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ON REPRESENTATION BY EXIT LAWS FOR SOME BOCHNER SUBORDINATED SEMIGROUPS

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Abstract. This paper is devoted to the integral representation of potentials by exit laws in the framework of sub-Markovian semigroups of kernels acting on $L^2(m)$. We mainly investigate subordinated semigroups in Bochner sense by means of subordinators with complete Bernstein functions. As application, we give a representation for the original semigroup.

1. Introduction

Let $\mathbb{P} = (\mathbb{P}_t)_{t\geq 0}$ be a sub-Markovian semigroup of kernels on $L^2(m)$. A $\mathbb{P}$-exit law is a family $\varphi = (\varphi_t)_{t>0}$ of $L^2_+(m)$ satisfying the functional equation

$$P_s\varphi_t = \varphi_{s+t}, \quad s,t > 0. \tag{1.1}$$

This notion is first introduced by Dynkin [5] in the framework of potential theory without reference measure. Then, the integral representation of potentials by exit laws was investigated in many papers (cf. [1] and [6–14]). As it is known, this allows explicit formulas for the energy and the capacity (cf. [6–8] and the related references).

Now, let $\beta = (\beta_t)_{t>0}$ be a Bochner subordinator, that is a vaguely continuous convolution semigroup of sub-probability measures on $[0, +\infty[$. The present

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paper is devoted to the representation by $\mathbb{P}^\beta$-exit laws, where $\mathbb{P}^\beta$ is the subordinated semigroup of $\mathbb{P}$ by means of β, i.e

$$P_t^\beta f := \int_0^\infty P_s f\, \beta_t(ds), \quad f \in L^2(m), t > 0. \tag{1.2}$$

More precisely, we suppose that β admits a complete Bernstein function f of the form

$$f(r) = \int_0^\infty (1 - e^{-sr})\, \ell(s)\, ds$$

for some function ℓ on $]0, \infty[$ (cf. Section 3.2). This family of subordinators contains many interesting examples and it is considered in [5].

In this context, we prove the following integral representation: Let h be a $\mathbb{P}^\beta$-potential, i.e $h \geq 0$, $P_t^\beta h \leq h$, $\lim_{t\to 0} P_t^\beta h = h$ and $P_t^\beta h \in D(A^\beta)$ the domain of the generator A^β of $\mathbb{P}^\beta$. Then there exists a unique $\mathbb{P}^\beta$-exit law $\psi = (\psi_t)_{t>0}$ such that

$$h = \int_0^\infty \psi_s\, ds. \tag{1.3}$$

If f is bounded then ψ_t is explicitly given by

$$\psi_t = \int_0^\infty (P_t^\beta h - P_s P_t^\beta h)\ell(s) ds.$$

By considering the one-sided stable subordinators of order $\alpha \in]0, 1[$, we deduce a representation for the original semigroup. Namely, if u is a $\mathbb{P}$-potential, then there exist a unique $\mathbb{P}$-exit law φ such that

$$u = \int_0^\infty \varphi_s\, ds. \tag{1.4}$$

A similar problem is studied in [12] by considering $\mathcal{C}^1$-subordinators instead of subordinators with complete Bernstein functions.

The integral representation (1.4) is already obtained in many papers, but under an additional hypothesis on $\mathbb{P}$. For example, $\mathbb{P}$ is supposed to be (almost) symmetric in [6–8], absolutely continuous in [9–10] and lattice in [11,13].

2. Preliminaries

Let $(E, \mathcal{E})$ be a measurable space and let m be a σ-finite positive measure on $(E, \mathcal{E})$. We denote by $L^2(m)$ the Banach space of square integrable (classes of) functions defined on E, by $\|.\|_2$ the associated norm and by $L^2_+(m)$ the positive elements of $L^2(m)$. Moreover, in the sequel, equality and inequality holds always m-a.e. (i.e. almost everywhere with respect to m).

In this section we summarize some known results (cf. [2], [3] and [15–18]).

2.1. Sub-Markovian semigroup

A *kernel* on E is a mapping $N : E \times \mathcal{E} \to [0, \infty[$ such that
1. $x \to N(x, A)$ is measurable for each $A \in \mathcal{E}$.
2. $A \to N(x, A)$ is a measure on $(E, \mathcal{E})$ for each $x \in E$.

Let N be a kernel on E. For $f \in L^2(m)$, we define

$$Nf(x) := \int_E f(y)\, N(x, dy), \quad x \in E.$$

If $N(L^2(m)) \subset L^2(m)$, we say that N is a kernel on $L^2(m)$. If $N1 \le 1$, N is said to be sub-Markovian.

A *sub-Markovian semigroup* on E is a family $\mathbb{P} := (P_t)_{t \ge 0}$ of sub-Markovian kernels on $L^2(m)$ such that $P_0 = I$,
1. $P_s P_t = P_{s+t}$ for all $s, t > 0$,
2. $\lim_{t \to 0} \|P_t f - f\|_2 = 0$ for every $f \in L^2(m)$,
3. $\|P_t f\|_2 \le \|f\|_2$ for each $t > 0$ and $f \in L^2(m)$.

Let $\mathbb{P}$ be a sub-Markovian semigroup on E. The associated $L^2(m)$-*generator* A is defined by

$$Af := \lim_{t \to 0} \frac{1}{t}(P_t f - f)$$

on its domain $D(A)$ which is the set of all functions $f \in L^2(m)$ for which this limit exists in $L^2(m)$. It is known that:
1. $D(A)$ is dense in $L^2(m)$ and A is closed.
2. If $u \in D(A)$ then $P_t u \in D(A)$ and $A(P_t u) = P_t A u$, for each $t > 0$.

2.2. Potentials and exit laws

Let $\mathbb{P}$ be a sub-Markovian semigroup on $L^2(m)$. A positive measurable function u is said to be $\mathbb{P}$-*excessive* if
(i) $P_t u \leq u$ for each $t > 0$,
(ii) $\lim_{t\to 0} P_t u = u$, m-a.e.
A $\mathbb{P}$-excessive function u is called a $\mathbb{P}$-*pseudo-potential* if
(iii) $P_t u \in L^2(m)$ for every $t > 0$.
A $\mathbb{P}$-excessive function u is called a $\mathbb{P}$-*potential* if
(iv) $P_t u \in D(A)$ for every $t > 0$.
A $\mathbb{P}$-*exit law* is a family $\varphi := (\varphi_t)_{t>0}$ of elements of $L^2_+(m)$ satisfying the *exit equation*:

$$P_s \varphi_t = \varphi_{s+t}, \quad s, t > 0. \tag{2.1}$$

In what follows, we consider $\mathbb{P}$-exit laws satisfying

$$\int_t^\infty \varphi_s \, ds \in L^2(m), \quad t > 0. \tag{2.2}$$

As it is discussed in our paper [14], condition (2.2) is in fact not restrictive.

The following general results are proved in [12].

THEOREM 1. *Let $\mathbb{P}$ be a sub-Markovian semigroup on $L^2(m)$ and let φ be a $\mathbb{P}$-exit law such that* (2.2) *holds.*

1. *The function*

$$u := \int_0^\infty \varphi_s \, ds \tag{2.3}$$

is a $\mathbb{P}$-potential and

$$\varphi_t = -AP_t u, \quad t > 0. \tag{2.4}$$

2. *There exists a unique $\mathbb{P}$-exit law φ such that* (2.3) *holds.*
3. *Let u be a $\mathbb{P}$-potential and for $t > 0$, let φ_t be defined by* (2.4). *Then $\varphi = (\varphi_t)_{t>0}$ is a $\mathbb{P}$-exit law.*

3. Representation in the subordinated structure

3.1. Bochner subordination

For the following standard notions, we refer the reader to [2–4] and [15–18]. We consider $\mathbb{R}$ endowed with its Borel field, we denote by λ the Lebesgue measure on $[0,\infty[$ and by ε_t the Dirac measure at point t. Moreover, for each bounded measure μ on $[0,\infty[$, $\mathcal{L}$ denotes its Laplace transform, i.e. $\mathcal{L}(\mu)(r) := \int_0^\infty \exp(-rs)\,\mu(ds)$ for all $r > 0$.

A *Bochner subordinator* is a convolution semigroup $\beta = (\beta_t)_{t>0}$ of sub-probability measures on $\mathbb{R}$ such that, for each $t > 0$, we have $\beta_t \neq \varepsilon_0$ and β_t is supported by $[0,\infty[$.

Let β be a Bochner subordinator.

1. The associated potential measure is defined by $\kappa := \int_0^\infty \beta_s\,ds$. Following [2, Proposition 14.1], κ is a Borel measure.
2. The associated *Bernstein function* f is defined by the Laplace transform $\mathcal{L}(\beta_t)(r) = \exp(-tf(r))$ for all $r, t > 0$. It is known that f admits the representation (cf. [2, Theorem 9.8])

$$f(r) = a + br + \int_0^\infty (1 - \exp(-rs))\,\nu(ds), \quad r > 0, \tag{3.1}$$

where $a, b \geq 0$ and ν is a measure on $]0,\infty[$ verifying $\int_0^\infty \frac{s}{s+1}\,\nu(ds) < \infty$. Moreover, a, b and ν are uniquely determined. They are called *parameters* of β or of f. ν is called Levy measure of β.

Let $\mathbb{P}$ be a sub-Markovian semigroup and let β be a Bochner subordinator. For every $t > 0$ and for every $u \in L^2(m)$, we may define

$$P_t^\beta u := \int_0^\infty P_s u\,\beta_t(ds). \tag{3.2}$$

Then $\mathbb{P}^\beta := (P_t^\beta)_{t>0}$ is a sub-Markovian semigroup. It is said to be *subordinated* to $\mathbb{P}$ in the sense of Bochner by means of β.

Let A^β be the generator of $\mathbb{P}^\beta$. The following two remarks will be used throughout this paper.

1. $D(A)$ is a subset of $D(A^\beta)$ (cf. [15, p. 269]) and

$$A^\beta u = -au + bAu + \int_0^\infty (P_t u - u)\,\nu(dt), \quad u \in D(A), \tag{3.3}$$

where a, b and ν are given in (3.1).

2. Each $\mathbb{P}$-potential is a $\mathbb{P}^\beta$-potential (For the proof, we can adapt the arguments of [3, p. 185]).

3.2. The class $\mathcal{H}$ of subordinators

Let β be a Bochner subordinator with Bernstein function f. The function f is said to be *complete* if the Levy measure ν is absolutely continuous with respect to λ and the density ℓ is of the form $\ell(s) := \int_0^\infty \exp(-ts)\,\varrho(dt)$ for some measure ϱ satisfying $\int_0^\infty \frac{1}{t(t+1)}\,\varrho(dt) < \infty$ (cf. [17, Definition 1.4 and Theorem 1.5]). In this case, we write $\nu = \ell.\lambda$ and (3.1) becomes

$$f(r) = a + br + \int_0^\infty (1 - \exp(-rs))\ell(s)\,ds, \quad r > 0. \tag{3.4}$$

In what follows, we consider the set, denoted by $\mathcal{H}$, of subordinators with Bernstein function of the form

$$f(r) = \int_0^\infty (1 - e^{-rs})\ell(s)\,ds, \quad r > 0 \tag{3.5}$$

(i.e. $a = b = 0$ in (3.4)).

Before we continue, let us give some examples:

1. One-sided stable subordinator: For each $\alpha \in]0,1[$ and $t > 0$, let η_t^α be the unique probability measure on $[0,\infty[$ such that the Laplace transform is $\mathcal{L}\,(\eta_t^\alpha)(r) = \exp(-tr^\alpha)$ for $r > 0$. Then $\eta^\alpha := (\eta_t^\alpha)_{t>0}$ is a convolution semigroup on $[0,\infty[$ called *the one-sided stable (or fractional power) of subordinator of index* α. Following [2, p.71], the associated Bernstein function is given by

$$f(r) = r^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-rs}) \frac{ds}{s^\alpha + 1}, \quad r > 0.$$

   In fact $\eta^\alpha \in \mathcal{H}$ for each $\alpha \in]0,1[$ (cf. [4] or [17] for more details).
2. Γ-subordinator: For $t > 0$, let $g_t(s) := 1_{]0,\infty[}(s)(1/\Gamma(t))\,s^{t-1}\exp(-s)$ and $\gamma_t := g_t \cdot \lambda$. Then $\gamma := (\gamma_t)_{t>0}$ is a subordinator, called Γ-*subordinator.* Following [2, p.71], the associated Bernstein function is given by

$$f(r) = \ln(1+r) = \int_0^\infty (1 - e^{-sr}) \frac{e^{-s}}{s}\,ds, \quad r > 0.$$

   Moreover $\gamma \in \mathcal{H}$ (cf. [17, p. 373]).

3. Poisson subordinator: For $t > 0$ and $c > 0$, let $\tau_t := \exp(-ct) \sum_{n=0}^{\infty} \frac{(ct)^n}{n!} \varepsilon_n$. The semigroup $\tau := (\tau_t)_{t>0}$ is a subordinator called *Poisson subordinator of jump c*. Moreover following [2, p. 71], the associated Bernstein function is given by

$$f(r) = 1 - \exp(-cr), \quad r > 0.$$

Then $\tau \notin \mathcal{H}$ (cf. [17, p. 373]).
4. Let $c > 0$. Following [17, p. 373] or [15, p. 276], the Bochner subordinator associated to the Bernstein function

$$f(r) := \frac{r}{r+c}, \quad r > 0,$$

belongs to $\mathcal{H}$.
5. Let $\beta \in \mathcal{H}$ with Bernstein function f and let $c > 0$. Then the subordinator with Bernstein function $\frac{f}{f+c}$, belongs to $\mathcal{H}$ (cf. [15, p. 276]).
6. For each subordintator β, the subordinator $(\beta_t * \varepsilon_t)_{t>0} \notin \mathcal{H}$.

3.3. The bounded case

Let β be a subordinator, let $\kappa := \int_0^\infty \beta_t \, dt$ be the potential measure of β and let ν be the Levy measure of β.

In this subsection, we suppose that the Bernstein function f of β is bounded and is of the form (3.5).

Lemma 1. *We have* $\nu(]0,\infty[) < \infty$ *and*

$$\kappa * (\nu(]0,\infty[)\varepsilon_0 - \nu) = \varepsilon_0. \tag{3.6}$$

Proof. By (3.5) and Fatou's Lemma

$$\nu(]0,\infty[) \leq \liminf_{r\to\infty} \int_0^\infty (1 - \exp(-rs))\nu(ds) = \sup_{r>0} f(r) < \infty.$$

Moreover, for $r > 0$, we have

$$\begin{aligned}\mathcal{L}(\kappa * (\nu(]0,\infty[)\varepsilon_0 - \nu))(r) &= \mathcal{L}(\kappa)(r)\mathcal{L}((\nu(]0,\infty[)\varepsilon_0 - \nu))(r) \\ &= \frac{1}{f(r)}(\nu(]0,\infty[) - \int_0^\infty \exp(-sr)\nu(ds) \\ &= \frac{1}{f(r)} f(r) = 1 = \mathcal{L}(\varepsilon_0)(r).\end{aligned}$$

We deduce (3.6) by the injectivity of Laplace transform. □

PROPOSITION 1. *For each* $u \in L^2(m)$

$$P_t u = \int_0^\infty P_t \psi_s \, ds, \quad t > 0, \tag{3.7}$$

where

$$\psi_s := \int_0^\infty (P_s^\beta u - P_r P_s^\beta u)\nu(dr), \quad s > 0. \tag{3.8}$$

PROOF. Let $u \in L^2(m)$. By Lemma 1, we have

$$\| \int_0^\infty (P_s^\beta u - P_r P_s^\beta u)\nu(dr)\|_2 \leq 2(\int_0^\infty \nu(ds))\|u\|_2 < \infty.$$

Hence, ψ_s is well defined by (3.8). Moreover using (3.6), we get

$$\begin{aligned}
\int_0^\infty P_t\psi_s ds &= \int_0^\infty (\int_0^\infty \int_0^\infty (P_{r+t}u - P_{r+q+t}u)\nu(dq)\beta_s(dr))\, ds \\
&= \int_0^\infty \int_0^\infty (P_{s+t}u - P_{r+s+t}u)\nu(dr)\,\kappa(ds) \\
&= \int_0^\infty P_{s+t}u\Big((\int_0^\infty \nu(ds))\varepsilon_0 - \nu) * \kappa\Big)(ds) \\
&= \int_0^\infty P_{s+t}u\,\varepsilon_0(ds) = P_t u.
\end{aligned}$$

Hence, (3.7) holds. □

PROPOSITION 2. *For each* $\mathbb{P}^\beta$*-pseudo-potential* h*, there exist a unique* $\mathbb{P}^\beta$*-exit law* $\psi := (\psi_t)_{t>0}$ *such that*

$$h = \int_0^\infty \psi_t \, dt. \tag{3.9}$$

Moreover, ψ_t *is explicitly given by*

$$\psi_t := \int_0^\infty (P_t^\beta h - P_s P_t^\beta h)\,\nu(ds), \tag{3.10}$$

where ν *is the Levy measure of* β*.*

PROOF. Let h be a $\mathbb{P}^\beta$-pseudo-potential. Since $P_t^\beta h \in L^2(m)$ for all $t > 0$, then by Lemma 1, we have

$$\| \int_0^\infty (P_t^\beta h - P_s P_t^\beta h)\nu(ds)\|_2 \leq 2(\int_0^\infty \nu(ds))\|P_t^\beta h\|_2 < \infty, \quad t > 0.$$

Hence ψ_t given by (3.10), is well defined. Moreover by (3.10), Fubini's theorem and the semigroup property, we get

$$P_t^\beta \psi_s = \int_0^\infty (P_{t+s}^\beta h - P_r P_{t+s}^\beta h)\nu(dr) = \psi_{t+s}, \quad s, t > 0. \tag{3.11}$$

By Proposition 1, (3.2), (3.6) and the dominated convergence theorem, we have

$$P_s P_t^\beta h = \int_0^\infty P_s \psi_{r+t}\, dr = \int_t^\infty P_s \psi_r\, dr, \quad s, t > 0.$$

By integration with respect to β_s, $s > 0$, Proposition 1 and the dominated convergence theorem imply that

$$P_{s+t}^\beta h = \int_t^\infty P_s^\beta \psi_r\, dr = \int_t^\infty \psi_{r+s}\, dr = \int_{s+t}^\infty \psi_r\, dr.$$

If we let $s \to 0$, we obtain

$$P_t^\beta h = \int_t^\infty \psi_r\, dr, \quad t > 0, \tag{3.12}$$

in $L^2(m)$. We deduce (3.9) m-a.e. by letting $t \to 0$ in (3.12).

Since h is a $\mathbb{P}^\beta$-excessive, then $t \to P_t^\beta h$ is decreasing. This implies by (3.12) that

$$\frac{1}{t}\int_s^{s+t} \psi_r\, dr = \frac{1}{t}(P_t^\beta h - P_{t+s}^\beta h) \geq 0, \quad s, t > 0.$$

By letting $t \to 0$, we find $\psi_s \geq 0$, m-a.e. for $s > 0$ and therefore $\psi := (\psi_t)_{t>0}$ is a $\mathbb{P}^\beta$-exit law.

Now, let us prove the uniqueness of ψ. Let ξ a $\mathbb{P}^\beta$-exit law such that $h = \int_0^\infty \xi_s\, ds$. Then by (3.11) and the dominated convergence theorem, we have

$$P_t^\beta h = \int_t^\infty \psi_s\, ds = \int_t^\infty \xi_s\, ds.$$

Hence for all $s, t > 0$, we get

$$\frac{1}{t}\int_0^t \psi_{r+s}\,dr = \frac{1}{t}(\int_t^\infty \psi_{r+s}\,dr - \int_0^\infty \psi_{r+s}\,dr)$$
$$= \frac{1}{t}(\int_t^\infty \xi_{r+s}\,dr - \int_0^\infty \xi_{r+s}\,dr) = \frac{1}{t}\int_0^t \xi_{r+s}\,dr$$

and, by letting $t \to 0$, we obtain $\psi_s = \xi_s$ for all $s > 0$. □

3.4. The general case

Let β be a subordinator, let $\kappa := \int_0^\infty \beta_t\,dt$ be the potential measure of β and let ν be the Levy measure of β.

In this subsection, we suppose that $\beta \in \mathcal{H}$ but the Bernstein function f of β is not necessarily bounded. Following [17], we approximate f by a sequence of bounded complete Bernstein functions $(f_n)_{n\in\mathbb{N}}$ as follows

$$f_n(r) := \int_0^\infty (1 - \exp(-tr))\ell_n(t)dt, \quad r > 0, n \in \mathbb{N}, \tag{3.13}$$

where

$$\ell_n(t) := \int_0^n \exp(-st)\varrho(ds), \quad r > 0, n \in \mathbb{N}. \tag{3.14}$$

Note that $f(r) = \lim_{n\to\infty} f_n(r)$ for all $r > 0$. Moreover, we index by “n” all entities associated to f_n. In particular κ_n is the potential measure, $\nu_n := \ell_n.\lambda$ is the Levy measure and β^n is the associated subordinator.

The proof of the following lemma is given in [17, Lemma 2.3].

Lemma 2. *Let β be in $\mathcal{H}$ with Bernstein function f and let $(f_n)_{n\in\mathbb{N}}$ be defined by* (3.13)*. Then for all $n \in \mathbb{N}$*

$$\gamma_n := ((\nu_n(]0,\infty[))\varepsilon_0 - \nu_n) * \kappa \tag{3.15}$$

is a positive measure on $[0,\infty[$. Moreover,

$$\gamma_n * \kappa_n = \kappa. \tag{3.16}$$

The next useful lemma can be deduced from [17, Theorem 2.8 and Corollary 2.9].

LEMMA 3. *Let $\mathbb{P}$ a sub-Markovian semigroup, let $\beta \in \mathcal{H}$ and let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ by means of β. For each $u \in D(A^\beta)$, we have*

$$\int_0^\infty P_s A^\beta u \, \gamma_n(ds) = \int_0^\infty (P_s u - u)\nu_n(ds), \quad n \in \mathbb{N},$$

where γ_n is defined by (3.15).

THEOREM 2. *Let $\mathbb{P}$ be a sub-Markovian semigroup, let β be in $\mathcal{H}$ and let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ by means of β. Then for each $\mathbb{P}^\beta$-potential h, there exists a unique $\mathbb{P}^\beta$-exit law $\psi = (\psi_t)_{t>0}$ such that*

$$h = \int_0^\infty \psi_s \, ds. \tag{3.17}$$

PROOF. Let h be a $\mathbb{P}^\beta$-potential, define

$$\psi_t := -A^\beta P_t^\beta h, \quad t > 0. \tag{3.18}$$

As in Theorem 1, $\psi_t \in L^2(m)$ and $(\psi_t)_{t>0}$ satisfy the $\mathbb{P}^\beta$-exit equation.

On the other hand, let $n \in \mathbb{N}$, f_n be as in (3.13) and $t > 0$. Since f_n is bounded, $\beta^n \in \mathcal{H}$ and $P_t^\beta h \in L^2(m)$ then by Proposition 1, we have

$$P_s(P_t^\beta h) = \int_0^\infty P_s \psi_r^{n,t} dr, \quad s > 0, \tag{3.19}$$

where

$$\psi_r^{n,t} := \int_0^\infty \Big(P_r^{\beta^n}(P_t^\beta h) - P_q P_r^{\beta^n}(P_t^\beta h)\Big)\nu_n(dq), \quad r > 0. \tag{3.20}$$

Moreover, since $P_t^\beta h \in L^2(m)$ and

$$\| \int_0^\infty (P_t^\beta h - P_s P_t^\beta h)\nu_n(ds)\|_2 \le 2(\int_0^\infty \nu_n(ds))\|P_t^\beta h\|_2 < \infty,$$

then the following function is well defined and belongs to $L^2(m)$

$$\varphi^{n,t} := \int_0^\infty (P_t^\beta h - P_r P_t^\beta h)\nu_n(dr). \tag{3.21}$$

By (3.18), (3.21) and Lemma 3, we have

$$\varphi^{n,t} = -\int_0^\infty P_r A^\beta P_t^\beta h \, \gamma_n(dr) = \int_0^\infty P_r \psi_t \, \gamma_n(dr). \tag{3.22}$$

Using (3.19) and (3.21), it is clear to see that $\psi_r^{n,t} = P_r^{\beta^n}\varphi^{n,t}$ and

$$P_s P_t^\beta h = \int_0^\infty P_{s+r}\varphi^{n,t}\kappa_n(dr). \tag{3.23}$$

It follows from (3.16), (3.22) and (3.23), that

$$\begin{aligned}
P_s P_t^\beta h &= \int_0^\infty P_{s+r}\varphi^{n,t}\kappa_n(dr)\\
&= \int_0^\infty \int_0^\infty P_{r+s+q}\psi_t\,\gamma_n(dq)\kappa_n(dr) = \int_0^\infty P_{r+s}\psi_t\,(\gamma_n * \kappa_n)(dr)\\
&= \int_0^\infty P_{s+r}\psi_t\,\kappa(dr) = \int_0^\infty \int_0^\infty P_{s+r}\psi_t\,\beta_q(dr)\,dq\\
&= \int_0^\infty P_s P_q^\beta \psi_t\,dq = \int_0^\infty P_s\psi_{q+t}\,dq = \int_t^\infty P_s\psi_q\,dq.
\end{aligned}$$

By integration with respect to β_s, $s > 0$ and by using (3.16) and the dominated convergence theorem, we obtain

$$P_{s+t}^\beta h = \int_t^\infty P_s^\beta \psi_r\,dr = \int_t^\infty \psi_{r+s}\,dr = \int_{s+t}^\infty \psi_r\,dr.$$

Hence, if we let $s \to 0$, we get

$$P_t^\beta h = \int_t^\infty \psi_r dr, \quad t > 0,$$

and we deduce (3.17) by letting $t \to 0$.

Finally, we complete the proof exactly as the proof of Proposition 2. □

Corollary 1. *Let $\mathbb{P}$ be a sub-Markovian semigroup, let β be in $\mathcal{H}$ with the Levy measure ν and let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ by means of β. Then for each $\mathbb{P}^\beta$-pseudo-potential h such that*

$$\int_0^1 \|P_t^\beta h - P_s P_t^\beta h\|_2\,\nu(ds) < \infty, \quad t > 0, \tag{3.24}$$

there exist a unique $\mathbb{P}^\beta$-exit law $\psi = (\psi_t)_{t>0}$ such that

$$h = \int_0^\infty \psi_s\,ds. \tag{3.25}$$

Moreover, ψ_t is explicitly given by

$$\psi_t = \int_0^\infty (P_t^\beta h - P_s P_t^\beta h)\,\nu(ds). \tag{3.26}$$

PROOF. Let h be a $\mathbb{P}^\beta$-potential such that (3.24) holds. Fix $t > 0$ and put

$$\O_t := \int_0^\infty \|P_t^\beta h - P_s P_t^\beta h\|_2\,\nu(ds) = \int_0^\infty \|P_t^\beta h - P_s P_t^\beta h\|_2\,\ell(s)\,ds. \tag{3.27}$$

By the contraction property of $\mathbb{P}$, we have

$$\int_1^\infty \|P_t^\beta h - P_s P_t^\beta h\|_2\,\nu(ds) \leq 2\nu([1,\infty[)\|P_t^\beta h\|_2 < \infty$$

and by (3.24), we get

$$\O_t = \int_0^1 \|P_t^\beta h - P_s P_t^\beta h\|_2\,\nu(ds) + \int_1^\infty \|P_t^\beta h - P_s P_t^\beta h\|_2\,\nu(ds) < \infty. \tag{3.28}$$

Using (3.28), it is easy to see that $\psi = (\psi_t)_{t>0}$ given by (3.26), is well defined and lies in $L^2(m)$. Now, let $n \in \mathbb{N}$ and let $\nu_n := \ell_n.\lambda$ be as in (3.14). Since $\ell_n \uparrow \ell$ then

$$\|P_s P_t^\beta h - P_t^\beta h\|_2(\ell(s) - \ell_n(s)) \leq 2\|P_s P_t^\beta h - P_t^\beta h\|_2 \ell(s), \quad s > 0. \tag{3.29}$$

From the definition (3.26) of ψ_t, we have

$$\begin{aligned}\|\psi_t + \int_0^\infty (P_s P_t^\beta h - P_t^\beta h)\nu_n(ds)\|_2 &= \|\int_0^\infty (P_s P_t^\beta h - P_t^\beta h)(\ell_n(s) - \ell(s))ds\|_2 \\ &\leq \int_0^\infty \|P_s P_t^\beta h - P_t^\beta h\|_2(\ell(s) - \ell_n(s))\,ds.\end{aligned}$$

Combining (3.27), (3.28), (3.29) and using the dominated convergence theorem, we obtain

$$\psi_t = -\lim_{n\to\infty} \int_0^\infty (P_s P_t^\beta h - P_t^\beta h)\nu_n(ds) \quad \text{in } L^2(m). \tag{3.30}$$

Recall from [17, Theorem 2.8 and Corollary 2.10] that

$$D(A^\beta) = \{u \in L^2(m) : \lim_{n\to\infty} \int_0^\infty (P_s u - u)\nu_n(ds) \text{ exists in } L^2(m)\} \tag{3.31}$$

and

$$A^\beta u = \lim_{n\to\infty} \int_0^\infty (P_s u - u)\nu_n(ds), \quad u \in D(A^\beta). \tag{3.32}$$

Hence, we conclude from (3.30) that $P_t^\beta h \in D(A^\beta)$ and $\psi_t = -A^\beta P_t^\beta h$. So, the remainder of the proof is an immediate consequence of Theorem 2. □

4. Application to the initial semigroup

For each $\alpha \in]0,1[$, let η_t^α be the one-sided stable subordinator of index α. We denote by κ^α (resp. ν^α) the associated potential (resp. Levy) measure.

4.1. Representation in terms of the fractional power

Proposition 3. *Let $\mathbb{P}$ be a sub-Markovian semigroup and let h be a $\mathbb{P}$-potential. Then for each $\alpha \in]0,1[$, there exist a unique $\mathbb{P}$-exit law ϕ^α such that*

$$P_t h = \int_0^\infty \phi_{s+t}^\alpha \, \kappa^\alpha(ds), \quad t > 0. \tag{4.1}$$

Proof. Let η^α be the one-sided stable subordinator of index $\alpha \in]0,1[$. Since $P_t h \in D(A)$ then by (3.3), the following function is well defined and belongs to $L^2(m)$

$$\phi_t^\alpha := -A^{\eta^\alpha} P_t h = \int_0^\infty (P_t h - P_{t+r}h)\nu^\alpha(dr), \quad t > 0. \tag{4.2}$$

By Fubini's theorem and (4.2), we have

$$P_t \phi_s^\alpha = \int_0^\infty (P_{s+t}h - P_{s+r+t}h)\nu^\alpha(dr) = \phi_{s+t}^\alpha, \quad s,t > 0. \tag{4.3}$$

Now since $t \to P_t h$ is decreasing then by (4.2), $\phi^\alpha := (\phi_t^\alpha)_{t>0}$ is positive and by (4.6), ϕ^α is a $\mathbb{P}$-exit law. On the other hand, let $t > 0$. As $P_t h$ is $\mathbb{P}$-potential then $P_t h$ is also $\mathbb{P}^\beta$-potential. Moreover since $P_t h \in D(A) \subset D(A^{\eta^\alpha})$ then by Theorem 2, we get

$$P_t h = \int_0^\infty \psi_s^{\alpha,t} \, ds, \tag{4.4}$$

where

$$\psi_s^{\alpha,t} := -A^{\eta^\alpha} P_s^{\eta^\alpha}(P_t h), \quad s > 0. \tag{4.5}$$

Using (4.3) and (4.5), we find

$$\psi_s^{\alpha,t} = -\int_0^\infty A^{\eta^\alpha} P_{r+t} h \, \eta_s^\alpha(dr) = \int_0^\infty \phi_{r+t}^\alpha \, \eta_s^\alpha(dr), \quad s > 0. \tag{4.6}$$

Finally, (4.1) is immediate from (4.4) and (4.6). □

NOTATION. In the following, let $\mathcal{K}(\mathbb{P})$ be the set of all functions $u \in L^2(m)$ such that $\Upsilon(u)$ and $\Upsilon^2(u) := \Upsilon(\Upsilon(u))$ lies in $L^2(m)$, where Υ is defined by

$$\Upsilon(g) := \int_0^1 (P_r g - g) \, r^{-3/2} \, dr, \quad g \in L^2(m).$$

LEMMA 4. *Let $\mathbb{P}$ be a sub-Markovian semigroup satisfying $P_t(L^2(m)) \subset D(A)$, then $P_t(L^2(m)) \subset \mathcal{K}(\mathbb{P})$.*

PROOF. Let $u \in L^2(m)$ and $t > 0$. For each $v \in D(A)$, we have

$$\begin{aligned}
\|\Upsilon(v)\|_2 &\le \int_0^1 \|P_s v - v\|_2 \, s^{-3/2} \, ds \\
&\le \int_0^1 \| \int_0^s P_r A v \, dr \|_2 \, s^{-3/2} \, ds \\
&\le (\int_0^1 s \, s^{-3/2} \, ds) \|Av\|_2 \\
&< \infty.
\end{aligned}$$

Hence, $\Upsilon(v) \in L^2(m)$. Now, let $u \in L^2(m)$ and $t > 0$. For $v = P_t u$, we have $\Upsilon(P_t u) \in L^2(m)$ and for $v = \Upsilon(P_t u) = P_{t/2}\Upsilon(P_{t/2} u) \in D(A)$, we have $\Upsilon^2(P_t u) \in L^2(m)$. So, $P_t u \in \mathcal{K}(\mathbb{P})$ and $P_t(L^2(m)) \subset \mathcal{K}(\mathbb{P})$. □

EXAMPLES

Let $\mathbb{P}$ be a sub-Markovian semigroup with generator A.

1. If the operator A is bounded, then $\mathcal{K}(\mathbb{P}^\beta) = L^2(m)$.

   In particular, for each sub-Markovian semigroup $\mathbb{Q}$ and each Bochner subordinator β with bounded Bernstein function, we have $\mathcal{K}(\mathbb{Q}^\beta) = L^2(m)$ (cf. [17], Lemma 2.1).

2. Suppose that $\mathbb{P}$ verifies the *sector condition*, i.e. there exists a constant $M > 0$ such that for all $f, g \in D(A)$

$$|\langle -Au, v\rangle| \leq M\langle -Au, u\rangle^{1/2}\langle -Av, v\rangle^{1/2},$$

(where $\langle ., .\rangle$ is the inner product in $L^2(m)$). According to [7], we have $\mathcal{K}(\mathbb{P}^\beta) \subset L^2(m)$. For example, if $\mathbb{P}$ is m-symmetric, i.e.

$$\langle P_t u, v\rangle = \langle u, P_t v\rangle, \quad t > 0, u, v \in L^2(m),$$

the sector condition is fulfilled for $M = 1$.

Proposition 4. *Let $\mathbb{P}$ be a sub-Markovian semigroup and let $\eta^{1/2}$ be the one-sided stable subordinator of index $\frac{1}{2}$. Let $u \in \mathcal{K}(\mathbb{P})$ be $\mathbb{P}$-excessive. Then*

$$A^{\eta^{1/2}} u = \frac{1}{2\sqrt{\pi}} \int_0^\infty (P_s u - u) s^{-3/2}\, ds \tag{4.7}$$

and

$$Au = -\frac{1}{4\pi} \int_0^\infty \int_0^\infty (P_{s+r}u + u - P_r u - P_s u) r^{-3/2} s^{-3/2}\, ds\, dr. \tag{4.8}$$

In particular, we have

$$\mathcal{K}(\mathbb{P}) \subset D(A).$$

Proof. Let $u \in \mathcal{K}(\mathbb{P})$. Recall that the Levy measure associated to $\eta^{1/2}$ is given by

$$\nu^{\eta^{1/2}}(ds) := \frac{1}{2\sqrt{\pi}}\, s^{-3/2}\, ds.$$

Using the contraction property of $\mathbb{P}$, we get

$$\begin{aligned} \|\int_0^\infty (P_s u - u) s^{-3/2}\, ds\|_2 &\leq \|\Upsilon(u)\|_2 + \|\int_1^\infty (P_s u - u) s^{-3/2}\, ds\|_2 \\ &\leq \|\Upsilon(u)\|_2 + \int_1^\infty \|P_s u - u\|_2 s^{-3/2}\, ds \\ &\leq \|\Upsilon(u)\|_2 + 2\|u\|_2 \int_1^\infty s^{-3/2}\, ds \\ &< \infty. \end{aligned}$$

Hence the following function is well defined and belongs to $L^2(m)$

$$\emptyset(u) := \frac{1}{2\sqrt{\pi}} \int_0^\infty (P_s u - u) s^{-3/2}\, ds. \tag{4.9}$$

Now, let $n \in \mathbb{N}$ and let $\nu_n^{\eta^{1/2}} := \ell_n^{\eta^{1/2}}.\lambda$ be as in (3.14). Since $\ell_n^{\eta^{1/2}} \uparrow \ell^{\eta^{1/2}}$ and $P_t u \le u$ then

$$0 \le -\int_0^\infty (P_s u - u)\, \nu_n^{\eta^{1/2}}(ds) \le -\emptyset(u), \quad s > 0.$$

Using (3.31), (3.32), and the monotone convergence theorem, we have $u \in D(A^{\eta^{1/2}})$ and

$$A^{\eta^{1/2}} u = \frac{1}{2\sqrt{\pi}} \int_0^\infty (P_s u - u) s^{-3/2}\, ds. \tag{4.10}$$

On the other hand, by (4.10), Fubini's theorem and the contraction property of $\mathbb{P}$, we obtain

$$\begin{aligned}
\|\Upsilon(A^{\eta^{1/2}} u)\|_2 &= \| \int_0^1 (P_r A^{\eta^{1/2}} u - A^{\eta^{1/2}} u) r^{-3/2} dr \|_2 \\
&= \frac{1}{2\sqrt{\pi}} \| \int_0^1 \int_0^1 (P_{s+r} u + u - P_r u - P_s u) r^{-3/2} s^{-3/2}\, ds\, dr \|_2 \\
&\quad + \frac{1}{2\sqrt{\pi}} \| \int_0^1 \int_1^\infty (P_{s+r} u + u - P_r u - P_s u) r^{-3/2} s^{-3/2}\, ds\, dr \|_2 \\
&\le \frac{1}{2\sqrt{\pi}} \|\Upsilon^2(u)\|_2 + \frac{1}{2\sqrt{\pi}} \| \int_1^\infty (P_s \Upsilon(u) - \Upsilon(u)) s^{-3/2}\, ds \|_2 \\
&\le \frac{1}{2\sqrt{\pi}} \|\Upsilon^2(u)\|_2 + \frac{1}{\sqrt{\pi}} \|\Upsilon(u)\|_2 \int_1^\infty s^{-3/2}\, ds \|_2 \\
&< \infty.
\end{aligned}$$

Since $P_t A^{\eta^{1/2}} u \le A^{\eta^{1/2}} u$ for all $t > 0$, then by (4.10), we have $A^{\eta^{1/2}} u \in D(A^{\eta^{1/2}})$ and

$$\begin{aligned}
A^{\eta^{1/2}}(A^{\eta^{1/2}} u) &= \frac{1}{2\sqrt{\pi}} \int_0^\infty (P_s A^{\eta^{1/2}} u - A^{\eta^{1/2}} u) s^{-3/2}\, ds \\
&= \frac{1}{4\pi} \int_0^\infty \int_0^\infty (P_{s+r} u + u - P_r u - P_s u) r^{-3/2} s^{-3/2}\, ds\, dr.
\end{aligned}$$

According to [17, Remark 4.2], we have $A = -A^{\eta^{1/2}} A^{\eta^{1/2}}$ and

$$D(A) = D(A^{\eta^{1/2}} A^{\eta^{1/2}}) := \{u \in D(A^{\eta^{1/2}}) : A^{\eta^{1/2}} u \in D(A^{\eta^{1/2}})\}.$$

Therefore, $\mathcal{K}(\mathbb{P}) \subset D(A)$ and (4.8) holds. □

4.2. Representation for the initial semigroup

THEOREM 3. *Let $\mathbb{P}$ be a sub-Markovian semigroup and let h be a $\mathbb{P}$-potential h such that $P_t h \in \mathcal{K}(\mathbb{P})$ for all $t > 0$. Then, there exist a unique $\mathbb{P}$-exit law φ such that*

$$h = \int_0^\infty \varphi_s \, ds. \tag{4.11}$$

Moreover, φ_t is explicitly given by

$$\varphi_t := \frac{1}{4\pi} \int_0^\infty \int_0^\infty (P_{s+t+r} h + P_t h - P_{r+t} h - P_{s+t} h) r^{-3/2} s^{-3/2} \, ds \, dr. \tag{4.12}$$

PROOF. Let h be a measurable function such that $P_t h \in \mathcal{K}(\mathbb{P})$ for all $t > 0$ and let

$$\varphi_t = -A P_t h = -\frac{\partial}{\partial t} P_t h, \quad t > 0. \tag{4.13}$$

Using Theorem 1 and Proposition 4, we deduce that $\varphi = (\varphi_t)_{t>0}$ is a $\mathbb{P}$-exit law.

On the other hand, let $\eta^{1/2}$ be the one-sided stable subordinator of index $\frac{1}{2}$. From Proposition 3, we have

$$P_t h = \int_0^\infty \phi^{1/2}_{s+t} \kappa^{\eta^{1/2}}(ds), \quad t > 0, \tag{4.14}$$

where

$$\phi^{1/2}_s := -A^{\eta^{1/2}} P_s h = \int_0^\infty (P_s h - P_{r+s} h) \nu^{\eta^{1/2}}(dr), \quad s > 0. \tag{4.15}$$

Therefore, by (4.14), (4.15), Fubini's theorem, and Proposition 4, we get

$$\begin{aligned}
\phi_t^{1/2} &= -A^{\eta^{1/2}} P_t h = -A^{\eta^{1/2}} \Big(\int_0^\infty \phi_{s+t}^{1/2} \, \kappa\eta^{1/2}(ds) \Big) \\
&= -\int_0^\infty A^{\eta^{1/2}} \phi_{s+t}^{\eta^{1/2}} \, \kappa^{\eta^{1/2}}(ds) \\
&= -\int_0^\infty A^{\eta^{1/2}} (\phi_{s+t}^{\eta^{1/2}}) \, \kappa^{\eta^{1/2}}(ds) \\
&= \int_0^\infty A^{\eta^\alpha} (A^{\eta^{1/2}} P_{s+t} h) \, \kappa^{\eta^{1/2}}(ds) \\
&= \int_0^\infty -A P_{s+t} h \, \kappa^{\eta^{1/2}}(ds).
\end{aligned}$$

This implies that

$$\phi_t^\alpha = \int_0^\infty \varphi_{s+t} \, \kappa^{\eta^{1/2}}(ds), \quad t > 0. \tag{4.16}$$

Now, since $\mathcal{L}(\kappa^{\eta^{1/2}})(r) = r^{-1/2}$, it follows by considering the Laplace transform, that $\kappa^{\eta^{1/2}} * \kappa^{\eta^{1/2}}(ds) = ds$. Hence by (4.14), (4.15), (4.16) and the dominated convergence theorem, we get

$$\begin{aligned}
P_t h &= \int_0^\infty \phi_{s+t}^{1/2} \, \kappa^{\eta^{1/2}}(ds) \\
&= \int_0^\infty \int_0^\infty \varphi_{r+s+t} \, \kappa^{\eta^{1/2}}(dr) \, \kappa^{\eta^{1/2}}(ds) \\
&= \int_0^\infty \varphi_{s+t} \, (\kappa^{\eta^{1/2}} * \kappa^{\eta^{1/2}})(ds) \\
&= \int_0^\infty \varphi_{s+t} \, ds = \int_t^\infty \varphi_s \, ds.
\end{aligned}$$

As usually, we deduce (4.11) by letting $t \to 0$. □

REMARK. In Theorem 3, we can replace $\mathcal{K}(\mathbb{P})$ by $D(A)$, but we do not have an explicit formula for φ.

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