then it is weakly measurable. If X is separable and φ is weakly measurable and closed-valued, then $\operatorname{Gr}\varphi \in \mathcal{T} \otimes \mathcal{B}(X)$. A function $h\colon T \to X$ is a *measurable selector of* φ if it is measurable

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and $h(t) \in \varphi(t)$ for all $t \in T$. A countable family (h_n) of measurable selectors of φ such that for each $t \in T$ the set $\{h_n(t) : n \in \mathbb{N}\}$ is dense in $\varphi(t)$ is called a *Castaing representation of* φ. We shall use the following measurable selection theorems (cf. [10] and [11]):

THEOREM 1.1. *Let X be separable and φ a weakly measurable multifunction from T to X with complete values. Then φ has a Castaing representation.*

THEOREM 1.2. *Suppose $\mathcal{T}$ is complete, X is separable and complete, and φ is a multifunction from T to X. If $\operatorname{Gr}\varphi \in \mathcal{T} \otimes \mathcal{B}(X)$, then φ admits a Castaing representation.*

Assume that the metric space X is locally compact and separable. Denote by $C(X,Y)$ the space of all continuous functions $u\colon X \to Y$ endowed with the compact-open topology. There a sequence of compact sets (X_n) such that $X = \bigcup_{n=1}^{\infty} X_n$ and $X_n \subset \operatorname{int} X_{n+1}$. The compact-open topology of $C(X,Y)$ is metrizable by the metric

$$\rho(u,v) = \sum_{n=1}^{\infty} \frac{\rho_n(u,v)}{2^n(1+\rho_n(u,v))},$$

where $\rho_n(u,v) = \sup\{\rho_Y(u(x),v(x)) : x \in X_n\}$, $n \in \mathbb{N}$, and ρ_Y is the metric of Y (see, e.g., [9], 44.VII). A sequence (u_n) converges in this metric iff it converges uniformly on each compact subset of X. It is known that if Y is separable (complete) then $C(X,Y)$ is also separable (complete) (cf. [9], 44.VII, Theorem 3).

2. Carathéodory functions

Let f be a function from $T \times X$ to Y. We associate with f a new function F defined on T by $F(t)(x) = f(t,x)$. The following theorem is well known:

THEOREM 2.1. *Suppose X is locally compact and separable and Y is separable. Then $f\colon T \times X \to Y$ is measurable in t and continuous in x iff F is $C(X,Y)$-valued and measurable as a function from T to $C(X,Y)$.*

Appel and Väth gave a version of such theorem with $(T,\mathcal{T},\mu)$ being a measure space and the Bochner measurability ([1], Theorem 1). We are going to prove an analogue of Theorem 2.1 for the case when the domain of f is a subset of $T \times X$.

If the space X is separable and a function $f\colon T\times X\to Y$ is measurable in t and continuous in x, then f is product-measurable. It is not the case when f is defined on a subset of $T\times X$.

Let $D\subset T\times X$ satisfying $\mathrm{proj}_T D=T$ be fixed for the rest of the paper. By D_t and D^x we denote, respectively, t-sections and x-sections of D. A function $f\colon D\to Y$ such that for each $t\in T$, $f(t,\cdot)$ is continuous on D_t, and for each $x\in \mathrm{proj}_X D$, $f(\cdot,x)$ is $D^x\cap\mathcal{T}$-measurable need not be $D\cap\mathcal{T}\otimes\mathcal{B}(X)$-measurable (see e.g., [8], p. 304). This observation motivates the following definition: A function $f\colon D\to Y$ is *Carathéodory* if it is $D\cap\mathcal{T}\otimes\mathcal{B}(X)$-measurable and for each $t\in T$, $f(t,\cdot)$ is continuous on D_t (cf. [8]).

Remark 2.1. Note that if X is separable, $g\colon T\times X\to Y$ is measurable in t and continuous in x, then g is Carathéodory in the above sense. Moreover, for each $D\subset T\times X$ the function $g|_D$ is Carathéodory.

Now let $f\colon D\to Y$ be continuous in x, i.e., for each $t\in T$, $f(t,\cdot)$ is continuous on D_t. As previous, we associate with f the function F defined by $F(t)(x)=f(t,x)$, $x\in D_t$. For each $t\in T$, $F(t)$ is an element of the space $C(D_t,Y)$. How can we define the measurability of such a function F?

Suppose X is locally compact and separable, and (X_n) is a sequence of compact sets such as in Section 1. We shall assume that the sections D_t are closed and the multifunction $t\mapsto D_t$, $t\in T$, is measurable. Since D is the graph of this multifunction, it implies $D\in\mathcal{T}\otimes\mathcal{B}(X)$. The space $C(D_t,Y)$ is endowed with the metric ρ_t defined by

$$\rho_t(v,w)=\sum_{n=1}^{\infty}\frac{\rho_{nt}(v,w)}{2^n(1+\rho_{nt}(v,w))},\quad v,w\in C(D_t,Y),$$

where $\rho_{nt}(v,w)=0$ if $D_t\cap X_n=\emptyset$, and $\rho_{nt}(v,w)=\sup\{\rho_Y(v(x),w(x)) : x\in D_t\cap X_n\}$ if $D_t\cap X_n\neq\emptyset$.

We say that the function F defined as above is *measurable* if for each $u\in C(X,Y)$ the real-valued function $t\mapsto\rho_t(u|_{D_t},F(t))$, $t\in T$, is measurable.

Remark 2.2. If $D_t=X$ for all $t\in T$, then the metric ρ_t does not depend on t, and coincides with ρ defined in Section 1. In this case our definition says that for each $u\in C(X,Y)$ the function $t\mapsto\rho(u,F(t))$, $t\in T$, is measurable. If Y is separable then $C(X,Y)$ is also separable, and the last condition is equivalent to the measurability of F as a function from T to $C(X,Y)$.

3. Main results

In this section X and D satisfy assumptions stated above, before the definition of the metric ρ_t on $C(D_t, Y)$. Let $f\colon D \to Y$ be continuous in x, and F associated to f. We shall study relations among the following three conditions:

(i) f is Carathéodory;
(ii) F is measurable;
(iii) f can be extended to a Carathéodory function $g\colon T \times X \to Y$.

It follows from Remark 2.1 that (iii) implies (i).

THEOREM 3.1. *If Y is separable then (i) $\Rightarrow$ (ii).*

PROOF. Let $T_n = \{t \in T : D_t \cap X_n \neq \emptyset\}$, $n \in \mathbb{N}$. Under our assumptions, $T_n \in \mathcal{T}$. Fix $n \in \mathbb{N}$ such that $T_n \neq \emptyset$ and $u \in C(X, Y)$. It suffices to prove that the function $t \mapsto \rho_{nt}(u|_{D_t}, F(t))$, $t \in T_n$, is measurable.

Note that the multifunction $t \mapsto D_t \cap X_n$, $t \in T_n$, is measurable and compact-valued. Thus it admits the Castaing representation, i.e., there exists a sequence of measurable functions $h_k\colon T_n \to X$, $k \in \mathbb{N}$, such that $\mathrm{cl}\{h_k(t) : k \in \mathbb{N}\} = D_t \cap X_n$ for each $t \in T_n$. Hence,

$$\sup\{\rho_Y(u(x), f(t,x)) : x \in D_t \cap X_n\} = \sup\{\rho_Y(u(h_k(t)), f(t, h_k(t)) : k \in \mathbb{N}\}.$$

The functions $t \mapsto u(h_k(t))$, $t \in T_n$, and $t \mapsto f(t, h_k(t))$, $t \in T_n$, are measurable. Thus $t \mapsto \rho_Y(u(h_k(t)), f(t, h_k(t))$, $t \in T_n$, is also measurable. Consequently, the function $t \mapsto \rho_{nt}(u|_{D_t}, F(t))$, $t \in T$, is measurable, which completes the proof. □

REMARK 3.1. In some interesting cases the measurability of the multifunction $t \mapsto D_t$ follows from $D \in \mathcal{T} \otimes \mathcal{B}(X)$. Note that D is the graph of this multifunction. If $\mathcal{T}$ is complete, X is a complete and separable metric space, and $D \in \mathcal{T} \otimes \mathcal{B}(X)$, then $t \mapsto D_t$, $t \in T$, is measurable. It follows from the Projection Theorem (see e.g. [3], Theorem 1.3). If T and X are complete and separable metric spaces, $\mathcal{T} = \mathcal{B}(T)$ and $D \in \mathcal{B}(T \times X)$ has σ-compact t-sections, then the multifunction $t \mapsto D_t$ is Borel-measurable. This is a consequence of the Arsenin–Kunugui–Novikov Theorem (see [6], Theorem 18.18).

THEOREM 3.2. *If the σ-field $\mathcal{T}$ is complete and Y is a separable Banach space, then (ii) $\Rightarrow$(iii).*

PROOF. Let the multifunction $\Phi\colon T \to 2^{C(X,Y)}$ be defined by

$$\Phi(t) = \{v \in C(X,Y) : v|_{D_t} = F(t)\}, \quad t \in T.$$

By the Dugundji Extension Theorem, $\Phi(t) \neq \emptyset$. If $G\colon T \to C(X,Y)$ is a measurable selector of Φ, then for each $t \in T$, $G(t)|_{D_t} = F(t)$. Let $g(t,x) = G(t)(x)$. Such g is measurable in t, continuous in x, and $g(t,x) = F(t)(x) = f(t,x)$ for $x \in D_t$. It means that g is a required extension of f. Hence, it suffices to prove that Φ has a measurable selector.

In order to apply Theorem 1.2, we show that $\mathrm{Gr}\,\Phi \in \mathcal{T} \otimes \mathcal{B}(C(X,Y))$. We have $\mathrm{Gr}\,\Phi = \{(t,v) \in T \times C(X,Y) : \rho_t(v|_{D_t}, F(t)) = 0\}$. The function $(t,v) \mapsto \rho_t(v|_{D_t}, F(t))$, $(t,v) \in T \times C(X,Y)$, is continuous in v and measurable in t, by the measurability of F. Since Y is separable, $C(X,Y)$ is separable too, and this function is $\mathcal{T} \otimes \mathcal{B}(X)$-measurable. Hence $\mathrm{Gr}\,\Phi \in \mathcal{T} \otimes \mathcal{B}(X)$, and Φ has a measurable selector. It completes the proof. □

COROLLARY. *Suppose $\mathcal{T}$ is complete and Y is a separable Banach space. Then (i) ⇔ (ii) ⇔ (iii).*

REMARK 3.2. The problem of the extension of Carathéodory functions defined on $D \subset T \times X$ was studied by DeBlasi and Myjak [4], Kucia [7],[8], and Brown and Schreiber [2]. It is already known that under assumptions of Theorem 3.2 the implication (i)⇒(iii) holds (see [7], Corollary and [8], Corollary 3).

PROBLEMS:

1. It would be interesting to know if the implication (ii) ⇒ (i) holds without completeness of $\mathcal{T}$ and linear structure of Y. Of course, in this case we can not expect the implication (ii) ⇒ (iii).

2. Does Theorem 3.2 hold for an arbitrary σ-field $\mathcal{T}$? In fact, we ask if the multifunction Φ defined in the proof admits a measurable selector.

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