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ON APPROXIMATION OF APPROXIMATELY QUADRATIC MAPPINGS BY QUADRATIC MAPPINGS

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Abstract. In this paper we establish an approximation of approximately quadratic mappings by quadratic mappings, which solves the pertinent Ulam stability problem.

Introduction

In 1940 S. M. Ulam [34] proposed before the Mathematics Club of the University of Wisconsin a number of interesting open problems, one of which is the following problem: Give conditions in order for a linear mapping near an approximately linear mapping to exist. In 1968 S. M. Ulam [34] proposed the general problem: When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true. In 1978 P. M. Gruber [7] proposed the *Ulam type problem*: Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this object by objects, satisfying the property exactly? According to P. M. Gruber [7] this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982–2000 we ([15]–[28]) solved the above-mentioned Ulam problem, or equivalently the Ulam type problem for linear mappings as well as for quadratic, cubic and quartic mappings and established analogous stability problems. In this paper we introduce the

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following quadratic functional equation

$$(*) \qquad Q(a_1x_1+a_2x_2)+Q(a_2x_1-a_1x_2)=(a_1^2+a_2^2)[Q(x_1)+Q(x_2)]$$

with quadratic mappings $Q: X \to Y$ satisfying condition $Q(0)=0$ if $m = a_1^2+a_2^2 > 0$ such that X and Y are real linear spaces, and then establish an approximation of approximately quadratic mappings $f: X \to Y$, with $f(0) = 0$ (if $m = 1$), by quadratic mappings $Q: X \to Y$, such that the corresponding approximately quadratic functional inequality
(**)
$$\|f(a_1x_1+a_2x_2)+f(a_2x_1-a_1x_2)-(a_1^2+a_2^2)[f(x_1)+f(x_2)]\| \le c\|x_1\|^{r_1}\|x_2\|^{r_2}$$

holds with a constant $c \ge 0$ (independent of $x_1, x_2 \in X$), and any fixed pair (a_1, a_2) of reals $a_i \neq 0$ $(i = 1, 2)$ and (r_1, r_2) of reals $r_i \neq 0$ $(i = 1, 2)$:

$$I_1 = \{(r,m) \in \mathbb{R}^2 : r < 2, m > 1 \text{ or } r > 2, 0 < m < 1\},$$
$$I_2 = \{(r,m) \in \mathbb{R}^2 : r < 2, 0 < m < 1 \text{ or } r > 2, m > 1\},$$

or

$$I_3 = \{(r,m) \in \mathbb{R}^2 : r < 2, m = 1 = 2a^2 : a_1 = a_2 = a = 2^{-\frac{1}{2}}\}$$

hold, where $m = a_1^2 + a_2^2 > 0$ and $r = r_1 + r_2 \neq 0$. However, we have established the following case: $r_i = 0$ $(i = 1, 2)$ such that $r = 0$ [23].

Note that $m^{r-2} < 1$ if $(r,m) \in I_1$, $m^{2-r} < 1$ if $(r,m) \in I_2$, and $2^{r-2} < 1$ if $(r, m = 1) \in I_3$.

It is useful for the following, to observe that, from (*) with $x_1 = x_2 = 0$, and $0 < m \neq 1$ we get

$$2(m-1)Q(0) = 0,$$

or

$$(1) \qquad Q(0) = 0.$$

DEFINITION 1.1. Let X and Y be real linear spaces. Then a mapping $Q: X \to Y$ is called *quadratic*, if (*) holds for every vector $(x_1, x_2) \in X^2$.

For every $x \in \mathbb{R}$ set $Q(x) = x^2$. Then the mapping $Q: \mathbb{R} \to \mathbb{R}$ is quadratic. Finally let $F: X^2 \to Y$ be a bilinear mapping. Set $Q(x) = F(x,x)$ for every $x \in X$. Then $Q: X \to Y$ is quadratic.

Denote

(2)
$$\overline{Q}(x) = \begin{cases} \frac{Q(a_1x)+Q(a_2x)}{a_1^2+a_2^2}, & \text{if } (r, m = a_1^2+a_2^2) \in I_1 \\ (a_1^2+a_2^2)\left[Q\left(\frac{a_1}{a_1^2+a_2^2}x\right)+Q\left(\frac{a_2}{a_1^2+a_2^2}x\right)\right], & \text{if } (r, m = a_1^2+a_2^2) \in I_2 \end{cases}$$

for all $x \in X$.

Now, claim that for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$

$$Q(x) = \begin{cases} m^{-2n} Q(m^n x), & \text{if } (r, m) \in I_1, \\ m^{2n} Q(m^{-n} x), & \text{if } (r, m) \in I_2, \\ 2^{-n} Q(2^{n/2} x), & \text{if } (r, m = 1) \in I_3 \end{cases} \tag{3}$$

for all $x \in X$ and $n \in \mathbb{N}$.

For $n = 0$, it is trivial. From (1), (2) and (*), with $x_i = a_i x$ $(i = 1, 2)$, we obtain

$$Q(mx) = m[Q(a_1 x) + Q(a_2 x)],$$

or

$$\overline{Q}(x) = m^{-2} Q(mx), \tag{4}$$

if I_1 holds. Besides from (1), (2) and (*), with $x_1 = x, x_2 = 0$, we get

$$Q(a_1 x) + Q(a_2 x) = mQ(x),$$

or

$$\overline{Q}(x) = Q(x), \tag{5}$$

if I_1 holds. Therefore from (4) and (5) we have

$$Q(x) = m^{-2} Q(mx), \tag{6}$$

which is (3) for $n = 1$, if I_1 holds. Similarly, from (1), (2) and (*), with $x_i = \frac{a_i}{m} x$ $(i = 1, 2)$, we obtain

$$Q(x) = \overline{Q}(x) \tag{7}$$

if I_2 holds. Besides from (1), (2) and (*), with $x_1 = \frac{x}{m}, x_2 = 0$, we get

$$Q\left(\frac{a_1}{m} x\right) + Q\left(\frac{a_2}{m} x\right) = mQ(m^{-1} x),$$

or

$$\overline{Q}(x) = m^2 Q(m^{-1} x) \tag{8}$$

if I_2 holds. Therefore from (7) and (8) we have

$$Q(x) = m^2 Q(m^{-1} x), \tag{9}$$

which is (3) for $n = 1$, if I_2 holds. Also, with $x_1 = x_2 = x$ in (*) and $a_1 = a_2 = a = 2^{-1/2}$, we obtain

$$Q(2^{1/2}x) = 2Q(x),$$

or

$$Q(x) = 2^{-1}Q(2^{1/2}x), \tag{10}$$

which is (3) for $n = 1$, if I_3 holds.

Assume (3) is true and from (6), with $m^n x$ on place of x, we get:

$$Q(m^{n+1}x) = m^2 Q(m^n x) = m^2(m^n)^2 Q(x) = (m^{n+1})^2 Q(x). \tag{11}$$

Similarly, with $m^{-n}x$ on place of x, we get:

$$\begin{aligned} Q\left(m^{-(n+1)}x\right) &= m^{-2}Q(m^{-n}x) = m^{-2}(m^{-n})^2 Q(x) \\ &= \left(m^{-(n+1)}\right)^2 Q(x). \end{aligned} \tag{12}$$

Also, with $(2a)^n x$ $(= 2^{n/2}x)$ on place of x, we get:

$$\begin{aligned} Q\left(2^{\frac{n+1}{2}}x\right) &= Q\left((2a)^{n+1}x\right) = 2^1 Q\left((2a)^n x\right) \\ &= 2^1(2^n)Q(x) = 2^{n+1}Q(x) = \left(2^{\frac{n+1}{2}}\right)^2 Q(x). \end{aligned} \tag{13}$$

These formulas (11), (12) and (13) by induction, prove formula (3), ([1]–[6], [8]–[14], [29]–[33]).

Quadratic functional stability

THEOREM 2.1. *Let X and Y be normed linear spaces. Assume that Y is complete. Assume in addition that $f : X \to Y$ satisfies functional inequality (**), such that $f(0) = 0$ (if $m > 0$).*

Define

$$f_n(x) = \begin{cases} m^{-2n} f(m^n x), & \text{if } (r, m) \in I_1 \\ m^{2n} f(m^{-n} x), & \text{if } (r, m) \in I_2 \\ 2^{-n} f(2^{n/2} x), & \text{if } (r, m = 1) \in I_3 \end{cases}$$

for all $x \in X$ and $n \in \mathbb{N}$.

Then the limit

(14) $$Q(x) = \lim_{n\to\infty} f_n(x)$$

exists for all $x \in X$ and $Q : X \to Y$ is the unique quadratic mapping, such that $Q(0) = 0$ (if $m > 1$) and

(15) $$\|f(x) - Q(x)\| \le \|x\|^r \begin{cases} \gamma c/(m^2 - m^r), & \text{if } (r, m) \in I_1 \\ \gamma c/(m^r - m^2), & \text{if } (r, m) \in I_2 \\ c/(2 - 2^{r/2}), & \text{if } (r, m = 1) \in I_3 \end{cases}$$

holds for all $x \in X, c \ge 0$ (constant independent of $x \in X$) and $\gamma = |a_1|^{r_1}|a_2|^{r_2} > 0$.

Existence

Proof. It is useful for the following, to observe that, from (**) with $x_1 = x_2 = 0$ and $0 < m \neq 1$, we get

$$2|m - 1|\|f(0)\| \le 0,$$

or

(16) $$f(0) = 0.$$

Now claim that for $n \in \mathbb{N}$

(17)
$$\|f(x) - f_n(x)\| \le \|x\|^r \begin{cases} \frac{\gamma c}{m^2 - m^r}\left(1 - m^{n(r-2)}\right), & \text{if } (r, m) \in I_1 : m^{r-2} < 1 \\ \frac{\gamma c}{m^r - m^2}\left(1 - m^{n(2-r)}\right), & \text{if } (r, m) \in I_2 : m^{2-r} < 1 \\ \frac{c}{2 - 2^{r/2}}\left(1 - 2^{n(r-2)/2}\right), & \text{if } (r, m = 1) \in I_3 : 2^{r-2} < 1. \end{cases}$$

For $n = 0$, it is trivial.

Denote

(18)
$$\overline{f}(x) = \begin{cases} \frac{f(a_1x) + f(a_2x)}{a_1^2 + a_2^2}, & \text{if } (r, m = a_1^2 + a_2^2) \in I_1 \\ (a_1^2 + a_2^2)\left[f\left(\frac{a_1}{a_1^2 + a_2^2}x\right) + f\left(\frac{a_2}{a_1^2 + a_2^2}x\right)\right], & \text{if } (r, m = a_1^2 + a_2^2) \in I_2 \end{cases}$$

for all $x \in X$.

From (16), (18) and (**), with $x_i = a_i x$ $(i = 1, 2)$, we obtain

$$\|f(mx) - m[f(a_1x) + f(a_2x)]\| \le \gamma c\|x\|^r,$$

or

$$\|m^{-2}f(mx) - \overline{f}(x)\| \le \frac{\gamma c}{m^2}\|x\|^r, \tag{19}$$

if I_1 holds. Besides from (16), (18) and (**), with $x_1 = x, x_2 = 0$, we get

$$\|f(a_1x) + f(a_2x) - mf(x)\| \le 0,$$

or

$$\overline{f}(x) = f(x), \tag{20}$$

if I_1 holds. Therefore from (19) and (20) we have

$$\|f(x) - m^{-2}f(mx)\| \le \frac{\gamma c}{m^2}\|x\|^r = \frac{\gamma c}{m^2 - m^r}(1 - m^{r-2})\|x\|^r, \tag{21}$$

which is (17) for $n = 1$, if I_1 holds.

Similarly, from (16), (18) and (**), with $x_i = \frac{a_i}{m}x$ $(i = 1, 2)$, we obtain

$$\|f(x) - \overline{f}(x)\| \le \frac{\gamma c}{m^r}\|x\|^r, \tag{22}$$

if I_2 holds. Besides from (16), (18) and (**), with $x_1 = \frac{x}{m}, x_2 = 0$, we get

$$\left\|f\left(\frac{a_1}{m}x\right) + f\left(\frac{a_2}{m}x\right) - mf(m^{-1}x)\right\| \le 0,$$

or

$$\overline{f}(x) = m^2 f(m^{-1}x), \tag{23}$$

if I_2 holds. Therefore from (22) and (23) we have

$$\left\|f(x) - m^2 f(m^{-1}x)\right\| \le \frac{\gamma c}{m^r}\|x\|^r = \frac{\gamma c}{m^r - m^2}(1 - m^{2-r})\|x\|^r, \tag{24}$$

which is (17) for $n = 1$, if I_2 holds.

Also, with $x_1 = x_2 = x$ in (**) and $a_1 = a_2 = a = 2^{-1/2}$, we obtain

$$\left\|f(2ax) - 2f(x)\right\| \le c\|x\|^r,$$

or

$$\begin{aligned} \left\|f(x) - 2^{-1}f(2^{1/2}x)\right\| &= \left\|f(x) - 2^{-1}f((2a)^1x)\right\| \\ &\le \frac{c}{2}\|x\|^r = \frac{c}{2 - 2^{r/2}}[1 - 2^{(r-2)/2}]\|x\|^r, \end{aligned} \tag{25}$$

which is (17) for $n = 1$, if I_3 holds.

Assume (17) is true if $(r, m) \in I_1$. From (21), with $m^n x$ on place of x, and the triangle inequality, we have

(26)
$$\begin{aligned}\|f(x) - f_{n+1}(x)\| &= \|f(x) - m^{-2(n+1)} f(m^{n+1}x)\| \\ &\le \|f(x) - m^{-2n} f(m^n x)\| + \|m^{-2n} f(m^n x) - m^{-2(n+1)} f(m^{n+1}x)\| \\ &\le \frac{\gamma c}{m^2 - m^r}\left[(1 - m^{n(r-2)}) + m^{-2n}(1 - m^{r-2})m^{nr}\right]\|x\|^r \\ &= \frac{\gamma c}{m^2 - m^r}\left(1 - m^{(n+1)(r-2)}\right)\|x\|^r,\end{aligned}$$

if I_1 holds.

Similarly assume (17) is true if $(r, m) \in I_2$. From (24), with $m^{-n}x$ on place of x, and the triangle inequality, we have

(27)
$$\begin{aligned}\|f(x) - f_{n+1}(x)\| &= \|f(x) - m^{2(n+1)} f(m^{-(n+1)}x)\| \\ &\le \|f(x) - m^{2n} f(m^{-n}x)\| + \|m^{2n} f(m^{-n}x) - m^{2(n+1)} f(m^{-(n+1)}x)\| \\ &\le \frac{\gamma c}{m^r - m^2}\left[(1 - m^{n(2-r)} + m^{2n}(1 - m^{2-r})m^{-nr}\right]\|x\|^r \\ &= \frac{\gamma c}{m^r - m^2}(1 - m^{(n+1)(2-r)})\|x\|^r,\end{aligned}$$

if I_2 holds.

Also, assume (17) is true if $(r, m = 1) \in I_3$. From (25), with $(2a)^n x$ $(= 2^{n/2}x)$ on place of x, and the triangle inequality, we have

(28)
$$\begin{aligned}\|f(x) - f_{n+1}(x)\| &= \left\|f(x) - 2^{-(n+1)} f\left(2^{\frac{n+1}{2}} x\right)\right\| \\ &= \|f(x) - 2^{-(n+1)} f((2a)^{n+1}x)\| \\ &\le \|f(x) - 2^{-n} f((2a)^n x)\| + \|2^{-n} f((2a)^n x) - 2^{-(n+1)} f((2a)^{n+1}x)\| \\ &\le \frac{c}{2 - 2^{r/2}}\left\{\left[1 - 2^{n(r-2)/2}\right] + 2^{-n}\left[1 - 2^{(r-2)/2}\right](2a)^{nr}\right\}\|x\|^r \\ &= \frac{c}{2 - 2^{r/2}}[1 - 2^{(n+1)(r-2)/2}]\|x\|^r,\end{aligned}$$

if I_3 holds.

Therefore inequalities (26), (27) and (28) prove inequality (17) for any $n \in \mathbb{N}$.

Claim now that the sequence $\{f_n(x)\}$ converges.

To do this it suffices to prove that it is a Cauchy sequence. Inequality

(17) is involved if $(r,m) \in I_1$. In fact, if $i > j > 0$, and $h_1 = m^j x$, we have:

$$\begin{aligned} \|f_i(x) - f_j(x)\| &= \|m^{-2i} f(m^i x) - m^{-2j} f(m^j x)\| \\ &= m^{-2j} \|m^{-2(i-j)} f(m^{i-j} h_1) - f(h_1)\| = \\ &= m^{-2j} \|f_{i-j}(h_1) - f(h_1)\| \leq \\ &= m^{-2j} \|h_1\|^r \frac{\gamma c}{m^2 - m^r} (1 - m^{(i-j)(r-2)}) \\ &= m^{(r-2)j} \frac{\gamma c}{m^2 - m^r} (1 - m^{(i-j)(r-2)}) \|x\|^r \\ &< \frac{\gamma c}{m^2 - m^r} m^{(r-2)j} \|x\|^r \underset{j \to \infty}{\longrightarrow} 0, \end{aligned} \tag{29}$$

if I_1 holds: $m^{r-2} < 1$.

Similarly, if $h_2 = m^{-j} x$ in I_2, we have:

$$\begin{aligned} \|f_i(x) - f_j(x)\| &= \|m^{2i} f(m^{-i} x) - m^{2j} f(m^{-j} x)\| \\ &= m^{2j} \|m^{2(i-j)} f(m^{-(i-j)} h_2) - f(h_2)\| \\ &\leq m^{(2-r)j} \frac{\gamma c}{m^r - m^2} (1 - m^{(i-j)(2-r)}) \|x\|^r \\ &< \frac{\gamma c}{m^r - m^2} m^{(2-r)j} \|x\|^r \underset{j \to \infty}{\longrightarrow} 0. \end{aligned} \tag{30}$$

if I_2 holds: $m^{2-r} < 1$.

Also, if $h_3 = 2^{j/2} x$ in I_3, we have:

(31)

$$\begin{aligned} \|f_i(x) - f_j(x)\| &= \|2^{-i} f(2^{i/2} x) - 2^{-j} f(2^{j/2} x)\| \\ &= 2^{-j} \|2^{-(i-j)} f(2^{(i-j)/2} h_3) - f(h_3)\| \\ &= 2^{-j} \|f_{i-j}(h_3) - f(h_3)\| \leq 2^{-j} \|h_3\|^r \frac{c}{2 - 2^{r/2}} (1 - 2^{(i-j)(r-2)/2}) \\ &= 2^{-j/2} \frac{c}{2 - 2^{r/2}} (1 - 2^{(i-j)(r-2)/2}) \|x\|^r < \frac{c}{2 - 2^{r/2}} 2^{-j/2} \|x\|^r \underset{j \to \infty}{\longrightarrow} 0, \end{aligned}$$

if I_3 holds: $2^{r-2} < 1$.

Then inequalities (29), (30) and (31) define a mapping $Q : X \to Y$, given by (14).

Claim that from (**) and (14) we can get (*), or equivalently that the afore-mentioned well-defined mapping $Q : X \to Y$ is *quadratic*.

In fact, it is clear from the functional inequality (**) and the limit (14) for $(r,m) \in I_1$ that the following functional inequality

$$\begin{aligned} m^{-2n} \|f(a_1 m^n x_1 + a_2 m^n x_2) + f(a_2 m^n x_1 - a_1 m^n x_2) \\ - (a_1^2 + a_2^2)[f(m^n x_1) + f(m^n x_2)]\| \\ \leq m^{-2n} c \|m^n x_1\|^{r_1} \|m^n x_2\|^{r_2}, \end{aligned}$$

holds for all vectors $(x_1, x_2) \in X^2$, and all $n \in \mathbb{N}$ with $f_n(x) = m^{-2n} f(m^n x)$: I_1 holds. Therefore

$$\left\| \lim_{n\to\infty} f_n(a_1x_1 + a_2x_2) + \lim_{n\to\infty} f_n(a_2x_1 - a_1x_2) - (a_1^2 + a_2^2)\left[\lim_{n\to\infty} f_n(x_1) + \lim_{n\to\infty} f_n(x_2)\right]\right\| \leq \left(\lim_{n\to\infty} m^{n(r-2)}\right) c\|x_1\|^{r_1}\|x_2\|^{r_2} = 0,$$

because $m^{r-2} < 1$ or

(32) $\|Q(a_1x_1 + a_2x_2) + Q(a_2x_1 - a_1x_2) - (a_1^2 + a_2^2)[Q(x_1) + Q(x_2)]\| = 0,$

or mapping Q satisfies the quadratic equation (*).

Similarly, from (**) and (14) for $(r, m) \in I_2$ we get that

$$m^{2n} \|f(a_1m^{-n}x_1 + a_2m^{-n}x_2) + f(a_2m^{-n}x_1 - a_1m^{-n}x_2) - (a_1^2 + a_2^2)\left[f(m^{-n}x_1) + f(m^{-n}x_2)\right]\| \leq m^{2n}c\|m^{-n}x_1\|^{r_1}\|m^{-n}x_2\|^{r_2},$$

holds for all vectors $(x_1, x_2) \in X^2$, and all $n \in \mathbb{N}$ with $f_n(x) = m^{2n} f(m^{-n}x)$: I_2 holds. Thus

$$\left\| \lim_{n\to\infty} f_n(a_1x_1 + a_2x_2) + \lim_{n\to\infty} f_n(a_2x_1 - a_1x_2) - (a_1^2 + a_2^2)\left[\lim_{n\to\infty} f_n(x_1) + \lim_{n\to\infty} f_n(x_2)\right]\right\| \leq \left(\lim_{n\to\infty} m^{n(2-r)}\right) c\|x_1\|^{r_1}\|x_2\|^{r_2} = 0,$$

because $m^{2-r} < 1$, or (32) holds or mapping Q satisfies (*).

Also, from (**) and (14) for $(r, m = 1) \in I_3$ we obtain that

$$2^{-n} \left\|f(a_12^{n/2}x_1 + a_22^{n/2}x_2) + f(a_22^{n/2}x_1 - a_12^{n/2}x_2) - (a_1^2 + a_2^2)\left[f(2^{n/2}x_1) + f(2^{n/2}x_2)\right]\right\| \leq 2^{-n}c\|2^{n/2}x_1\|^{r_1}\|2^{n/2}x_2\|^{r_2},$$

holds for all vectors $(x_1, x_2) \in X^2$, and all $n \in \mathbb{N}$ with $f_n(x) = 2^{-n} f(2^{n/2}x)$: I_3 holds. Hence

$$\left\| \lim_{n\to\infty} f_n(a_1x_1 + a_2x_2) + \lim_{n\to\infty} f_n(a_2x_1 - a_1x_2) - (a_1^2 + a_2^2)\left[\lim_{n\to\infty} f_n(x_1) + \lim_{n\to\infty} f_n(x_2)\right]\right\| \leq \left(\lim_{n\to\infty} 2^{n(r-2)/2}\right) c\|x_1\|^{r_1}\|x_2\|^{r_2} = 0,$$

because $2^{r-2} < 1$, or (32) holds or mapping Q satisfies $(*)$.

Therefore (32) holds if I_j $(j = 1, 2, 3)$ hold or mapping Q satisfies $(*)$, completing the proof that Q is a quadratic mapping in X.

It is now clear from (17) with $n \to \infty$, as well as formula (14) that inequality (15) holds in X. This completes the existence proof of the above theorem 2.1.

Uniqueness

Let $Q' : X \to Y$ be a quadratic mapping satisfying (15), as well as Q. Then $Q' = Q$.

PROOF. Remember both Q and Q' satisfy (3) for $(r, m) \in I_1$, too. Then for every $x \in X$ and $n \in \mathbb{N}$,

(33)

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \|m^{-2n}Q(m^n x) - m^{-2n}Q'(m^n x)\| \\ &\leq m^{-2n}\{\|Q(m^n x) - f(m^n x)\| + \|Q'(m^n x) - f(m^n x)\|\} \\ &\leq m^{-2n}\frac{2\gamma c}{m^2 - m^r}\|m^n x\|^r = m^{n(r-2)}\frac{2\gamma c}{m^2 - m^r}\|x\|^r \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

if I_1 holds: $m^{r-2} < 1$.

Similarly for $(r, m) \in I_2$, we establish

(34)

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \|m^{2n}Q(m^{-n}x) - m^{2n}Q'(m^{-n}x)\| \\ &\leq m^{2n}\{\|Q(m^{-n}x) - f(m^{-n}x)\| + \|Q'(m^{-n}x) - f(m^{-n}x)\|\} \\ &\leq m^{2n}\frac{2\gamma c}{m^r - m^2}\|m^{-n}x\|^r = m^{n(2-r)}\frac{2\gamma c}{m^r - m^2}\|x\|^r \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

if I_2 holds: $m^{2-r} < 1$.

Also for $(r, m = 1) \in I_3$, we get

(35)

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \|2^{-n}Q(2^{n/2}x) - 2^{-n}Q'(2^{n/2}x)\| \\ &\leq 2^{-n}\left\{\|Q(2^{n/2}x) - f(2^{n/2}x)\| + \|Q'(2^{n/2}x) - f(2^{n/2}x)\|\right\} \\ &\leq 2^{-n}\frac{2c}{2 - 2^{r/2}}\|2^{n/2}x\|^r = 2^{n(r-2)/2}\frac{2c}{2 - 2^{r/2}}\|x\|^r \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

if I_3 holds: $2^{r-2} < 1$. Thus from (33), (34) and (35) we find $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of uniqueness and the stability of equation $(*)$.

Query. What is the situation in the above theorem 2.1 either in case $r = 2$ or for $m = 1$ when $a_1 \neq a_2$?

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