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STRING STABILITY OF SINGULARLY PERTURBED STOCHASTIC SYSTEMS

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Abstract. **The sufficient conditions of string stability of singularly perturbed, nonlinear stochastic systems are established. The excitations are assumed to be parametric white noise. In this case the objective is to analyze composite systems in their lower order subsystems and in terms of their interconnecting structure and the perturbation parameter ε. An example is given to illustrate the results.**

1. Introduction

The problem of string stability of interconnected deterministic systems was studied earlier for vehicle–following applications , for instance, in [1], [3] and recently in [8]. In particular, there have been several unprecise definitions for string stability, for instance, [1]. Recently, the precise definition of string stability was given by Swaroop and Hedrick [8]. The string stability analysis of nonlinear composite stochastic systems has not been completed yet. The sufficient conditions of exponential string stability for a few classes of nonlinear interconnected stochastic systems was given by Socha [7].

The stability analysis of large-scale stochastic singularly perturbed systems has been considered in [5] and [6].

The aim of this paper is to solve a problem of exponential string stability of singularly perturbed, nonlinear stochastic systems. To derive the sufficient conditions of exponential mean-square string stability of these systems the idea of the exponential stability of singularly, perturbed stochastic systems presented in [6] is combined with the concept of string stability of singularly perturbed interconnected deterministic systems (see [8]).

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2. Definitions and some auxiliary facts

We consider the nonlinear autonomous interconnected stochastic system described by Ito equation:

$$(1)\quad dx^i = F(x^i, x^{i-1}, \dots, x^{i-r+1})dt + \sum_{m=1}^{M} G_m(x^i)d\omega^m, \qquad x^i(0) = x^{i0},$$

where $i \in \mathbb{N}, t \in [0, +\infty), x^i$ is the state of each subsystems, $x^i \in R^k$ and we take $x^{i-j} \equiv 0$ for all $i \leqslant j$.

We assume that $F : \underbrace{R^k \times \dots \times R^k}_{r \text{ times}} \to R^k, G_m : R^k \to R^k, m = 1, \dots, M$ are nonlinear deterministic vector functions $F = [F_1, \dots, F_k], G_m = [G_{m1}, \dots, G_{mk}]$ and ω^m, $m = 1, \dots, M$, are independent standard Wiener processes.

We denote by $\mathcal{L}^*_{(1)}$ the operator associated with (1)

$$(2)\quad \begin{aligned} \mathcal{L}^*_{(1)}(\cdot) = \frac{\partial(\cdot)}{\partial t} &+ \sum_{j=1}^{k} F_j(x^i, x^{i-1}, \dots, x^{i-r+1})\frac{\partial(\cdot)}{\partial x^i_j} \\ &+ \frac{1}{2}\sum_{j=1}^{k}\sum_{l=1}^{k}\sum_{m=1}^{M} \sigma_{G_{mjl}}(x^i)\frac{\partial^2(\cdot)}{\partial x^i_j \partial x^i_l}, \end{aligned}$$

where $\sigma_{G_{mjl}}(x^i) = G_{mj}(x^i) \cdot G_{ml}(x^i)$.

We use the following notations:

$|\cdot|$ is Euclidean norm; for all $p < +\infty$ $\;||f(0)||^p_\infty$ denotes $\sup\limits_{i \in \mathbb{N}} E[|f^i(0)|^p]$ and $||f^i||^p_\infty = ||f^i(\cdot)||^p_\infty$ denotes $\sup\limits_{t \geqslant 0} E[|f^i(t)|^p]$.

To derive stability criteria we recall the following definitions (see [7]).

Definition 1. The equilibrium $x^i = 0, i \in \mathbb{N}$ of system (1) is p–mean string stable if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$(3)\quad ||x(0)||^p_\infty < \delta \Longrightarrow \sup_{i \in \mathbb{N}} ||x^i(\cdot)||^p_\infty < \varepsilon.$$

Definition 2. The origin $x^i = 0, i \in \mathbb{N}$ of system (1) is exponentially string p–stable if it is p–mean string stable and if there exist positive constants c_i and α_i, such that

$$(4)\quad E[|x^i(t)|^p] < c_i|x_0|^p \exp\{-\alpha_i(t - t_0)\}$$

for all $i \in \mathrm{N}$. In particular case for $p = 1$ and $p = 2$ it is called exponential mean and mean–square string stability.

In the sequel we will use the following lemmas.

LEMMA 1 [1]. *Consider any symmetric matrix $S(\epsilon) = [s_{ij}(\epsilon)]$, i, j=1, 2, in which the function $s_{ij} : (0, +\infty) \to R$ satisfy*

$$\lim_{\epsilon \to 0} s_{11}(\epsilon) = \lambda_0, \quad \lim_{\epsilon \to 0} s_{22}(\epsilon) = +\infty, \quad \lim_{\epsilon \to 0} \frac{s_{12}^2(\epsilon)}{s_{22}(\epsilon)} = 0$$

then $\lim\limits_{\epsilon \to 0} \lambda_{\min}(S(\epsilon)) = \lambda_0$, where $\lambda_{\min}(S)$ is the minimal eigenvalue of matrix S.

LEMMA 2 [7]. *Let $V^i = V^i(x^i(t)) \geqslant 0$ for all $i \in \mathrm{N}$, $t \geqslant 0$, $x^i \in R^k$ and if*

$$\mathcal{L}^*_{(1)} V^i(t) \leqslant -\beta_0 V^i(t) + \sum_{j=1}^{\infty} \beta_j V^{i-j}(t),$$

where $\beta_0 > 0$ and $\beta_j \geqslant 0$ for all $j = 1, 2, \ldots$ and $\beta_0 > \sum\limits_{j=1}^{\infty} \beta_j$, $V^j(t) = 0$ for all $j \leqslant 0$.

Then given any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$||V(0)||_\infty^1 < \delta \Longrightarrow \sup_{i \in \mathrm{N}} ||V^i(\cdot)||_\infty^1 < \epsilon.$$

3. System description

Let us consider the autonomous, interconnected singularly perturbed stochastic system described by Ito equations:

$$dx^i = f(x^i, z^i, x^{i-1}, \ldots, x^{i-r+1})dt + q_1(x^i, z^i)d\omega^1, \qquad x^i(0) = x^{i0}, \tag{5}$$

$$\epsilon dz^i = g(x^i, z^i)dt + \sqrt{\epsilon} q_2(x^i, z^i)d\omega^2, \qquad z^i(0) = z^{i0}, \tag{6}$$

where $i \in \mathrm{N}, t \in R^+$ is the time, $x^i \in R^n, x^{i-j} \equiv 0$ for all $i \leqslant j$,$z^i \in R^m$ and $\epsilon > 0$ is the singular perturbation parameter.

We assume that $f : R^n \times R^m \times \underbrace{R^n \times \ldots \times R^n}_{(r-1)\text{ times}} \to R^n$, $q_1 : R^n \times R^m \to R^n$ and $g, q_2 : R^n \times R^m \to R^m$ are nonlinear continuous functions such that

$$f(0,..,0) = q_1(0,0) = 0, \qquad g(0,0) = q_2(0,0) = 0$$

and ω^1, ω^2 are independent standard Wiener processes.

For convenience, we assume that the initial conditions $x^{i0} \in R^n$, $z^{i0} \in R^m$, $i \in \mathbf{N}$ are deterministic.

We introduce the following assumptions.

ASSUMPTION 1. The equation $g(x^i, z^i) = 0$ has a unique solution $z^i = h(x^i)$, where h is continuously twice differentiable, $h(0) = 0$ and a positive constant M exists such that for all $x^i \in R^n$ and $j = 1, \ldots, n$, $k = 1, \ldots, m$, $|\frac{\partial h_k}{\partial x^i_j}| \leqslant M$.

This assumption defines the complete reduced-order system by setting $z^i = h(x^i)$ in (5) as follows

$$dx^i = f(x^i, h(x^i), x^{i-1}, \ldots, x^{i-r+1})dt + q_1(x^i, h(x^i))d\omega^1. \tag{7}$$

We introduce a new variable

$$y^i = z^i - h(x^i) \tag{8}$$

called the boundary-layer state.

In the new coordinates the full–order interconnected system is

$$dx^i = F(x^i, y^i, x^{i-1}, \ldots, x^{i-r+1})dt + Q_{11}(x^i, y^i)d\omega^1, \qquad x^i(0) = x^{i0}, \tag{9}$$

$$\epsilon dy^i = G(x^i, y^i, x^{i-1}, \ldots, x^{i-r+1})dt + Q_{21}(x^i, y^i)d\omega^1 + Q_{22}(x^i, y^i)d\omega^2, \tag{10}$$

$y^i(0) = z^i(x^{i0}) - h(x^{i0})$, where j–th and l–th components of F, G, Q_{11}, Q_{21} and Q_{22}, for $j = 1, \ldots, n$, $l = 1, \ldots, m$, have the form

$$F_j(x^i, y^i, x^{i-1}, \ldots, x^{i-r+1}) = f_j(x^i, y^i + h(x^i), x^{i-1}, \ldots, x^{i-r+1}),$$

$$Q_{11l}(x^i, y^i) = q_{1l}(x^i, y^i + h(x^i)),$$

$$G_l(x^i, y^i, x^{i-1}, \ldots, x^{i-r+1})$$
$$= g_l(x^i, y^i + h(x^i)) - \epsilon \sum_{j=1}^{n} \frac{\partial h_l}{\partial x^i_j} f_j(x^i, y^i + h(x^i), x^{i-1}, \ldots, x^{i-r+1})$$

$$-\frac{1}{2}\epsilon\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 h_l}{\partial x_j^i \partial x_k^i}\sigma_{q_{1jk}}(x^i, y^i + h(x^i)),$$

$$Q_{21l}(x^i, y^i) = -\epsilon\sum_{j=1^n}\frac{\partial h_l}{\partial x_j^i}q_{1j}(x^i, y^i + h(x^i)),$$

$$Q_{22l}(x^i, y^i) = \sqrt{\epsilon}q_{2l}(x^i, y^i + h(x^i)).$$

System (9), (10) is treated as an interconnection of isolated subsystems described by

$$dx^i = F(x^i, y^i, 0, \ldots, 0)dt + Q_{11}(x^i, y^i)d\omega^1, \tag{11}$$

$$\epsilon dy^i = G(x^i, y^i, 0, \ldots, 0)dt + Q_{21}(x^i, y^i)d\omega^1 + Q_{22}(x^i, y^i)d\omega^2, \tag{12}$$

Intuitively, the origin of the perturbed interconnected stochastic system will be mean string stable if the origin of every perturbed subsystem (11), (12) is stable and the origin of the "reduced" interconnected system (7) is mean string stable.

This observation leads us to the following assumptions.

ASSUMPTION 2. There exists a positive definite function $V^i = V(x^i)$, $i \in \mathbf{N}$ continuously twice differentiable with respect to x^i and there exist positive constants $\gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha_3, \alpha_{1j}$, $j = 1, \ldots, r$, such that the following inequalities are satisfied:

$$\gamma_1|x^i|^2 \leqslant V(x^i) \leqslant \gamma_2|x^i|^2,$$

$$\mathcal{L}^*_{(7)}V(x^i) \leqslant -\alpha_1|x^i|^2 + \sum_{j=2}^{r}\alpha_{1j}|x^{i-j+1}|^2, \qquad \alpha_1 > \frac{\gamma_2}{\gamma_1}\sum_{j=2}^{r}\alpha_{1j},$$

$$\left|\frac{\partial V^i}{\partial x_j^i}\right| < \alpha_2|x^i|, \qquad \left|\frac{\partial^2 V^i}{\partial x_j^i \partial x_k^i}\right| \leqslant \alpha_3, \qquad j, k = 1, \ldots, n, \ i \in \mathbf{N}.$$

These conditions imply the mean string stability of reduced-order systems (see [7]).

ASSUMPTION 3. There exist a positive definite function $W^i = W(x^i, y^i)$, $i \in \mathbf{N}$ continuously twice differentiable with respect to x^i, y^i and there exist positive constants η_i, $i = 1, \ldots, 4$, s_1, s_2 and a continuous function $s_3 : (0, +\infty) \to R^+$ such that the following inequalities are satisfied:

$$\eta_1|y^i|^2 \leqslant W(x^i, y^i) \leqslant \eta_2|y^i|^2,$$

$$\left|\frac{\partial W^i}{\partial y^i_j}\right| < \eta_3 |y^i|, \qquad \left|\frac{\partial W^i}{\partial x^i_j}\right| \leqslant \eta_4 |y^i|, \qquad j = 1, \ldots, n, \; i \in \mathbb{N}.$$

$$\mathcal{L}^*_{(11,12)} W(x^i, y^i) \leqslant s_1 |x^i|^2 + s_2 |x^i||y^i| - s_3(\epsilon)|y^i|^2,$$

where $\lim\limits_{\epsilon \to 0} s_3(\epsilon) = +\infty$.

4. Main result

Now we give the sufficient conditions of string stability of the full–order system.

THEOREM. *Suppose that Assumptions 1–3 hold and additionally the following conditions are satisfied:*

ASSUMPTION 4. *Functions f, q_1 are globally Lipschitz in their arguments, i.e.*

$$\begin{aligned} &|f(x^i, z^i, x^{i-1}, \ldots, x^{i-r+1}) - f(y^i, z, y^{i-1}, \ldots, y^{i-r+1})| \\ &\leqslant \beta_1 |z^i - z| + \sum_{j=1}^{r} k^f_j |x^{i-j+1} - y^{i-j+1}|, \end{aligned}$$

$$|q_1(x^i, z^i) - q_1(y^i, z)| \leqslant k^{q_1}_1 |x^i - y^i| + k^{q_1}_2 |z^i - z|.$$

Then there exists a positive constant ϵ^* such that for each $\epsilon \in (0, \epsilon^*)$ the full-order interconnected system (9), (10) *is exponentially mean–square string stable.*

PROOF. First we remark that from Assumptions 1 and 4 it follows for $j, k = 1, \ldots, n, \; i \in \mathbb{N}$

$$\begin{aligned} &|q_{1j}(x^i, y^i + h(x^i)) q_{1k}(x^i, y^i + h(x^i)) - q_{1j}(x^i, h(x^i)) q_{1k}(x^i, h(x^i))| \\ &\leqslant k^{q_1}_2 |y^i| (2k^{q_1}_1 |x^i| + 2k^{q_1}_2 M |x^i| + k^{q_1}_2 |y^i|) \end{aligned} \tag{13}$$

for all $x^i, y^i \in R^n$.

We calculate $\mathcal{L}^*(V(x^i))$ for the full-order interconnected system (9), (10)

$$
\begin{aligned}
&\mathcal{L}^*_{(9,10)}(V(x^i)) \\
&= \mathcal{L}^*_{(9,10)}(V^i) = \mathcal{L}^*_{(9)}(V^i) \\
&= \sum_{j=1}^{n} \frac{\partial V^i}{\partial x^i_j} f_j(x^i, y^i + h(x^i), x^{i-1}, \dots, x^{i-r+1}) \\
&+ \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 V^i}{\partial x^i_j \partial x^i_k} \sigma_{q_{1jk}}(x^i, y^i + h(x^i)) \\
&= \sum_{j=1}^{n} \frac{\partial V^i}{\partial x^i_j} f_j(x^i, h(x^i), x^{i-1}, \dots, x^{i-r+1}) \\
&+ \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 V^i}{\partial x^i_j \partial x^i_k} \sigma_{q_{1jk}}(x^i, h(x^i)) \\
&+ \sum_{j=1}^{n} \frac{\partial V^i}{\partial x^i_j} [f_j(x^i, y^i + h(x^i), x^{i-1}, \dots, x^{i-r+1}) \\
&- f_j(x^i, h(x^i), x^{i-1}, \dots, x^{i-r+1})] \\
&+ \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 V^i}{\partial x^i_j \partial x^i_k} [\sigma_{q_{1jk}}(x^i, y^i + h(x^i)) - \sigma_{q_{1jk}}(x^i, h(x^i))],
\end{aligned}
$$

where $\sigma_{q_{1jk}} = q_{1j} \cdot q_{1k}$.

From Assumptions 2, 4 and (13) we find

$$
\begin{aligned}
\mathcal{L}^*_{(9)}(V^i) \leqslant -\alpha_1 |x^i|^2 &+ [n\alpha_2\beta_1 + n^2\alpha_3 k_2^{q_1}(k_1^{q_1} + k_2^{q_1} M)] |x^i||y^i| \\
&+ \frac{1}{2}\alpha_3 n^2 (k_2^{q_1})^2 |y^i|^2 + \sum_{j=2}^{r} \alpha_{1j} |x^{i-j+1}|^2.
\end{aligned}
$$

Defining s'_{12} and s'_{22} by

$$
\begin{aligned}
s'_{12} &:= n\alpha_2\beta_1 + n^2\alpha_3 k_2^{q_1}(k_1^{q_1} + k_2^{q_1} M), \\
s'_{22} &:= \frac{1}{2}\alpha_3 n^2 (k_2^{q_1})^2,
\end{aligned}
$$

we obtain

$$
\mathcal{L}^*_{(9)}(V^i) \leqslant -\alpha_1 |x_i|^2 + s'_{12} |x^i||y^i| + s'_{22} |y^i|^2 + \sum_{j=2}^{r} \alpha_{1j} |x^{i-j+1}|^2. \tag{14}
$$

We calculate $\mathcal{L}^*W(x^i, y^i)$ for the full-order system (9), (10)

$$\begin{aligned}
\mathcal{L}^*_{(9,10)}W(x^i,y^i) &= \mathcal{L}^*_{(9,10)}(W^i) = \frac{1}{\epsilon}\sum_{k=1}^{m}\frac{\partial W^i}{\partial y^i_k}G_k(x^i,y^i,x^{i-1},..,x^{i-r+1}) \\
&+ \frac{1}{2\epsilon^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 W^i}{\partial y^i_k \partial y^i_l}\sigma_{Q_{21kl}}(x^i,y^i) + \frac{1}{2\epsilon^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 W^i}{\partial y^i_k \partial y^i_l}\sigma_{Q_{22kl}}(x^i,y^i) \\
&+ \sum_{k=1}^{n}\frac{\partial W^i}{\partial x^i_k}F_j(x^i,y^i,x^{i-1},..,x^{i-r+1}) + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 W^i}{\partial x^i_k \partial x^i_l}\sigma_{Q_{11kl}}(x^i,y^i) \\
&+ \frac{1}{\epsilon}\sum_{l=1}^{n}\sum_{k=1}^{m}\frac{\partial^2 W^i}{\partial x^i_l \partial y^i_k}\sigma_{Q_{11}Q21lk}(x^i,y^i) \\
&= \mathcal{L}^*_{(11,12)}(W^i) \\
&+ \frac{1}{\epsilon}\left[\sum_{k=1}^{m}\frac{\partial W^i}{\partial y^i_k}\left(G_k(x^i,y^i,x^{i-1},..,x^{i-r+1}) - G_k(x^i,y^i,0,..,0)\right)\right] \\
&+ \sum_{k=1}^{n}\frac{\partial W^i}{\partial x^i_k}\left(F_k(x^i,y^i,x^{i-1},..,x^{i-r+1}) - F_k(x^i,y^i,0,..,0)\right).
\end{aligned}$$

From Assumptions 1, 3 and 4 we find

$$\begin{aligned}
\mathcal{L}^*_{(9,10)}(W^i) \leqslant s_1|x^i|^2 + s_2|x^i||y^i| - s_3(\epsilon)|y^i|^2 + nmM\eta_3|y^i|\sum_{j=2}^{r}k^f_j|x^{i-j+1}| \\
+ n\eta_4|y^i|\sum_{j=2}^{r}k^f_j|x^{i-j+1}|.
\end{aligned}$$

Using the inequality $xy \leqslant \frac{x^2+y^2}{2}$, the above equation results in

$$\begin{aligned}
\mathcal{L}^*_{(9,10)}(W^i) \leqslant s_1|x^i|^2 + s_2|x^i||y^i| - (s_3(\epsilon) - \beta\sum_{j=2}^{r}k^f_j)|y^i|^2 \\
+ \beta\sum_{j=2}^{r}k^f_j|x^{i-j+1}|^2,
\end{aligned} \tag{15}$$

where $\beta := \frac{mnM\eta_3 + n\eta_4}{2}$.

Let us consider a function described by:

$$L^i = L(x^i,y^i) = \frac{1}{2}[V(x^i) + kW(x^i,y^i)],$$

where

$$k = \min\left\{\frac{\gamma_2}{\eta_2}, \frac{\alpha_1}{2s_1}, \frac{\alpha_1\gamma_1 - \gamma_2\sum\limits_{j=1}^{r}\alpha_{1j}}{2(s_1\gamma_1 + \gamma_2\beta\sum\limits_{j=2}^{r}k_j^f)}\right\}. \tag{16}$$

From Assumptions 2 and 3 we have

$$\frac{\gamma_1|x^i|^2 + k\eta_1|y^i|^2}{2} \leqslant L^i \leqslant \frac{\gamma_2|x^i|^2 + k\eta_2|y^i|^2}{2}. \tag{17}$$

We calculate $\mathcal{L}^*_{(9,10)}L^i = \frac{1}{2}[\mathcal{L}^*_{(9,10)}V^i + k\mathcal{L}^*_{(9,10)}W^i]$.
Taking into account (14), (15), we obtain

(18) $\mathcal{L}^*_{(9,10)}(L^i)$

$$\leqslant -\left[|x^i|^2\left(\frac{\alpha_1}{2} - \frac{k}{2}s_1\right) - \frac{s_{12}' + ks_2}{2}|x^i||y^i| + \frac{k(s_3(\epsilon) - \beta\sum\limits_{j=2}^{r}k_j^f) - s_{22}'}{2}|y^i|^2\right]$$
$$+\sum_{j=2}^{r}\frac{\alpha_{1j} + k\beta k_j^f}{2}|x^{i-j+1}|^2.$$

Finally, we have

$$\mathcal{L}^*_{(9,10)}(L^i) \leqslant -\lambda_{\min}(\epsilon)[|x^i|^2 + |y^i|^2] + \sum_{j=2}^{r}\frac{\alpha_{1j} + k\beta k_j^f}{2}|x^{i-j+1}|^2,$$

where $\lambda_{\min}$ is the minimal eigenvalue of matrix $N = [n_{ij}]$, $i, j = 1, 2$ and

$$n_{11} = \frac{\alpha_1}{2} - \frac{k}{2}s_1,$$
$$n_{12} = n_{21} = -\frac{s_{12}' + ks_2}{4},$$
$$n_{22} = \frac{k(s_3(\epsilon) - \beta\sum\limits_{j=2}^{r}k_j^f) - s_{22}'}{2}.$$

Clearly from Lemma 1 and (16), we have

$$\lim_{\epsilon\to 0}\lambda_{\min}(\epsilon) = \frac{\alpha_1}{2} - \frac{k}{2}s_1 > 0. \tag{19}$$

From (16), (17) we obtain

$$\frac{\gamma_1}{2}|x^i|^2 \leqslant L(x^i, y^i) \leqslant \frac{\gamma_2(|x^i|^2 + |y^i|^2)}{2}. \tag{20}$$

Taking into account (20), inequality (18) results in

$$\mathcal{L}_{(9,10)}(L^i) \leqslant -\frac{2}{\gamma_2}\lambda_{\min}(\epsilon)L^i + \sum_{j=2}^{r}\frac{(\alpha_{1j} + k\beta k_j^f)}{\gamma_1}\mathrm{L}^{i-j+1}.$$

We define a continuous function $H(\epsilon)$ as follows

$$H(\epsilon) = \frac{2}{\gamma_2}\lambda_{\min}(\epsilon) - \frac{1}{\gamma_1}\sum_{j=2}^{r}(\alpha_{1j} + k\beta k_j^f).$$

From (16), (19) we obtain

$$\begin{aligned}\lim_{\epsilon\to 0} H(\epsilon) &= \frac{\alpha_1 - ks_1}{\gamma_2} - \frac{1}{\gamma_1}\sum_{j=2}^{r}\alpha_{1j} - \frac{1}{\gamma_1}k\beta\sum_{j=2}^{r}k_j^f \\ &= \frac{\alpha_1}{\gamma_2} - \frac{1}{\gamma_1}\sum_{j=2}^{r}\alpha_{1j} - k\left(\frac{s_1}{\gamma_2} + \frac{\beta}{\gamma_1}\sum_{j=2}^{r}k_j^f\right) > 0.\end{aligned}$$

There exists ϵ^* such that for all $\epsilon \in (0, \epsilon^*)$, $H(\epsilon) > 0$. From Lemma 2 we obtain that the interconnection of singularly perturbed stochastic system is mean-square string stable. Using similar arguments as in [7] one can show that $E[|L^i(t)|] \to 0$ exponentially.

EXAMPLE. We consider the following two-dimensional system:

$$dx^i = (-a_1x^i + a_2z^i + \sum_{j=1}^{r-1}c_jx^{i-j})dt + (a_3x^i + a_4z^i)d\omega^1, \tag{21}$$

$$\epsilon dz^i = (b_1x^i - b_2z^i)dt + \sqrt{\epsilon}(b_3x^i + b_4z^i)d\omega^2, \tag{22}$$

where a_i, b_i $(i = 1, 2, 3, 4)$ are constant parameters and ϵ is a perturbation parameter. We assume $b_1b_4 = -b_2b_3$. Repeating consideration given in Section 3, we obtain:

$$h(x^i) = \frac{b_1}{b_2}x^i, \qquad y^i = z^i - \frac{b_1}{b_2}x^i$$

and (21), (22) after transformation have the form:

(23) $$dx^i = (-A_1x^i + a_2y^i + \sum_{j=1}^{r-1} c_jx^{i-j})dt + (A_3x^i + a_4y^i)d\omega^1,$$

(24) $$\epsilon dy^i = \epsilon(B_1x^i - B_2y^i + \sum_{j=1}^{r-1} d_jx^{i-j})dt + \epsilon(B_3x^i + B_4y^i)d\omega^1 + \sqrt{\epsilon}b_4y^i d\omega^2,$$

where

$$A_1 = a_1 - \frac{b_1}{b_2}, \qquad A_3 = a_3 + \frac{b_1}{b_2},$$

$$B_1 = \frac{b_1a_1}{b_2} - \frac{b_1^2a_2}{b_2^2}, \qquad B_2 = B_2(\epsilon) = \frac{b_2}{\epsilon} + \frac{b_1a_2}{b_2},$$

$$B_3 = -\frac{b_1}{b_2}(a_3 + \frac{b_1}{b_2}), \qquad B_4 = -\frac{b_1a_4}{b_2}, \qquad d_j = -\frac{b_1}{b_2}c_j,\ j = 1,..,r-1.$$

The complete reduced-order system is:

$$dx^i = (-A_1x^i + \sum_{j=1}^{r-1} c_jx^{i-j})dt + A_3x^i d\omega^1.$$

The isolated subsystems are described by:

$$dx^i = (-A_1x^i + a_2y^i)dt + (A_3x^i + a_4y^i)d\omega^1,$$

$$\epsilon dy^i = \epsilon(B_1x^i - B_2y^i)dt + \epsilon(B_3x^i + B_4y^i)d\omega^1 + \sqrt{\epsilon}b_4y^i d\omega^2.$$

We propose the Lyapunov functions $V(x^i)$, $W(x^i, y^i)$ in the form:

$$V(x^i) = (x^i)^2, \quad W(x^i, y^i) = (y^i)^2.$$

Then

$$\mathcal{L}^*_{(23,24)}V(x^i) \leqslant -(2A_1 - A_3^2 - \sum_{j=1}^{r-1} c_j)(x^i)^2 + \sum_{j=1}^{r-1} c_j(x^{i-j})^2$$

and

$$\mathcal{L}^*_{(23,24)}W(x^i, y^i) \leqslant B_3^2(x^i)^2 + 2(B_1 + B_3B_4)|x^i||y^i| - \left(2B_2(\epsilon) - B_4^2 - \frac{b_4}{\epsilon}\right)(|y^i|)^2.$$

Then, the Assumptions 2 and 3 are satisfied if:

$$A_1 - \frac{1}{2}A_3^2 > \sum_{j=1}^{r-1} c_j \qquad \text{and} \quad b_2 > \frac{b_4^2}{2}. \tag{25}$$

Then from the theorem it follows that the full-order interconnected system (23), (24) is exponentially mean-square string stable for sufficiently small ϵ if the conditions (25) are satisfied.

5. Conclusion and final remarks

In this paper the problem of string stability of singularly perturbed, nonlinear stochastic systems has been studied. The sufficient conditions of exponential string stability for a class of interconnected stochastic systems and their robustness to small singular perturbation were presented. It is also possible to derive similarly stability criteria for the following system

$$dx^i = f(x^i, z^i, x^{i-1}, \ldots, x^{i-r+1})dt + q_1(x^i, z^i, x^{i-1}, \ldots, x^{i-r+1})d\omega^1,$$

$$x^i(0) = x^{i0}, \qquad \epsilon dz^i = g(x^i, z^i)dt + \sqrt{\epsilon} q_2(x^i, z^i)d\omega^2, \qquad z^i(0) = z^{i0}.$$

The further extensions can be done for the string systems as well with Gaussian excitations as with wideband noises (described by Stratonovich equations).

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