Annales Mathematicae Silesianae **24** (2010), 71–85
Prace Naukowe Uniwersytetu Śląskiego nr 2874, Katowice

A KNESER THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

Marc Mitschele

Abstract. We show that the set of solutions of the initial-value problem

$$u(\tau) = a, \quad u'(t) = g(t, u(t)) + k(t, u(t)), \quad \tau \leq t \leq T,$$

in a Banach space is compact and connected, whenever g and k are bounded and continuous functions such that g is one-sided Lipschitz and k is Lipschitz with respect to the Kuratowski measure of noncompactness. The existence of solutions is already known from Sabina Schmidt [10].

1. Introduction

In the following let E be a Banach space with norm $\|\cdot\|$, and let τ, T be real numbers such that $\tau < T$. We consider the initial-value problem

$$u(\tau) = a, \quad u'(t) = f(t, u(t)), \quad \tau \leq t \leq T, \tag{1.1}$$

where $a \in E$, $f = g + k$, the functions $g, k\colon [\tau, T] \times E \to E$ being continuous and bounded, g one-sided Lipschitz and k an α-Lipschitz function. The last two conditions mean the following:

$$[x - y, g(t, x) - g(t, y)]_- \leq L\,\|x - y\|, \quad \tau \leq t \leq T,\ x, y \in E,$$

Received: 8.02.2010. Revised: 27.01.2011.
(2010) Mathematics Subject Classification: 34G20.
Key words and phrases: ordinary differential equations in Banach spaces, theorem of Sabina Schmidt, theorem of Hellmuth Kneser.

where generally $[x, y]_- = \lim_{h \uparrow 0} \frac{1}{h} (\|x + hy\| - \|x\|)$, $x, y \in E$;

$$\alpha(k([\tau, T] \times B)) \le K\alpha(B), \quad B \subseteq E,\ B \text{ bounded},$$

α denoting the Kuratowski measure of noncompactness.

It is known from Sabina Schmidt (1989, [10]) that, under these hypotheses, the initial-value problem (1.1) has at least one solution

$$u\colon [\tau, T] \to E. \tag{1.2}$$

The proof of this result can also be found in Peter Volkmann's survey [11].

The present paper shows that the set of solutions (1.2) of (1.1) is a compact and connected subset of the Banach space $C([\tau, T], E)$.

2. Notations and tools

We use $S(x, r)$ to denote the closed ball in E with center x and radius r, and $\overline{A}$ to denote the closed hull of a set $A \subseteq E$. As usual, the diameter $\operatorname{diam}(A)$ of a set $A \subseteq E$ means the number $\sup\{\|x - y\| : x, y \in A\}$, which for A empty (unbounded) is taken to be zero (resp. infinity). The *Kuratowski measure of noncompactness* $\alpha(A)$ of a bounded set $A \subseteq E$ is defined as

$$\inf\left\{\delta > 0 :\ A = \bigcup_{i=1}^{n} A_i,\ \operatorname{diam}(A_i) \le \delta,\ i = 1, \ldots, n,\ n \in \mathbb{N}\right\}.$$

We use the symbol $\mathbb{N}$ for the set of natural numbers $\{1, 2, \ldots\}$. Now we list some properties of α (cf. [1]): Let A and B be bounded subsets of E and $s \in \mathbb{R}$, then

$$A \subseteq B \text{ implies } \alpha(A) \le \alpha(B), \tag{2.1}$$

$$\alpha(\overline{A}) = \alpha(A), \tag{2.2}$$

$$\alpha(A + B) \le \alpha(A) + \alpha(B),\ \alpha(s \cdot A) = |s| \cdot \alpha(A), \tag{2.3}$$

$$\alpha(A) = 0 \text{ if and only if } A \text{ is relatively compact}, \tag{2.4}$$

$$\alpha(S(x, r)) = 2r \text{ if } \dim E = \infty. \tag{2.5}$$

Let (x_n) be a sequence in E, $x \in E$ and let (c_n) be a bounded sequence in $\mathbb{R}$ such that $\|x_n - x\| \le c_n$ for all $n \in \mathbb{N}$, then

$$\alpha(\{x_n : n \in \mathbb{N}\}) \le 2 \limsup_{n \to \infty} c_n. \tag{2.6}$$

The following lemma has been proved by S. Schmidt [10] for χ instead of α, where χ denotes the Hausdorff measure of noncompactness.

LEMMA (Schmidt). *Let (x_n) be a bounded sequence in E. Then for any $\varepsilon > 0$ there exists a subsequence (y_n) of (x_n), such that each infinite subset B of $\{y_n : n \in \mathbb{N}\}$ satisfies $2\alpha(B) \geq \alpha(\{x_n : n \in \mathbb{N}\}) - \varepsilon$.*

PROOF. Without loss of generality we assume $\alpha_0 := \alpha(\{x_n : n \in \mathbb{N}\}) > \varepsilon$. Then we can choose $x_{n_1} = x_1$ and $x_{n_2}, x_{n_3}, \ldots$ with $n_2 < n_3 < \ldots$ such that

$$x_{n_{k+1}} \notin \bigcup_{j=1}^{k} S(x_{n_j}, \frac{1}{2}(\alpha_0 - \varepsilon))$$

for all $k \in \mathbb{N}$. From this we obtain the sequence (y_k) by setting $y_k = x_{n_k}$ for all $k \in \mathbb{N}$. □

In the following $C([\tau, T], E)$ denotes the Banach space of all continuous functions $u\colon [\tau, T] \to E$, where $\|u\| = \max_{\tau \leq t \leq T} \|u(t)\|$. Let $\mathcal{F}$ be a family of functions in $C([\tau, T], E)$. We set $\mathcal{F}([\tau, T]) = \{u(t) : t \in [\tau, T],\ u \in \mathcal{F}\}$ and $\mathcal{F}(t) = \{u(t) :\ u \in \mathcal{F}\}$ for $t \in [\tau, T]$.

A. Ambrosetti's paper [2] contains a result on the relationship between the Kuratowski measures of noncompactness in E and in $C([\tau, T], E)$.

THEOREM (Ambrosetti). *Let $\mathcal{F}$ be a bounded and equicontinuous family of functions in $C([\tau, T], E)$. Then*

$$\alpha(\mathcal{F}) = \sup\{\alpha(\mathcal{F}(t)) :\ t \in [\tau, T]\} = \alpha(\mathcal{F}([\tau, T])).$$

The following approximation theorem goes back to J.R.L. Webb [13], again with χ instead of α.

THEOREM (Webb). *Let $k\colon [\tau, T] \times E \to E$ be a bounded, continuous and α-Lipschitz function with constant $K \geq 0$. Moreover let $\varepsilon > 0$ and $A \subseteq E$ be bounded. Then there exists a finite-dimensional subspace Y of E and a bounded continuous function $s\colon [\tau, T] \times A \to Y$ such that*

$$\|s(t, x) - k(t, x)\| \leq K\alpha(A) + \varepsilon, \quad \tau \leq t \leq T,\ x \in A.$$

In the next section we make use of the symbol $[x, y]_-$, which was defined in the introduction. It satisfies

$$[x, y + z]_- \leq [x, y]_- + \|z\|, \quad x, y, z \in E. \tag{2.7}$$

Moreover, if the function $u\colon [\tau,T] \to E$ has the left-hand derivative $u'_- : (\tau,T] \to E$, then the left-hand derivative $\|u(\cdot)\|'_- : (\tau,T] \to \mathbb{R}$ exists and

$$\|u(t)\|'_- = \big[u(t), u'_-(t)\big]_-, \quad \tau < t \le T. \tag{2.8}$$

For a proof see [9], for example.

R.H. Martin [8] investigated the solvabilty of initial-value problems under one-sided Lipschitz conditions.

THEOREM (Martin). *Let $g\colon [\tau,T]\times E \to E$ be a bounded, continuous and one-sided Lipschitz function. Then the problem*

$$u(\tau) = a, \quad u'(t) = g(t,u(t)), \quad \tau \le t \le T,$$

has a unique solution.

We finish this section with a result on differential inequalities. A proof can be found in [12].

LEMMA (On differential inequalities). *Let $\varphi,\psi\colon [\tau,T] \to \mathbb{R}$ be continuous functions, $\varphi(\tau) < \psi(\tau)$, and let*

$$\varphi'_-(t) - \rho(t,\varphi(t)) < \psi'_-(t) - \rho(t,\psi(t)), \quad \tau < t \le T,$$

be satisfied with some real-valued function ρ. Then the inequality $\varphi(t) < \psi(t)$ holds for all $t \in [\tau,T]$.

3. The theorem of Sabina Schmidt

In 1989 Sabina Schmidt [10] proved the following result.

THEOREM (Schmidt). *Let $a \in E$, and let $g,k : [\tau,T]\times E \to E$ be bounded and continuous functions, such that g is one-sided Lipschitz with constant L and k is α-Lipschitz with constant $K \ge 0$. Then the initial-value problem*

$$u(\tau) = a, \quad u'(t) = g(t,u(t)) + k(t,u(t)), \quad \tau \le t \le T, \tag{P}$$

has at least one solution

$$u\colon [\tau,T] \to E. \tag{S}$$

The present paper complements this result by showing that the set of solutions (S) of (P) is a compact and connected subset of the Banach space $C([\tau,T],E)$. For the proof we use the following type of approximate solutions.

Definition. Let $f\colon [\tau,T]\times E\to E$ be a continuous function and $a\in E$. We call a sequence (u_n) in $C^1([\tau,T],E)$ a sequence of *approximate solutions* for the initial-value problem

$$u(\tau)=a,\quad u'(t)=f(t,u(t)),\quad \tau\le t\le T,$$

if the sequence satisfies the conditions $u_n(\tau)\to a$ and

$$\|u_n'(t)-f(t,u_n(t))\|\le \varepsilon_n,\quad \tau\le t\le T,$$

where $\varepsilon_n\to 0$. Here $C^1([\tau,T],E)$ denotes the space of all continuously differentiable functions $u\colon[\tau,T]\to E$.

Now we prove Schmidt's theorem by using her procedure with some alterations appropriate for our purpose.

Proof of Schmidt's theorem. Without loss of generality let $L>0$.

Part 1. First, we prove the solvability of (P) under the additional condition that

$$\frac{1}{L}\left(e^{L(T-\tau)}-1\right)\le\frac{1}{8(K+1)}. \tag{3.1}$$

By means of a theorem of Lasota and Yorke [7] we obtain approximate solutions u_n for the problem (P) with the following properties:

$$u_n(\tau)=a+a_n,\quad a_n\in E,\quad a_n\to 0,$$

$$u_n'(t)=g(t,u_n(t))+k(t,u_n(t))+r_n(t),\quad \tau\le t\le T,\ n\in\mathbb{N}, \tag{3.2}$$

$$r_n\in C([\tau,T],E),\quad \|r_n\|\le\tfrac{1}{n},\quad n\in\mathbb{N}, \tag{3.3}$$

see Deimling [5], for example.

The family of functions $\mathcal{F}=\{u_n : n\in\mathbb{N}\}$ is bounded and equicontinuous in $C([\tau,T],E)$. We will show that $\alpha(\mathcal{F})=0$. Assuming the contrary we can choose $\varepsilon=\frac{1}{8}\alpha(\mathcal{F})>0$. The set $A=\mathcal{F}([\tau,T])=\{u_n(t) : t\in[\tau,T],\ n\in\mathbb{N}\}$ is bounded. Hence by Webb's theorem there exists a finite-dimensional subspace Y of E and a bounded and continuous function $s\colon[\tau,T]\times A\to Y$, such that

$$\|s(t,x)-k(t,x)\|\le K\alpha(A)+\varepsilon=K\alpha(\mathcal{F})+\varepsilon \tag{3.4}$$

for all $t \in [\tau, T]$ and $x \in A$. For the last equality see Ambrosetti's theorem. Using Schmidt's lemma we obtain a subsequence $(\overline{u}_n)$ of (u_n) such that

$$2\alpha(\mathcal{B}) \geq \alpha(\mathcal{F}) - \varepsilon \tag{3.5}$$

for each infinite subset $\mathcal{B} \subseteq \{\overline{u}_n : n \in \mathbb{N}\}$.

Now we define functions $z_n \colon [\tau, T] \to Y$ via

$$z_n(t) = \int_\tau^t s(\zeta, \overline{u}_n(\zeta))\, d\zeta, \quad \tau \leq t \leq T,\ n \in \mathbb{N}.$$

The family $\{z_n :\ n \in \mathbb{N}\}$ is a bounded and equicontinuous family of functions in $C([\tau, T], Y)$. Since Y is finite-dimensional, Ambrosetti's theorem implies

$$\alpha(\{z_n : n \in \mathbb{N}\}) = \alpha(\{z_n(t) : t \in [\tau, T],\ n \in \mathbb{N}\}) = 0. \tag{3.6}$$

Hence a subsequence of (z_n) converges in $C([\tau, T], Y)$ to a continuous function $z : [\tau, T] \to Y$. Without loss of generality we assume (z_n) to do this.

Now we consider the initial-value problem

$$v(\tau) = a, \quad v'(t) = g(t, v(t) + z(t)), \quad \tau \leq t \leq T.$$

The right side of this problem is bounded, continuous and one-sided Lipschitz on $[\tau, T] \times E$. Hence the solution $v \colon [\tau, T] \to E$ of this problem exists due to Martin's theorem.

For $n \in \mathbb{N}$ and $\tau \leq t \leq T$ we define

$$v_n(t) = \overline{u}_n(t) - z_n(t) - v(t)$$

and $w_n(t) = g(t, v(t) + z_n(t)) - g(t, v(t) + z(t)) + \overline{r}_n(t)$, where $(\overline{r}_n)$ denotes the subsequence of (r_n) corresponding to $(\overline{u}_n)$. Then (3.2) and (3.3) hold for $\overline{u}_n, \overline{r}_n$ instead of u_n, r_n. Therefore we obtain for all $\tau \leq t \leq T$ that

$$v_n'(t) = [g(t, \overline{u}_n(t)) - g(t, v(t) + z_n(t))] + [k(t, \overline{u}_n(t)) - s(t, \overline{u}_n(t))] + w_n(t).$$

Moreover (2.7), (2.8) and (3.4) admit the following estimations:

$$\begin{aligned} \|v_n(t)\|_-' &= [v_n(t),\ v_n'(t)]_- \\ &\leq [v_n(t),\ g(t, \overline{u}_n(t)) - g(t, v(t) + z_n(t))]_- \\ &\quad + \|k(t, \overline{u}_n(t)) - s(t, \overline{u}_n(t))\| + \|w_n(t)\| \\ &\leq L\|v_n(t)\| + \|w_n(t)\| + K\alpha(\mathcal{F}) + \varepsilon \end{aligned} \tag{3.7}$$

for all $t \in (\tau, T]$. Setting $\mu_n = \max_{\tau \le t \le T} \|w_n(t)\|$ we can verify $\mu_n \to 0$, and the last estimation of (3.7) leads to

$$\|v_n(t)\|'_- \le L\|v_n(t)\| + \mu_n + K\alpha(\mathcal{F}) + \varepsilon. \tag{3.8}$$

Now let $\eta > 0$ and let $(\overline{a}_n)$ denote the subsequence of (a_n), which corresponds to $(\overline{u}_n)$. The solution of the initial-value problem

$$\begin{aligned} &\psi_\eta(\tau) = \|\overline{a}_n\| + \eta, \\ &\psi'_\eta(t) = L\psi_\eta(t) + \mu_n + K\alpha(\mathcal{F}) + \varepsilon + \|\overline{a}_n\| + \eta, \quad \tau \le t \le T, \end{aligned} \tag{3.9}$$

is given by

$$\psi_\eta(t) = \big(\|\overline{a}_n\| + \eta\big)e^{L(t-\tau)} + \frac{1}{L}\Big(e^{L(t-\tau)} - 1\Big)\big(\mu_n + K\alpha(\mathcal{F}) + \varepsilon + \|\overline{a}_n\| + \eta\big).$$

Since $v_n(\tau) = \overline{a}_n$, the inequality $\|v_n(\tau)\| < \psi_\eta(\tau)$ holds. Using (3.8) and (3.9) we can apply the lemma on differential inequalities to the functions $\|v_n(\cdot)\|$ and ψ_η. Hence we obtain $\|v_n(t)\| \le \psi_\eta(t)$ for all $t \in [\tau, T]$. Since we have chosen $\eta > 0$ arbitrarily, the last inequality and $\eta \to 0$ leads to the following estimation for all $t \in [\tau, T]$:

$$\|v_n(t)\| \le \|\overline{a}_n\|e^{L(t-\tau)} + \frac{1}{L}\Big(e^{L(t-\tau)} - 1\Big)\big(\mu_n + K\alpha(\mathcal{F}) + \varepsilon + \|\overline{a}_n\|\big).$$

Therefore the further estimations are valid due to (3.1):

$$\begin{aligned} \|v_n\| &\le \|\overline{a}_n\|\, e^{L(T-\tau)} + \frac{1}{L}\Big(e^{L(T-\tau)} - 1\Big)\big(\mu_n + K\alpha(\mathcal{F}) + \varepsilon + \|\overline{a}_n\|\big) \\ &\le \frac{1}{8(K+1)}\,\mu_n + \frac{1}{8}\,\alpha(\mathcal{F}) + \frac{1}{8}\,\varepsilon + \|\overline{a}_n\|\Big(e^{L(T-\tau)} + \frac{1}{8(K+1)}\Big) =: c_n. \end{aligned}$$

Since $v_n = (\overline{u}_n - z_n) - v$ and $\lim_{n\to\infty} c_n = \frac{1}{8}\alpha(\mathcal{F}) + \frac{1}{8}\varepsilon$, we obtain from (2.6)

$$\alpha\big(\{\overline{u}_n - z_n : n \in \mathbb{N}\}\big) \le \frac{1}{4}\big(\alpha(\mathcal{F}) + \varepsilon\big).$$

According to (2.3), (3.5) and (3.6) we can estimate

$$\begin{aligned} \frac{1}{2}\big(\alpha(\mathcal{F}) - \varepsilon\big) &\le \alpha\big(\{\overline{u}_n : n \in \mathbb{N}\}\big) \\ &\le \alpha\big(\{\overline{u}_n - z_n : n \in \mathbb{N}\}\big) + \alpha\big(\{z_n : n \in \mathbb{N}\}\big) \le \frac{1}{4}\big(\alpha(\mathcal{F}) + \varepsilon\big). \end{aligned}$$

This means $\alpha(\mathcal{F}) \le 3\varepsilon$, which contradicts $\varepsilon = \frac{1}{8}\alpha(\mathcal{F})$, so $\alpha(\mathcal{F}) = 0$.

Due to (2.4) there exists a subsequence $(\widetilde{u}_n)$ of (u_n), which converges uniformly to an element $u \in C([\tau,T],E)$. Considering the integral equations, which correspond to (3.2) with $\widetilde{u}_n$ instead of u_n, we obtain u as solution of the initial-value problem (P).

PART 2. To complete the proof we choose $\delta > 0$ such that $\frac{1}{L}\left(e^{L\delta}-1\right) \leq \frac{1}{8(K+1)}$, compare (3.1). Moreover let $\tau = t_0 < t_1 < \ldots < t_{m-1} < t_m = T$ be a subdivision of the interval $[\tau,T]$, such that $t_i - t_{i-1} \leq \delta$ for all $i = 1,\ldots,m$. In the following we use the approximate solutions (u_n) from part 1, which do not depend on the choice of δ.

We consider the sequence of the restricted approximate solutions $(u_n|_{[t_0,t_1]})$. Due to part 1, a subsequence $(u_n^{(1)}|_{[t_0,t_1]})$ of $(u_n|_{[t_0,t_1]})$ converges uniformly on $[t_0,t_1]$ to a solution $u^{(1)}\colon [t_0,t_1] \to E$ of the initial-value problem

$$(\mathrm{P}_{[t_0,t_1]}) \qquad u(t_0) = a_0, \quad u'(t) = g(t,u(t)) + k(t,u(t)), \quad t_0 \leq t \leq t_1,$$

where $a_0 = a$. In the next step we restrict the unrestricted subsequence $(u_n^{(1)})$ to the interval $[t_1,t_2]$. Hence we obtain a sequence of approximate solutions $(u_n^{(1)}|_{[t_1,t_2]})$ for the initial-value problem

$$(\mathrm{P}_{[t_1,t_2]}) \qquad u(t_1) = a_1, \quad u'(t) = g(t,u(t)) + k(t,u(t)), \quad t_1 \leq t \leq t_2,$$

where $a_1 = u^{(1)}(t_1)$. Note that the sequence $(u_n^{(1)}|_{[t_1,t_2]})$ satisfies the conditions (3.2) and (3.3) with t_1 and t_2 instead of τ and T, and some subsequence $(r_n^{(1)})$.

Applying part 1 again leads to a subsequence $(u_n^{(2)}|_{[t_1,t_2]})$ of $(u_n|_{[t_1,t_2]})$ that converges uniformly on $[t_1,t_2]$ to a solution $\widetilde{u}^{(2)} : [t_1,t_2] \to E$ of $(\mathrm{P}_{[t_1,t_2]})$.

Additionally we conclude that the restrictions $u_n^{(2)}|_{[t_0,t_2]}\colon [t_0,t_2] \to E$ converge uniformly on $[t_0,t_2]$ to a solution $u^{(2)}\colon [t_0,t_2] \to E$ of

$$(\mathrm{P}_{[t_0,t_2]}) \qquad u(t_0) = a_0, \quad u'(t) = g(t,u(t)) + k(t,u(t)), \quad t_0 \leq t \leq t_2.$$

Note that

$$u^{(2)}(t) = \begin{cases} u^{(1)}(t), & t_0 \leq t \leq t_1, \\ \widetilde{u}^{(2)}(t), & t_1 \leq t \leq t_2. \end{cases}$$

By iteration we obtain a subsequence $(u_n^{(m)})$ of (u_n), that converges uniformly on $[t_0,t_m] = [\tau,T]$ to a solution $u^{(m)}$ of

$$(\mathrm{P}) \qquad u(\tau) = a, \quad u'(t) = g(t,u(t)) + k(t,u(t)), \quad \tau \leq t \leq T. \qquad \square$$

4. Compactness of the set of solutions

In the setting of Schmidt's theorem we can prove the following theorem.

Theorem 1. *Let $a \in E$, and let $g, k\colon [\tau, T] \times E \to E$ be bounded and continuous functions such that g is one-sided Lipschitz with constant L and k is α-Lipschitz with constant $K \geq 0$. Moreover let the initial-value problem*

$$\text{(P)} \qquad u(\tau) = a, \quad u'(t) = g(t, u(t)) + k(t, u(t)), \quad \tau \leq t \leq T,$$

be given. Then the set of solutions

$$\mathcal{S} = \{u \mid u\colon [\tau, T] \to E,\ u \text{ is a solution of (P) }\}$$

is a compact subset of the Banach space $C([\tau, T], E)$.

Proof. Let (u_n) be a sequence in $\mathcal{S}$. Since the u_n solve (P), they are obviously approximate solutions for problem (P) with exact initial value. As in part 2 of the proof of Schmidt's theorem we obtain a subsequence of (u_n), which converges in $C([\tau, T], E)$ to a solution u of (P). Hence $\mathcal{S}$ is compact. □

In general the set of solutions of an initial-value problem in a Banach space is not compact as the following example shows. It was motivated by an example in a paper of Chaljub-Simon, Lemmert, Schmidt and Volkmann [4].

Let l_∞ denote the Banach space of all bounded and real sequences $u = (u_n)$, where $\|u\| = \sup_{n \in \mathbb{N}} |u_n|$.

Example. Let the function $\varphi\colon \mathbb{R} \to \mathbb{R}$ be given by

$$\varphi(\xi) = \begin{cases} 0, & \xi \leq 0, \\ \sqrt{\xi}, & 0 \leq \xi \leq 4, \\ 2, & 4 \leq \xi. \end{cases}$$

We define the bounded and continuous function $f\colon [0, 1] \times l_\infty \to l_\infty$ by

$$f(t, u) = (\varphi(u_1), \varphi(u_2), \dots), \quad 0 \leq t \leq 1,\ u = (u_n) \in l_\infty.$$

Then it is easy to see that the set of solutions $\mathcal{S}$ of the initial-value problem

$$\text{(P)} \qquad u(0) = (0, 0, \dots), \quad u'(t) = f(t, u(t)), \quad 0 \leq t \leq 1,$$

is the set of all functions $u\colon [0,1] \to l_\infty$ with $u(t) = (u_n(t))$, where for each $n \in \mathbb{N}$ we have

$$u_n(t) = \begin{cases} 0, & t \in [0, a_n], \\ \dfrac{1}{4}(t - a_n)^2, & t \in [a_n, 1], \end{cases}$$

with some (arbitrary) $a_n \in [0,1]$. For $0 \le t \le 1$ we consider the set $\mathcal{S}(t) = \{u(t) :\ u \in \mathcal{S}\}$. It is easy to verify that the set $\mathcal{S}(t)$ is a ball in l_∞ with radius $\frac{1}{8}t^2$ and hence for $0 < t \le 1$ it is not compact. Therefore we conclude that $\mathcal{S}$ is not compact.

Another example for a noncompact solution set can be found in Binding's paper [3].

5. Connectedness of the set of solutions

We prove a theorem of Hellmuth Kneser (1923, [6]) in the setting of Schmidt's theorem.

Let $f\colon [\tau, T] \times E \to E$ be a continuous function. Then f is called locally Lipschitz, if for each $(t,x) \in [\tau, T] \times E$ there exist $L = L(t,x) \ge 0$, a neighbourhood I_t of t and a neighbourhood U_x of x, such that

$$\|f(s,x_1) - f(s,x_2)\| \le L\,\|x_1 - x_2\|\,, \quad s \in I_t \cap [\tau, T],\ x_1, x_2 \in U_x.$$

Lemma 1. *Let the function $f\colon [\tau, T] \times E \to E$ be bounded and locally Lipschitz, and let the continuous function $h\colon [\tau, T] \times [0,1] \to E$ satisfy*

$$\|h(t,\lambda) - h(t,\mu)\| \le C\,|\lambda - \mu|\,, \quad \tau \le t \le T,\ \lambda, \mu \in [0,1],$$

with some constant $C \ge 0$. Moreover, for each $\lambda \in [0,1]$ let u_λ denote the solution of the initial-value problem

$$(\mathrm{P}_\lambda) \qquad u(\tau) = a, \quad u'(t) = f(t, u(t)) + h(t, \lambda), \quad \tau \le t \le T.$$

Then the mapping $\Lambda\colon [0,1] \to C([\tau,T],E)$, $\lambda \mapsto u_\lambda$, is continuous.

Recall that the well-known theorem of Picard-Lindelöf guarantees the existence and uniqueness of the solution u_λ of (P_λ).

As usual we denote the graph of a function $u\colon [\tau,T]\to E$ by

$$\operatorname{graph}(u)=\{(t,u(t)) :\ \tau\le t\le T\}\subseteq[\tau,T]\times E.$$

We consider $[\tau,T]\times E$ as a metric space, where the metric ρ is given by

$$\rho((t_1,x_1),(t_2,x_2))=|t_1-t_2|+\|x_1-x_2\|,\quad (t_1,x_1),(t_2,x_2)\in[\tau,T]\times E.$$

The distance $\operatorname{dist}(A,B)$ between two nonempty sets A and B of a metric space means the number $\inf\{\rho(a,b) :\ a\in A,\ b\in B\}$.

Proof of Lemma 1. We fix $\lambda\in[0,1]$ and the solution u_λ of (P_λ). The graph of u_λ is a compact subset of $[\tau,T]\times E$ and f is locally Lipschitz. Hence, there exist $\delta>0$, $L>0$ and a neighbourhood U in $[\tau,T]\times E$ of $\operatorname{graph}(u_\lambda)$ such that

$$U=\big\{(t,x)\in[\tau,T]\times E :\ \operatorname{dist}\big(\{(t,x)\},\operatorname{graph}(u_\lambda)\big)<2\delta\big\}$$

and such that the function f satisfies

$$\|f(t,x)-f(t,y)\|\le L\,\|x-y\|,\quad (t,x),(t,y)\in U. \tag{5.1}$$

We show that the mapping Λ is continuous at λ. For this let $\varepsilon>0$ and such that $\varepsilon<\delta$. Note that λ is still fixed.

Let $\mu\in[0,1]$ be such that $|\lambda-\mu|<\frac{1}{(1+C)\left[e^{L(T-\tau)}-1\right]}\,L\varepsilon$ and let u_μ denote the solution of (P_μ). Then $\operatorname{graph}(u_\mu)\subseteq U$: Assuming the contrary, there exists

$$\bar t=\min\big\{t\in[\tau,T] :\ \operatorname{dist}(\{(t,u_\mu(t))\},\operatorname{graph}(u_\lambda))=2\delta\big\},$$

and $\bar t>\tau$ due to $u_\lambda(\tau)=u_\mu(\tau)=a$ and the continuity of both functions. Hence $(t,u_\mu(t))\in U$ for all $t\in[\tau,\bar t)$.

From (2.7) and (2.8) we obtain for $t\in(\tau,\bar t)$ the following estimations:

$$\begin{aligned}\|u_\lambda(t)-u_\mu(t)\|'_- &\le \|u'_\lambda(t)-u'_\mu(t)\|\\ &=\|f(t,u_\lambda(t))+h(t,\lambda)-f(t,u_\mu(t))-h(t,\mu)\|\\ &\le L\,\|u_\lambda(t)-u_\mu(t)\|+C|\lambda-\mu|.\end{aligned}$$

The last inequality holds due to (5.1) and since $(t,u_\lambda(t))\in U$ and $(t,u_\mu(t))\in U$ for all $t\in[\tau,\bar t)$.

Now let $\eta > 0$. Using $\|u_\lambda(\tau) - u_\mu(\tau)\| = 0$ and the lemma on differential inequalities, it is easy to see that

$$\|u_\lambda(t) - u_\mu(t)\| \leq \eta e^{L(t-\tau)} + \frac{1}{L}\Big(e^{L(t-\tau)} - 1\Big)\big(C|\lambda - \mu| + \eta\big)$$

for all $t \in [\tau, \bar{t})$. Moreover, for $\eta \to 0$ we obtain the estimation

$$\|u_\lambda(t) - u_\mu(t)\| \leq |\lambda - \mu| \frac{C}{L}\Big(e^{L(t-\tau)} - 1\Big), \quad \tau \leq t < \bar{t}.$$

Due to our choice of μ and since u_λ and u_μ are continuous, the last inequality leads to the following contradiction:

$$\begin{aligned} 2\delta &\leq \|u_\lambda(\bar{t}) - u_\mu(\bar{t})\| \\ &\leq |\lambda - \mu| \frac{C}{L}\left(e^{L(\bar{t}-\tau)} - 1\right) \\ &\leq \frac{C}{1+C} \frac{e^{L(\bar{t}-\tau)} - 1}{e^{L(T-\tau)} - 1} \varepsilon \leq \varepsilon < \delta. \end{aligned}$$

Therefore we have $(t, u_\mu(t)) \in U$ for all $t \in [\tau, T]$ and we obtain by the same arguments

$$\|u_\lambda(t) - u_\mu(t)\| \leq |\lambda - \mu| \frac{C}{L}\Big(e^{L(t-\tau)} - 1\Big), \quad \tau \leq t \leq T.$$

Moreover, we deduce $\|u_\lambda - u_\mu\| \leq \varepsilon$, which means that the mapping Λ is continuous at λ. □

Finally we prove that in Schmidt's theorem the solution set $\mathcal{S}$ of the initial-value problem (P) is a connected subset of the Banach space $C([\tau, T], E)$.

Theorem 2. *Let $a \in E$, and let $g, k : [\tau, T] \times E \to E$ be bounded and continuous functions, such that g is one-sided Lipschitz with constant L and k is α-Lipschitz with constant $K \geq 0$. Moreover let the initial-value problem*

$$\text{(P)} \qquad u(\tau) = a, \quad u'(t) = g(t, u(t)) + k(t, u(t)), \quad \tau \leq t \leq T,$$

be given. Then the set

$$\mathcal{S} = \{u \mid u : [\tau, T] \to E,\ u \text{ is a solution of (P)}\}$$

is a connected subset of the Banach space $C([\tau, T], E)$.

Proof. The set $\mathcal{S}$ is nonempty due to the theorem of Schmidt and compact due to Theorem 1. Suppose $\mathcal{S}$ is not connected. Then there exist nonempty, disjoint and compact sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq C([\tau,T],E)$ such that $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. Hence, $\beta = \operatorname{dist}(\mathcal{S}_1, \mathcal{S}_2) = \min\{\|s_1 - s_2\| : s_1 \in \mathcal{S}_1,\ s_2 \in \mathcal{S}_2\} > 0$.

The functional $\Phi\colon C([\tau,T],E) \to \mathbb{R}$ defined by $\Phi(u) = \operatorname{dist}(u,\mathcal{S}_1) - \operatorname{dist}(u,\mathcal{S}_2)$ is continuous. Moreover $\Phi(u) \le -\beta$ on $\mathcal{S}_1$ and $\Phi(u) \ge \beta$ on $\mathcal{S}_2$.

Now we prove the existence of some $u \in \mathcal{S}$ such that $\Phi(u) = 0$, which leads to a contradiction. For this we construct a sequence of approximate solutions (u_n) for the initial-value problem (P) with $\Phi(u_n) = 0$ for all $n \in \mathbb{N}$. Then, as in part 2 of the proof of Schmidt's theorem, a subsequence of (u_n) converges uniformly to a solution u of (P), and hence $\Phi(u) = 0$.

Let $\varepsilon > 0$. We define the function $f\colon [\tau,T] \times E \to E$ by

$$f(t,x) = g(t,x) + k(t,x), \quad \tau \le t \le T;\ x \in E.$$

Due to a theorem of Lasota and Yorke [7] there exists a locally Lipschitz function $l_\varepsilon\colon [\tau,T] \times E \to E$ satisfying $\|l_\varepsilon(t,x) - f(t,x)\| \le \varepsilon$ on $[\tau,T] \times E$.

Now let $s_1 \in \mathcal{S}_1,\ s_2 \in \mathcal{S}_2$. For $i = 1,2$ we consider the functions

$$f_\varepsilon^{(i)}(t,x) = l_\varepsilon(t,x) + f(t,s_i(t)) - l_\varepsilon(t,s_i(t)), \quad \tau \le t \le T,\ x \in E,$$

and for $\lambda \in [0,1]$ the functions

$$f_{\lambda,\varepsilon}(t,x) = f_\varepsilon^{(1)}(t,x) + \lambda \cdot \left[f_\varepsilon^{(2)}(t,x) - f_\varepsilon^{(1)}(t,x)\right], \quad \tau \le t \le T,\ x \in E.$$

For each $\lambda \in [0,1]$ the function $f_{\lambda,\varepsilon}$ is locally Lipschitz and

$$\|f_{\lambda,\varepsilon}(t,x) - f(t,x)\| \le 2\varepsilon, \quad \tau \le t \le T,\ x \in E. \tag{5.2}$$

Due to the theorem of Picard-Lindelöf there exist unique solutions $u_{\lambda,\varepsilon}$ of the initial-value problems

$$u(\tau) = a, \quad u'(t) = f_{\lambda,\varepsilon}(t,u(t)), \quad \tau \le t \le T. \tag{$\mathrm{P}_{\lambda,\varepsilon}$}$$

Using Lemma 1 we conclude that the mapping

$$\Lambda\colon [0,1] \to C([\tau,T],E), \quad \lambda \mapsto u_{\lambda,\varepsilon},$$

is continuous, and therefore the mapping

$$\Psi\colon [0,1] \to \mathbb{R}, \quad \Psi(\lambda) := \Phi(u_{\lambda,\varepsilon}) = (\Phi \circ \Lambda)(\lambda),$$

is continuous as well. Since $f_{0,\varepsilon}(t, s_1(t)) = f_\varepsilon^{(1)}(t, s_1(t)) = s_1'(t)$, we obtain $u_{0,\varepsilon} = s_1$ and in the same way $u_{1,\varepsilon} = s_2$. That means $\Psi(0) \leq -\beta$ and $\Psi(1) \geq \beta$, and there exists $\lambda(\varepsilon) \in (0, 1)$ such that $u_{\lambda(\varepsilon),\varepsilon}$ satisfies $\Phi(u_{\lambda(\varepsilon),\varepsilon}) = 0$.

Now let (ε_n) be a sequence of positive numbers, and $\varepsilon_n \to 0$. As before, to each ε_n we obtain the solution $u_n = u_{\lambda(\varepsilon_n),\varepsilon_n}$ of the initial-value problem $(\mathrm{P}_{\lambda(\varepsilon_n),\varepsilon_n})$. We set $r_n(t) = f_{\lambda(\varepsilon_n),\varepsilon_n}(t, u_n(t)) - f(t, u_n(t))$ for all $t \in [\tau, T]$. Then from inequality (5.2) it follows that $\|r_n\| \leq 2\varepsilon_n$. Moreover, u_n is a solution of the initial-value problem

$$u_n(\tau) = a, \quad u_n'(t) = f(t, u_n(t)) + r_n(t), \quad \tau \leq t \leq T,$$

and satisfies $\Phi(u_n) = 0$. Hence the sequence (u_n) is a sequence of approximate solutions for problem (P) with $\Phi(u_n) = 0$ for all $n \in \mathbb{N}$. □

Examples for disconnected solution sets in less restrictive situations can be found in [3].

References

[1] Akhmerov R.R., Kamenskiĭ M.I., Potapov A.S., Rodkina A.E., Sadovskiĭ B.N., *Measures of noncompactness and condensing operators*, Birkhäuser, Basel, 1992. Translation from the Russian.

[2] Ambrosetti A., *Un teorema di esistenza per le equazioni differenziali negli spazi di Banach*, Rend. Sem. Math. Univ. Padova **39** (1967), 349–361.

[3] Binding P., *On infinite-dimensional differential equations*, J. Diff. Eq. **24** (1977), 349–354.

[4] Chaljub-Simon A., Lemmert R., Schmidt S., Volkmann P., *Gewöhnliche Differentialgleichungen mit quasimonoton wachsenden rechten Seiten in geordneten Banachräumen.* General Inequalities 6, Internat. Ser. Numer. Math., Vol. 103, Birkhäuser, Basel, 1992, pp. 307–320. Addenda: J. Inequalities Pure Appl. Math. **4**(3), Art. 49 (2003), p. 27; `http://jipam.vu.edu.au`

[5] Deimling K., *Ordinary differential equations in Banach spaces*, Lecture Notes in Math. **596**, Springer, Berlin, 1977.

[6] Kneser H., *Über die Lösungen eines Systems gewöhnlicher Differentialgleichungen, das der Lipschitzschen Bedingung nicht genügt.* Sitz.ber. Preuß. Akad. Wiss., Phys.-Math. Kl. (1923), 171–174.

[7] Lasota A., Yorke J.A., *The generic property of existence of solutions of differential equations in Banach spaces*, J. Diff. Eq. **13** (1973), 1–12.

[8] Martin R.H.(jr.), *A global existence theorem for autonomous differential equations in a Banach space*, Proc. Amer. Math. Soc. **26** (1970), 307–314.

[9] Martin R.H.(jr.), *Nonlinear operators and differential equations in Banach spaces*, Wiley, New York, 1976.

[10] Schmidt S., *Existenzsätze für gewöhnliche Differentialgleichungen in Banachräumen*, Dissertation, Karlsruhe (1989); published in: Funkcial. Ekvac. **35** (1992), 199–222.

[11] Volkmann P., *Cinq cours sur les équations différentielles dans les espaces de Banach*, in: Topological Methods in Differential Equations and Inclusions, edited by Andrzej Granas and Marène Frigon, Kluwer, Dordrecht, 1995, pp. 501–520.
[12] Walter W., *Differential- und Integral-Ungleichungen*, Springer, Berlin, 1964.
[13] Webb J.R.L., *On a characterisation of k-set contractions*, Rend. Sci. fis. mat. nat. Accad. Lincei **50** (1971), 686–689.

Institut für Analysis
Karlsruher Institut für Technologie
76128 Karlsruhe
Germany
e-mail: marc.mitschele@kit.edu