THE D'ALEMBERT AND LOBACZEVSKI DIFFERENCE OPERATORS IN $\mathcal{F}_\lambda$ SPACES

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Abstract. Let X be a linear normed space, $\lambda \geq 0$, $n \in \mathbb{N}$. Let $\mathcal{F}_\lambda^{(n)}$ be a set defined by

$$\mathcal{F}_\lambda^{(n)} := \{g\colon X^n \to \mathbb{C} \mid |g(\overrightarrow{x})| \leq M_g \cdot e^{\lambda \sum_{k=1}^n \|x_k\|}, \quad \overrightarrow{x} \in X^n\},$$

where M_g is a constant depending on g. Moreover for all $g \in \mathcal{F}_\lambda^{(n)}$ we define

$$\|g\| := \sup_{\overrightarrow{x} \in X^n} \{e^{-\lambda \sum_{k=1}^n \|x_k\|} \cdot |g(\overrightarrow{x})|\}.$$

In the paper norms of the d'Alembert and Lobaczevski difference operators in the $\mathcal{F}_\lambda^n$ spaces are calculated (their Pexider type generalizations are also considered). Moreover it is proved that if $f\colon X \to \mathbb{C}$ is a function such that $A(f) \in \mathcal{F}_\lambda^{(2)}$, where A is the d'Alembert difference operator, then $f \in \mathcal{F}_\lambda$ or $A(f) = 0$.

1. Introduction

In the theory of functional equations and inequalities there are two related functional equations: the Cauchy additive functional equation $f(x+y) = f(x) + f(y)$ and the Pexider functional equation $f(x+y) = g(x) + h(y)$ (more details can be found in [2]). They are related, because the Pexider equation is a generalization of the Cauchy equation, therefore the Pexider equation shows the direction of generalization which can be considered in case of other functional equations. Moreover, using these equations mentioned above we can easily define the operators: the Cauchy difference operator $C(f)(x,y) = f(x+y) - f(x) - f(y)$ and the Pexider difference operator $P(f)(x,y) = f(x+y) - g(x) - h(y)$. It makes possible to establish some properties of these equations by the theory of linear operators. Obviously different vector spaces can be considered.

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In the paper we use the idea presented shortly above to find some properties of the d'Alembert and Lobaczevski functional equations. We define the d'Alembert and Lobaczevski difference operators in the same manner as it is made for the Cauchy functional equation. Next we provide a definition of normed vector spaces of functions and calculate norms of these operator in these spaces. Moreover, we consider the Pexider type generalizations of these equations.

Additionaly it was proved that the d'Alembert functional equation is superstable in the spaces provided in the text.

The Cauchy and Pexider difference operators were considered in [3]. The results obtained in that paper are cited in the text below.

2. Preliminaries

Let us recall the definition of a quadratic operator and its norm and the definition of a linear-quadratic operator (which is a sum of a linear and a quadratic operator) and its norm. In the next section we will prove that the Lobaczevski difference operator is quadratic and the d'Alembert difference operator is linear-quadratic. Let E, F be vector spaces over a field $\mathbb{K}$.

DEFINITION 1. An operator $Q\colon E \to F$ is called quadratic if it satisfies following equations:

(a) $\forall\ x, y \in E \qquad Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y),$

(b) $\forall\ k \in \mathbb{K} \quad \forall\ x \in E \quad Q(kx) = k^2 Q(x).$

DEFINITION 2. A quadratic operator $Q\colon E \to F$ is called bounded if

$$\exists\ c > 0\ \forall\ x \in E \quad \|Q(x)\| \leq c\|x\|^2.$$

A norm of a quadratic operator $Q\colon E \to F$ is defined by

$$\|Q\| := \inf\{c > 0 \mid \|Q(x)\| \leq c\|x\|^2,\ \ x \in E\}. \tag{1}$$

If such a number c does not exist we define $\|Q\| := \infty$.
By $\mathcal{B}_Q(E, F)$ we denote a set of all quadratic operators $Q\colon E \to F$ such that $\|Q\| < \infty$.

REMARK 1. *Analogously as for a bounded linear operator one can prove an alternative definition:*

$$\|Q\| := \sup\{\|Q(x)\| \mid x \in E, \|x\| = 1\}. \tag{2}$$

Let us notice that the $(\mathcal{B}_Q(E, F), \|\cdot\|)$ space is a linear normed space. Now we are ready to define a linear-quadratic operator.

DEFINITION 3. By $\mathcal{B}_{\mathcal{L}Q}$ we denote the set

$$\mathcal{B}_{\mathcal{L}Q}(E,F) = \{T \in F^E \mid \exists\ L \in \mathcal{B}(E,F) \wedge \exists\ Q \in \mathcal{B}_Q(E,F)\ \ T = L + Q\}.$$

Moreover, for all $T = L + Q \in \mathcal{B}_{\mathcal{L}Q}(E,F)$ we define

$$\|T\| = \|L\| + \|Q\|.$$

An operator $T \in \mathcal{B}_{\mathcal{L}Q}(E,F)$ is called a bounded linear-quadratic operator.

Let us notice that the space $(\mathcal{B}_{\mathcal{L}Q}, \|\cdot\|)$ is a linear normed space.

3. The d'Alembert and Lobaczevski difference operators

A standard symbol $\mathbb{C}$ denotes the set of complex numbers, for a set X a symbol $\mathbb{C}^X$ denotes a set of all functions $f\colon X \to \mathbb{C}$.

DEFINITION 4. Let X be a linear normed space. The Lobaczevski difference operator $\mathcal{L}\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ is defined by:

(3) $$\mathcal{L}(f)(x,y) := f^2\left(\frac{x+y}{2}\right) - f(x)f(y), \quad x,y \in X.$$

LEMMA 1. *The Lobaczevski difference operator $\mathcal{L}\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ defined above is a quadratic operator.*

DEFINITION 5. Let X be a linear normed space. The d'Alembert difference operator $A\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ is defined by:

(4) $$A(f)(x,y) := f(x+y) + f(x-y) - 2f(x)f(y), \quad x,y \in X.$$

LEMMA 2. *Let $A\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ be the d'Alembert difference operator, then there exist a linear operator L_A and a quadratic operator Q_A such that*

(5) $$A(f)(x,y) = L_A(f)(x,y) + Q_A(f)(x,y), \qquad x,y \in X.$$

PROOF. Let $L_A\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ and $Q_A\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ are defined by:

$$L_A(f)(x,y) := f(x+y) + f(x-y),$$
$$Q_A(f)(x,y) := -2f(x)f(y),$$

Therefore $A = L_A + Q_A$. □

4. The d'Alembert and Lobaczevski difference operators in $\mathcal{F}_\lambda$ spaces

4.1. The $\mathcal{F}_\lambda^{(n)}$ spaces

DEFINITION 6 ([1], [3], see also [2]). Let X be a linear normed space, $\lambda \geq 0$, $n \in \mathbb{N}$. Let $\mathcal{F}_\lambda^{(n)}$ be a set defined by

$$\mathcal{F}_\lambda^{(n)} := \{g\colon X^n \to \mathbb{C} \mid |g(\overrightarrow{x})| \leq M_g \cdot e^{\lambda \sum_{k=1}^n \|x_k\|}, \ \ \overrightarrow{x} \in X^n\},$$

where M_g is a constant depending on g. Moreover for all $g \in \mathcal{F}_\lambda^{(n)}$ we define

$$\|g\| := \sup_{\overrightarrow{x} \in X^n} \{e^{-\lambda \sum_{k=1}^n \|x_k\|} \cdot |g(\overrightarrow{x})|\}.$$

Clearly the following lemma holds.

LEMMA 3. *The $(\mathcal{F}_\lambda^n, \|\cdot\|)$ space, where $\|\cdot\|$ is the norm defined above is a linear normed space for every $n \in \mathbb{N}$.*

We denote $\mathcal{F}_\lambda := \mathcal{F}_\lambda^{(1)}$.

LEMMA 4. *Let $\mathcal{L}\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ be the Lobaczevski difference operator. Then*

$$\forall \ f \in \mathcal{F}_\lambda \quad \mathcal{L}(f) \in \mathcal{F}_\lambda^{(2)}.$$

PROOF. Let $f \in \mathcal{F}_\lambda$. Then we obtain

$$\begin{aligned} |\mathcal{L}(f)(x,y)| &\leq |f(\tfrac{x+y}{2})|^2 + |f(x)f(y)| \\ &\leq M_f^2 e^{2\lambda \|\frac{x+y}{2}\|} + M_f^2 e^{\lambda(\|x\|+\|y\|)} \leq 2M_f^2 e^{\lambda(\|x\|+\|y\|)}, \end{aligned}$$

thus $\mathcal{L}(f) \in \mathcal{F}_\lambda^{(2)}$ as claimed. □

LEMMA 5. *Let $A\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ be the d'Alembert difference operator. Then*

$$\forall f \in \mathcal{F}_\lambda \quad A(f) \in \mathcal{F}_\lambda^{(2)}.$$

PROOF. Let $f \in \mathcal{F}_\lambda$. Then we obtain

$$\begin{aligned} |A(f)(x,y)| &\leq |f(x+y)| + |f(x-y)| + 2|f(x)f(y)| \\ &\leq M_f e^{\lambda\|x+y\|} + M_f e^{\lambda\|x-y\|} + 2M_f^2 e^{\lambda(\|x\|+\|y\|)} \\ &\leq N_f e^{\lambda(\|x\|+\|y\|)}, \end{aligned}$$

where $N_f = \max\{M_f, 2M_f^2\}$. Thus the lemma holds. □

4.2. Norms of the d'Alembert and the Lobaczevski difference operators

We will prove the following theorem.

THEOREM 1. *The Lobaczevski difference operator* $\mathcal{L}\colon \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\lambda^{(2)}$ *defined by* (3) *belongs to the* $\mathcal{B}_Q(\mathcal{F}_\lambda, \mathcal{F}_\lambda^{(2)})$ *space and for all* $f \in \mathcal{F}_\lambda$ *we have*

$$\|\mathcal{L}(f)\| \leq 2\|f\|^2.$$

PROOF. We have

$$\begin{aligned}\|\mathcal{L}(f)\| &\leq \sup_{x,y\in X}\{e^{-2\lambda\|\frac{x+y}{2}\|}|f(\tfrac{x+y}{2})|^2\} + \sup_{x,y\in X}\{e^{-\lambda\|x\|}|f(x)|e^{-\lambda\|y\|}|f(y)|\}\\ &\leq (\sup_{x,y\in X}\{e^{-\lambda\|\frac{x+y}{2}\|}|f(\tfrac{x+y}{2})|\})^2 + \sup_{x\in X}\{e^{-\lambda\|x\|}|f(x)|\}\sup_{y\in X}\{e^{-\lambda\|y\|}|f(y)|\}\\ &\leq \|f\|^2 + \|f\|^2 = 2\|f\|^2.\end{aligned}$$

Thus $\mathcal{L} \in \mathcal{B}_Q(\mathcal{F}_\lambda, \mathcal{F}_\lambda^{(2)})$ as claimed. □

THEOREM 2. *The d'Alembert difference operator* $A\colon \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\lambda^{(2)}$ *defined by* (4) *belongs to the* $\mathcal{B}_{\mathcal{L}Q}(\mathcal{F}_\lambda, \mathcal{F}_\lambda^{(2)})$ *space and for all* $f \in \mathcal{F}_\lambda$ *we have*

$$\|A(f)\| \leq 2\|f\| + 2\|f\|^2.$$

PROOF. In view of (5) we have $A = L_A + Q_A$, where the linear operator $L_A\colon \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\lambda^{(2)}$ and the quadratic operator $Q_A\colon \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\lambda^{(2)}$ are given by:

$$\begin{aligned}L_A(f)(x,y) &:= f(x+y) + f(x-y),\\ Q_A(f)(x,y) &:= -2f(x)f(y).\end{aligned}$$

The operator L_A is linear and for all $f \in \mathcal{F}_\lambda$ we obtain

$$\begin{aligned}\|L_A f\| &\leq \sup_{x,y\in X}\{e^{-\lambda(\|x\|+\|y\|)}|f(x+y)|\} + \sup_{x,y\in X}\{e^{-\lambda(\|x\|+\|y\|)}|f(x-y)|\}\\ &\leq \sup_{x,y\in X}\{e^{-\lambda(\|x+y\|)}|f(x+y)|\} + \sup_{x,y\in X}\{e^{-\lambda(\|x-y\|)}|f(x-y)|\} = 2\|f\|.\end{aligned}$$

Thus $L_A \in \mathcal{B}(\mathcal{F}_\lambda, \mathcal{F}_\lambda^{(2)})$. We shall show that Q_A is bounded and $\|Q_A\| = 2$. Let $f \in \mathcal{F}_\lambda$, then

$$\begin{aligned}\|Q_A(f)\| &= \sup_{x,y\in X}\{e^{-\lambda(\|x\|+\|y\|)}|2f(x)f(y)|\}\\ &= 2\sup_{x\in X}\{e^{-\lambda\|x\|}|f(x)|\}\sup_{y\in X}\{e^{-\lambda\|y\|}|f(y)|\} = 2\|f\|^2.\end{aligned}$$

The operator Q_A is quadratic and bounded so therefore $Q_A \in \mathcal{B}_Q(\mathcal{F}_\lambda, \mathcal{F}_\lambda^{(2)})$. Due to the fact that $A = L_A + Q_A$ we obtain that $A \in \mathcal{B}_{\mathcal{L}Q}(\mathcal{F}_\lambda, \mathcal{F}_\lambda^{(2)})$ and

$$\|A(f)\| = \|L_A f + Q_A(f)\| \leq \|L_A f\| + \|Q_A(f)\| \leq 2\|f\| + 2\|f\|^2.$$

□

In this part of the paper we will find norms of the d'Alembert and Lobaczevski difference operators.

THEOREM 3. *If* $\mathcal{L}\colon \mathcal{F}_\lambda \to \mathcal{F}_\lambda^{(2)}$ *is defined by* (3), *then*

$$\|\mathcal{L}\| = 2.$$

PROOF. Let $u \in X$. Let us define a function h by

$$h(x) := \begin{cases} -e^{\lambda\|u\|}, & x = u, \\ e^{2\lambda\|u\|}, & x = 2u, \\ e^{\frac{3}{2}\lambda\|u\|}, & x = \frac{3}{2}u, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly we have

$$|h(x)| \le e^{\lambda\|u\|}e^{\lambda\|x\|}, \ \ x \in X,$$

therefore $h \in \mathcal{F}_\lambda$. Moreover,

$$e^{-\lambda\|x\|}|h(x)| = \begin{cases} 1, \ x \in \{u, 2u, \frac{3}{2}u\}, \\ 0, \ \text{otherwise} \end{cases}.$$

Then $\|h\| = 1$ and

$$\begin{aligned} \|\mathcal{L}(h)\| &\ge e^{-3\lambda\|u\|}|h^2(\tfrac{3}{2}u) - h(u)h(2u)| \\ &= e^{-3\lambda\|u\|}|e^{3\lambda\|u\|} + e^{\lambda\|u\|}e^{2\lambda\|u\|}| = 2, \end{aligned}$$

whence

$$\|\mathcal{L}\| := \sup\{\|\mathcal{L}(f)\| \mid f \in \mathcal{F}_\lambda, \|f\| = 1\} \ge \|\mathcal{L}(h)\| \ge 2.$$

In view of Theorem 1, $\|\mathcal{L}\| \le 2$, thus $\|\mathcal{L}\| = 2$. □

THEOREM 4. *If* $A\colon \mathcal{F}_\lambda \to \mathcal{F}_\lambda^{(2)}$ *is defined by* (4), *then*

$$\|A\| = 4.$$

PROOF. Due to the fact that $\|A\| = \|L_A\| + \|Q_A\|$, where L_A and Q_A are defined above, we will find $\|L_A\|$ (it was proved before that $\|Q_A\| = 2$).

Let $x_n \in X$ for all $n \in \mathbb{N}$ be a sequence such that $\lim\limits_{n\to\infty} \|x_n\| = 0$. Let us define for $n \in \mathbb{N}$ a function f_n by

$$f_n(x) := \begin{cases} e^{2\lambda\|x_n\|}, & x \in \{0, 2x_n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly we have

$$|f_n(x)| \le e^{2\lambda\|x_n\|}e^{\lambda\|x\|}, \ \ x \in X,$$

therefore $f_n \in \mathcal{F}_\lambda$ for all $n \in \mathbb{N}$. Moreover,

$$e^{-\lambda\|x\|}|f_n(x)| = \begin{cases} e^{2\lambda\|x_n\|}, & x = 0, \\ 1, & x = 2x_n, \\ 0, & \text{otherwise}. \end{cases}$$

Because the sequence $\{\|x_n\|\}$ is a sequence of nonnegative numbers which is convergent to 0, we obtain that $e^{2\lambda\|x_n\|} > 1$, so $\|f_n\| = e^{2\lambda\|x_n\|}$ for all $n \in \mathbb{N}$. Moreover,

$$\|L_A f_n\| \geq e^{-2\lambda\|x_n\|}|f_n(2x_n) + f_n(0)| = e^{-2\lambda\|x_n\|}2e^{2\lambda\|x_n\|} = 2.$$

Thus $\|L_A f_n\| \geq 2$. Now let us suppose that $\|L_A\| < 2$, then there exists $\epsilon > 0$ such that

$$\|L_A f_n\| \leq (2 - \epsilon)\|f_n\|, \quad f_n \in \mathcal{F}_\lambda.$$

On the other hand, for $f_n \in \mathcal{F}_\lambda$ we have

$$2 \leq \|L_A f_n\| \leq (2 - \epsilon)e^{2\lambda\|x_n\|}.$$

Let us notice that if $n \to \infty$ then $\|x_n\| \to 0$ and $e^{2\lambda\|x_n\|} \to 1$, thus $(2-\epsilon)e^{2\lambda\|x_n\|} \to 2 - \epsilon$, so we get $2 \leq 2 - \epsilon$, where $\epsilon > 0$, which is impossible. Thus we obtain that $\|L_A\| = 2$.

Because $\|A\| = \|L_A\| + \|Q_A\|$, then we have $\|A\| = 4$. □

REMARK 2. *In the paper* [3] *Stefan Czerwik and Krzysztof Dłutek have proved that the Cauchy difference operator* $C : \mathcal{F}_\lambda \to \mathcal{F}_\lambda^{(2)}$ *defined by*

$$C(f)(x, y) = f(x + y) - f(x) - f(y), \quad x, y \in X$$

is a linear bounded operator and $\|C\| = 3$.

4.3. Superstability of the d'Alembert functional equation in the $\mathcal{F}_\lambda$ spaces

By direct calculations one can prove the following lemmas

LEMMA 6. *Let* $f \in \mathcal{F}_\lambda^{(2)}$, *then*

$$\forall\ y \in X \quad f(\cdot, y) \in \mathcal{F}_\lambda,$$
$$\forall\ y, u \in X \quad f(\cdot + u, y) \in \mathcal{F}_\lambda.$$

LEMMA 7. *Let* G *be an abelian group. Then for all* $x, u, v \in G$

$$\begin{aligned} 2f(x)[A(f)(u, v)] &= A(f)(x + u, v) - A(f)(x, u + v) - A(f)(x, u - v) \\ &\quad + A(f)(x - u, v) + 2f(v)A(f)(x, u), \end{aligned}$$

where $A(f)$ *is defined by* (4).

THEOREM 5. *Let $f\colon X \to \mathbb{C}$ be a function such that $A(f) \in \mathcal{F}_\lambda^{(2)}$. Then $f \in \mathcal{F}_\lambda$ or $A(f) = 0$.*

PROOF. Let us suppose that $f \notin \mathcal{F}_\lambda$, therefore for every $M \in \mathbb{R}_+$ there exists $x \in X$ such that

$$|f(x)| > Me^{\lambda\|x\|}.$$

From the equality from the previous lemma for all $x, u, v \in X$ we have

$$\begin{aligned} 2f(x)[A(f)(u,v)] = {} & A(f)(x+u,v) - A(f)(x,u+v) - A(f)(x,u-v) \\ & + A(f)(x-u,v) + 2f(v)A(f)(x,u). \end{aligned}$$

From the previous lemma and due to the fact that the $\mathcal{F}_\lambda$ is a linear space we obtain that the right-hand side of the equality belongs to the $\mathcal{F}_\lambda$ space as a function of x, therefore there exists $M_A \in \mathbb{R}$ such that

$$|f(x)[A(f)(u,v)]| \le M_A e^{\lambda\|x\|}, \quad x, u, v \in X.$$

Let us consider two cases:

1. $|A(f)(u,v)| \neq 0$ for some $u, v \in X$,
2. $|A(f)(u,v)| = 0$ for all $u, v \in X$.

In the first case we obtain

$$|f(x)| \le \frac{M_A}{|A(f)(u,v)|} e^{\lambda\|x\|}, \quad \text{for all } x \in X.$$

Under the assumption there exists $x_0 \in X$ such that

$$|f(x_0)| > \frac{M_A}{|A(f)(u,v)|} e^{\lambda\|x_0\|},$$

which causes a contradiction. Therefore the second case is true and

$$A(f)(u,v) = 0$$

for all $u, v \in X$ as claimed. □

5. Remarks about Pexider type generalizations

5.1. The $(\mathcal{F}_\lambda)^n$ spaces

DEFINITION 7. For $n > 1$ we define

$$(\mathcal{F}_\lambda)^n := \{(f_1, f_2, \dots, f_n) \mid \forall\ 1 \le i \le n\ \ f_i \in \mathcal{F}_\lambda\},$$

$$\|(f_1, f_2, \dots, f_n)\| := \max\{\|f_1\|, \|f_2\|, \dots, \|f_n\|\}.$$

Let us notice that for all $n > 1$, the $(\mathcal{F}_\lambda)^n$ spaces with norms provided above are vector normed spaces.

REMARK 3. *In the paper* [3] *Stefan Czerwik and Krzysztof Dłutek have proved that the Pexider difference operator* $P : (\mathcal{F}_\lambda)^3 \to \mathcal{F}_\lambda^{(2)}$ *defined by*

$$P((f,g,h))(x,y) = f(x+y) - g(x) - h(y), \quad x,y \in X$$

is a linear bounded operator and $\|P\| = 3$.

5.2. The Pexider type generalization of the Lobaczevski difference operator

DEFINITION 8. Let X be a linear normed space. The Pexider–Lobaczevski difference operator $\mathcal{L}_P\colon (\mathbb{C}^X)^4 \to \mathbb{C}^{X^2}$ is defined by

$$\mathcal{L}_P((f,g,h,k))(x,y) := f\left(\tfrac{x+y}{2}\right) g\left(\tfrac{x+y}{2}\right) - h(x)k(y), \quad x,y \in X. \tag{6}$$

This operator is not quadratic. For $f = g = h = k$ we obtain the Lobaczevski difference operator. We will prove the following theorem.

THEOREM 6. *For all* $u \in (\mathcal{F}_\lambda)^4$ *the Pexider–Lobaczevski difference operator* $\mathcal{L}_P\colon (\mathcal{F}_\lambda)^4 \to \mathcal{F}_\lambda^{(2)}$ *defined in the previous definition satisfies inequality*

$$\|\mathcal{L}_P(u)\| \le 2\|u\|^2.$$

PROOF. It is easy to show that $\forall\, u \in (\mathcal{F}_\lambda)^4 \;\; \mathcal{L}_P(u) \in \mathcal{F}_\lambda^{(2)}$. Take $u = (f,g,h,k)$, then we have by the definition

$$\begin{aligned}\|\mathcal{L}_P((f,g,h,k))\| &\le \sup_{x,y\in X} \{e^{-2\lambda\|\frac{x+y}{2}\|}|f(\tfrac{x+y}{2})|\cdot|g(\tfrac{x+y}{2})|\} \\ &\quad + \sup_{x,y\in X}\{e^{-\lambda\|x\|}|h(x)|e^{-\lambda\|y\|}|k(y)|\} \\ &\le \|f\|\|g\| + \|h\|\|k\| \\ &= 2(\max\{\|f\|,\|g\|,\|h\|,\|k\|\})^2 = 2\|u\|^2.\end{aligned}$$

□

COROLLARY 1. *If* $\mathcal{L}_P\colon (\mathcal{F}_\lambda)^4 \to \mathcal{F}_\lambda^{(2)}$ *is given by* (6), *then*

$$\inf\{c > 0 \mid \|\mathcal{L}_P(u)\| \le c\|u\|^2, u \in (\mathcal{F}_\lambda)^4\} = 2.$$

PROOF. Let us assume on the contrary that

$$d := \inf\{c > 0 \mid \|\mathcal{L}_P(u)\| \le c\|u\|^2, u \in (\mathcal{F}_\lambda)^4\} < 2.$$

Then for $f = g = h = k$, we get

$$\|\mathcal{L}_P(u)\| = \|\mathcal{L}(f)\| \le d\|(f,f,f,f)\|^2 = d\|f\|^2,$$

whence

$$\|\mathcal{L}(f)\| \le d\|f\|^2.$$

By the hypothesis, $d < 2$ and therefore we infer that $\|\mathcal{L}\| < 2$, which is impossible in view of the previous lemma. □

5.3. The Pexider type generalization of the d'Alembert difference operator

DEFINITION 9. Let X be a linear normed space. The Pexider–d'Alembert difference operator $A_P\colon (\mathbb{C}^X)^4 \to \mathbb{C}^{X^2}$ is defined by

$$A_P((f,g,h,k))(x,y) := f(x+y) + g(x-y) - 2h(x)k(y), \quad x,y \in X.$$

We shall prove the following theorem.

THEOREM 7. *For all $u \in (\mathcal{F}_\lambda)^4$ the Pexider–d'Alembert difference operator $A_P\colon (\mathcal{F}_\lambda)^4 \to \mathcal{F}_\lambda^{(2)}$ defined in the previous definition satisfies the inequality*

$$\|A_P(u)\| \le 2\|u\| + 2\|u\|^2.$$

PROOF. It is easy to show that $\forall\, u \in (\mathcal{F}_\lambda)^4 \;\; A_P(u) \in \mathcal{F}_\lambda^{(2)}$. Take $u = (f,g,h,k)$, then we have by the definition

$$\begin{aligned}\|A_P(u)\| &\le \sup_{x,y\in X}\{e^{-\lambda\|x+y\|}|f(x+y)|\} + \sup_{x,y\in X}\{e^{-\lambda\|x-y\|}|g(x-y)|\} \\ &\quad + 2\sup_{x,y\in X}\{e^{-\lambda\|x\|}|h(x)|e^{-\lambda\|y\|}|k(y)|\} \\ &\le \|f\| + \|g\| + 2\|h\|\|k\| \\ &= 2\max\{\|f\|,\|g\|,\|h\|,\|k\|\} + 2(\max\{\|f\|,\|g\|,\|h\|,\|k\|\})^2 \\ &= 2\|u\| + 2\|u\|^2.\end{aligned}$$

□

COROLLARY 2. *For all $u \in (\mathcal{F}_\lambda)^2$ the difference operator $L_P : (\mathcal{F}_\lambda)^2 \to \mathcal{F}_\lambda^{(2)}$ defined by*

$$L_P((f,g))(x,y) := f(x+y) + g(x-y), \qquad x,y \in X$$

is linear and satisfies the inequality

$$\|L_P(u)\| \le 2\|u\|.$$

Moreover $\|L_P\| = 2$.

PROOF. The first part of the proof is simple and analogous to the proof of the previous lemma. We shall prove that $\|L_P\| = 2$. Let us assume on the contrary that $\|L_P\| < 2$. Then for $f = g$, we get

$$\|L_P((f,g))\| = \|L_A f\| \leq \|L_P\| \cdot \|(f,f)\| = \|L_P\| \cdot \|f\|,$$

whence $\|L_A f\| \leq \|L_P\| \cdot \|f\|$. By the hypothesis, $\|L_P\| < 2$ and therefore we infer that $\|L_A\| < 2$, which is imposible in view of the previous lemma. □

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