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Report of Meeting

The First Katowice–Debrecen Winter Seminar on Functional Equations, February 7–10, 2001, Cieszyn, Poland

The First Katowice–Debrecen Winter Seminar on Functional Equations was held in Cieszyn, Poland from February 7 through February 10, 2001, at the Cieszyn Branch of the Silesian University.

20 participants came from the University of Debrecen (Hungary) and the Silesian University of Katowice (Poland) at 10 from each of the 2 cities.

Professor Roman Ger opened the Seminar and welcomed the participants to Cieszyn. He used this occasion to present briefly the idea of annual meetings of functional equationists representing the Debrecen and Katowice schools.

The scientific talks presented at the Seminar focused on the following topics: equations in a single and several variables, iteration theory and the theory of chaos, equations on algebraic structures, conditional equations, Hyers-Ulam stability, functional inequalities and mean values. Interesting discussions were generated by the talks.

There were two longer and very profitable Problem Sessions.

The social program included two festive dinners. There was also a well-received concert performed by the artists employed at the Cieszyn Branch of the Silesian University.

The clossing address was given by Professor Zsolt Páles. His invitation to hold the Second Debrecen–Katowice Winter Seminar on Functional Equations in February 2002 in Hungary was gratefully accepted.

Summaries of the talks in alphabetic order of the authors follow in section 1, problems and remarks in section 2, and the list of participants in the final section.

1. Abstracts of talks

BARON, KAROL: *On the existence of solutions of linear iterative equations in a class of distribution functions*

The equation

$$F(x) = \sum_{n=1}^{N} p_n F(\tau_n(x))$$

was considered and results on the existence of its solutions F in distribution function classes were presented.

BESSENYEI, MIHÁLY: *Hadamard-type inequalities*
(Joint work with Zsolt Páles)

The classical Hadamard-inequality provides the following lower and upper estimation for a convex function $f\colon [a,b] \to R$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Our goal is to generalize this inequality when $f\colon [a,b] \to R$ is supposed to be *n-monotone*, that is, for $a \leq x_0 < \ldots < x_n \leq b$:

$$(-1)^n \begin{vmatrix} f(x_o) & \ldots & f(x_n) \\ 1 & \ldots & 1 \\ x_0 & \ldots & x_n \\ \vdots & & \vdots \\ x_0^{n-1} & \ldots & x_n^{n-1} \end{vmatrix} \geq 0.$$

For smooth enough function Hadamard-type inequalities are proved by using orthogonal polynomial systems and mean-value theorems. For the general case, a smoothing technique is developed and applied.

For instance, for a 3-monotone function $f\colon [a,b] \to R$, one can deduce that

$$\frac{f(a)+3f\left(\frac{a+2b}{3}\right)}{4} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(b)+3f\left(\frac{2a+b}{3}\right)}{4}.$$

BOROS, ZOLTÁN: *Decomposition of strongly* $\mathbb{Q}$*-differentiable functions*

A real function is called strongly $\mathbb{Q}$-differentiable if, for every real number h, the limit of the ratio $(f(x+rh)-f(x))/r$ exists whenever x tends to any fixed real number and r tends to zero through the positive rationals. After examining the dependence of strong $\mathbb{Q}$-derivatives on their parameters, we prove that every strongly $\mathbb{Q}$-differentiable function can be represented as the sum of an additive mapping and a continuously differentiable function.

DARÓCZY, ZOLTÁN: *A functional equation on complementary means* (Joint work with Che Tat Ng)

We say that a function $M : [a,b]^2 \to [a,b]$ ($a < b; a, b \in \mathbb{R}$) is a *mean* on $[a,b]$ if it satisfies the following conditions

$$(M1) \qquad \min\{x,y\} \le M(x,y) \le \max\{x,y\} \quad \text{for all} \quad x, y \in [a,b],\ x \ne y;$$

$$(M2) \qquad M \quad \text{is continuous on} \quad [a,b]^2.$$

If M is a mean on $[a,b]$, then $M(x,x) = x$ for all $x \in [a,b]$, and the function defined by $\hat{M}(x,y) := x + y - M(x,y)$ $(x, y \in [a,b])$ is also a mean on [a,b]. The pair M and $\hat{M}$ satisfy $A(M, \hat{M}) = A$, where $A(x,y) := (x+y)/2$ is the *arithmetic* mean. In this sense, $\hat{M}$ is complementary to M with respect to the arithmetic mean.

Let M be a mean on $[a,b]$. A function $f : [a,b] \to \mathbb{R}$ is called *M-associate* if it possesses the following property

(MA) If $x, y \in [a,b]$ satisfy $M(x,y) = (x+y)/2$ and $f(x) = f((x+y)/2)$ then $f(y) = f(x)$.

We consider the functional equation

$$f(M(x,y)) = f(\hat{M}(x,y)) \qquad (x, y \in [a,b])$$

with f being M-associate and continuous.

GER, ROMAN: *Fischer-Muszély additivity of mappings between normed spaces*

Generalizing numerous earlier results (see e.g. P. Fischer & Gy Muszély [3], J. Dhombres [2], R. Ger [4], G. Berruti & F. Skof [1], P. Schöpf [6], and R. Ger & B. Koclęga [5]) we have obtained, among others, the following two theorems.

THEOREM 1. *Let $(X,+)$ be an Abelian group and let $(Y, \|\cdot\|_Y)$ be a real normed linear space. Let further $f : X \longrightarrow Y$ be a solution to the functional equation*

$$(1) \qquad \|f(x+y)\|_Y = \|f(x) + f(y)\|_Y, \quad x, y \in X.$$

Then there exists a nonempty set $T \subset \mathbb{R}^X$, an additive operator $A : X \longrightarrow B(T, \mathbb{R})$ (the Banach space of all bounded functions on T, equiped with the uniform convergence norm) and an odd isometry $I : A(X) \longrightarrow Y$ such that

$$f(x) = I(A(x)), \quad x \in X.$$

Conversely, for an arbitrary real normed linear space $(Z, \|\cdot\|_Z)$, any additive operator $A : X \longrightarrow Z$ and any odd isometry $I : A(X) \longrightarrow Y$ the superposition $f := I \circ A$ yields a solution of equation (1).

THEOREM 2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two real normed linear spaces. Let further $f : X \longrightarrow Y$ be a solution to the functional equation* (1) *such that the function $\varphi : X \longrightarrow \mathbb{R}$ defined by the formula*

$$\varphi(x) := \|f(x)\|_Y, \quad x \in X,$$

satisfies any regularity condition that forces a Jensen convex functional to be continuous. Then there exists a nonempty set $T \subset \mathbb{R}^X$, a continuous linear operator $L : X \longrightarrow B(T, \mathbb{R})$ and an odd isometry $I : L(X) \longrightarrow Y$ such that

$$f(x) = I(L(x)), \quad x \in X.$$

Conversely, for an arbitrary real normed linear space $(Z, \|\cdot\|_Z)$, any continuous linear operator $L : X \longrightarrow Z$ and any odd isometry $I : L(X) \longrightarrow Y$ the superposition $f := I \circ L$ yields a solution of equation (1) *and the corresponding function φ is continuous.*

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GILÁNYI, ATTILA: *On the Dinghas derivative and convex functions of higher order*

In this talk Jensen convex functions of higher order are characterized with the help of the lower Dinghas interval derivative.

HÁZY, ATTILA: *On approximately convex functions*
(Joint work with Zsolt Páles)

A function $f : D \to \mathbb{R}$ is called (ε, δ)–midconvex if

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x)+f(y))+\delta+\varepsilon \mid x-y \mid$$

for all $x, y \in D$.

Our main result shows that if f is locally bounded from above and (ε, δ) –midconvex, then f satisfies the following convexity inequality

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)+2\delta+\varepsilon\varphi(\lambda) \mid x-y \mid$$

for every $x, y \in D$ and $\lambda \in [0,1]$, where φ is defined by

$$\varphi(\lambda)=\begin{cases} -2\lambda\log_2\lambda & 0 \leq \lambda \leq \frac{1}{2}, \\ -2(1-\lambda)\log_2(1-\lambda) & \frac{1}{2} \leq \lambda \leq 1. \end{cases}$$

In the case $\varepsilon = 0$ the result reduces to that of Nikodem and Ng from 1993.

JARCZYK, WITOLD: *Continuous iteration semigroups and cocycles*
(Joint work with Grzegorz Guzik and Janusz Matkowski)

We reformulated four theorems of M. C. Zdun and gave a unified and simplified description of all continuous iteration semigroups defined on the product of $(0,\infty)$ and an arbitrary closed interval X contained in $[-\infty,\infty]$. Using this result we found all solutions $G : (0,\infty) \times X \to Y$ of the cocycle equation

$$G(s+t, x) = G(s,x)G(t, F(s,x)),$$

where $(Y, \cdot)$ is a given commutative group and $F : (0,\infty) \times X \to X$ is a continuous iteration semigroup. In the proof we also used a result on the form of solutions $\Delta : (0,\infty) \times (p,q) \to Y$ of a conditional triangular equation

$$\Delta(s+t, u) = \Delta(s,u)\Delta(t, s+u).$$

KAISER, ZOLTÁN: *The stability of the Cauchy equation in p–adic fields* (Joint work with Zoltán Boros)

It is proved that if f is a function from a vector space X over $\mathbb{Q}$ to the p–adic field $\mathbb{Q}_p$ satisfying

$$\|f(x+y)-f(x)-f(y)\|_p \leq K$$

for some fixed K and all $x, y \in X$ (where $\| \ \|_p$ is the p–adic norm in $\mathbb{Q}_p$), then there exists an additive function $g : X \to \mathbb{Q}_p$ for which

$$\|f(x)-g(x)\|_p \leq K$$

for all $x \in X$. A similar result is also established for the Jensen equation and for endomorphisms.

KAPICA, RAFAL: *Convergence of sequences of iterates of random–valued vector functions*

Given a probability space $(\Omega, \mathcal{A}, P)$ and a closed subset X of a Banach lattice we consider functions $f : X \times \Omega \to X$, their iterates $f^n : X \times \Omega^{\mathbb{N}} \to X$ defined by $f^1(x,\omega) = f(x,\omega_1)$, $f^{n+1}(x,\omega) = f(f^n(x,\omega),\omega_{n+1})$ and obtain theorems on the convergence (a.s., in L^1) of the sequence $(f^n(x,\cdot))$.

KOCLĘGA–KULPA, BARBARA: *On a functional inequality in normed spaces*

The functional inequality

$$\|f(x+y)\| \geq \|f(x)+f(y)\|, \quad x, y \in G \tag{1}$$

has been studied by Gy. Maksa and P. Volkmann (*Characterization of group homomorphisms having values in an inner product space*, Publ. Math. Debrecen **56**/1-2 (2000), 197–200) for f mapping a group $(G,+)$ into a real or complex inner product space $(X, \|\cdot\|)$. It was shown that the inequality (1) implies the Cauchy equation

$$f(x+y) = f(x)+f(y), \quad x, y \in G,$$

and it was asked if this statement was true also for a strictly convex normed space X.

At present, we deal with (1) assuming that X is an arbitrary normed space and the function $f : \mathbb{R} \to X$ satisfies some regularity conditions. We have the following:

THEOREM. *Let $(X, \|\cdot\|)$ be a real normed linear space and let $f : \mathbb{R} \longrightarrow X$ be a solution of the functional inequality* (1). *If the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by the formula $\varphi(x) := \| f(x) \|$, $x \in \mathbb{R}$ and*

$$(J) \qquad \left\{ \begin{array}{l} \varphi \text{ satisfies any regularity condition that} \\ \text{forces Jensen–convex function to be convex,} \end{array} \right.$$

then

$$(2) \qquad f(x) = \gamma I(x), \quad x \in \mathbb{R},$$

where $I : \mathbb{R} \longrightarrow X$ yields an odd isometry and γ is a real constant.

Conversely, for an arbitrary odd isometry $I : \mathbb{R} \longrightarrow X$ and for every constant $\gamma \in \mathbb{R}$, the function $f : \mathbb{R} \longrightarrow X$ given by the formula (2) *yields a solution to the inequality* (1) *and the corresponding function φ is continuous and convex.*

LAJKÓ, KÁROLY: *Functional Equations in Probability Theory (solved and unsolved problems)*

Functional equations have many interesting applications in the characterization problems of probability theory (e.g. in the characterizations of univariate probability distributions by independent statistics and in the characterizations of bivariate distributions from conditional distributions). In these characterizations the functional equations with measurable unknown functions are satisfied for all or for almost all pairs (x, s) from an open set of $\mathbb{R}^2$ (or $\mathbb{R}^n$) respectively. Several solved and unsolved problems were presented in this talk.

MAKSA, GYULA: *Hyperstability of a class of linear functional equations*
(Joint work with Zsolt Páles)

First we investigate the stability properties of the functional equation

$$(1) \qquad \psi(xy) = M(x)\psi(y) + M(y)\psi(x) \qquad (x, y \in]0, 1])$$

where M is a given multiplicative function which has a value greater than 1 and prove that the stability inequality

$$|\psi(xy) - M(x)\psi(y) - M(y)\psi(x)| \leq \varepsilon \qquad (x, y \in]0, 1])$$

(with any fixed $\varepsilon \geq 0$) implies (1). We say shortly that (1) is hyperstable.

Next we present the following generalization.

Let $S = (S, \cdot)$ and X denote a semigroup and a real normed space, respectively. In addition, let $\phi_1, \ldots, \phi_n : S \to S$ be pairwise distinct automorphisms of S such that the set $\{\phi_1, \ldots, \phi_n\}$ is a group with respect to the composition as group operation.

THEOREM. *Let $\varepsilon : S \times S \to \mathbb{R}$ be a function such that there exists a sequence $(u_k) : \mathbb{N} \to S$ satisfying*

$$\lim_{k\to\infty} \varepsilon(u_k s, t) = 0 \qquad (s, t \in S).$$

Assume that $f : S \to X$ satisfies

$$\left\| f(s) + f(t) - \frac{1}{n}\sum_{i=1}^{n} f(s\phi_i(t)) \right\| \le \varepsilon(s,t) \qquad (s, t \in S).$$

Then f is a solution of

$$f(s) + f(t) = \frac{1}{n}\sum_{i=1}^{n} f(s\phi_i(t)) \qquad (s, t \in S).$$

MATKOWSKI, JANUSZ: *A solution of a problem of H. Haruki and Th. M. Rassias*

We prove the following

THEOREM. *A function $f : (0,\infty) \times (0,\infty) \to \mathbb{R}$, continuous on the diagonal $\{(x,x) : x > 0\}$, satisfies the functional equation*

$$f\left(\frac{x+y}{2}, \frac{2xy}{x+y}\right) = f(x,y), \quad x, y > 0$$

if and only if there exists a single variable and continuous function $F : (0,\infty) \to \mathbb{R}$ such that

$$f(x,y) = F(xy), \quad x, y > 0.$$

This solves an open problem posed by H. Haruki and Th. M. Rassias in [1].

A k-dimensional generalization of this result is also presented.

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PÁLES, ZSOLT: *Stability of generalized monomial functional equations*

The stability problem and selection theorems concerning the generalized monomial functional equation

$$p_0 f(x) + p_1 f(xy) + \cdots + p_n f(xy^n) = g(y) \qquad (x, y \in S)$$

are investigated with the help of the so-called invariant mean technique, where $p_0, p_1, \dots, p_n \in \mathbb{R}$, $p_n \neq 0$, $p_0 + p_1 + \dots + p_n = 0$, S is a commutative semigroup, and f maps S into a locally convex space. If

$$p_0 + p_1 t + \dots + p_n t^n = (t-1)^n \qquad (t \in \mathbb{R}),$$

then the results reduce to that of obtained jointly with R. Badora, R. Ger, and L. Székelyhidi in some recent papers.

References

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Sablik, Maciej: *On compatibility of the social development indices*

We discuss the question of compatibility of some indices used by the United Nations Development Program to determine the level of human development. Our goal is to restrict the range of arbitrariness in choosing quasi-arithmetic means to measure the development in different countries.

Various indices used by UNDP are usually aggregated from some basic subindices with the help of quasiarithmetic means. However, the arbitrariness in choosing the aggregating means leads to a non-compatibility of two ways of determining the national index, first one consisting in aggregating subindices on the national scale, and the other in counting regional indices first, and then accumulating them into a national index. We show that compatibility assumption leads to a variant of generalized bisymmetry equation, which was solved in a pretty general setting by J. Aczél, Gy. Maksa and M. Taylor. However, since we are looking for means as solutions, we are able to get directly some results with assumptions slightly relaxed. In particular, we prove the following.

Theorem. *Let $N \geq 2$ be a positive integer, let $I \subset \mathbb{R}$ be a non-degenerate interval, and suppose that $M : I^3 \to I$, $S : I^3 \to I$, $A : I^N \to I$ and $B : I^N \to I$ are means satisfying*

$$\begin{aligned} &M(A(x_{1,1}, \dots, x_{1,N}), A(x_{2,1}, \dots, x_{2,N}),\\ &\quad S(A(t_1, \dots, t_N), A(u_1, \dots, u_N), A(v_1, \dots, v_N))) =\\ &\quad B(M(x_{1,1}, x_{2,1}, S(t_1, u_1, v_1)), \dots, M(x_{1,N}, x_{2,N}, S(t_N, u_N, v_N))), \end{aligned}$$

for all $x_{i,j}, t_j, u_j, v_j \in I$, $i \in \{1,2,3\}$, $j \in \{1, \dots, N\}$. If A, or B, or M is a quasi-arithmetic weighted mean with an increasing and continuous generating function φ then all the remaining means are also weighted quasi-arithmetic means with the same generating function φ.

Székelyhidi, László: *Functional Equations on Hypergroups*

The concept of DJS–hypergroup (according to the initials of C. F. Dunkl, R. I. Jewett and R. Spector) is due to R. Lasser (see e.g. [1]). One begins with a locally compact Haussdorff space K, with the space $\mathcal{M}(\mathcal{K})$ of all finite complex regular measures on K, and with the space $\mathcal{M}^\infty(\mathcal{K})$ of all probability measures in $\mathcal{M}(\mathcal{K})$. The point mass concentrated at x is denoted by δ_x. Suppose that we have the following:

- (H^*) There is a continuous mapping $(x,y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $\mathcal{M}^\infty(\mathcal{K})$, the latter endowed with the weak topology with respect to the space of compactly supported complex valued continuous functions on K. This mapping is called *convolution.*
- $(H^\vee)$ There is an involutive homeomorphism $x \mapsto x^\vee$ from K to K. This mapping is called *involution.*
- (He) There is a fixed element e in K. This element is called *identity.*

Identifying x by δ_x the mapping in (H^*) has a unique extension to a continuous bilinear mapping from $\mathcal{M}(\mathcal{K}) \times \mathcal{M}(\mathcal{K})$ to $\mathcal{M}(\mathcal{K})$. The involution on K extends to an involution on $\mathcal{M}(\mathcal{K})$. Then a DJS–hypergroup is a quadruple $(K, *, \vee, e)$ satisfying the axioms: for any x, y, z in K we have

- (H1) $\delta_x * (\delta_y * \delta_z) = (\delta_x * \delta_y) * \delta_z$;
- (H2) $(\delta_x * \delta_y)^\vee = \delta_{y^\vee} * \delta_{x^\vee}$;
- (H3) $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;
- (H4) e is in the support of $\delta_x * \delta_{y^\vee}$ if and only if $x = y$;
- (H5) the support of $\delta_x * \delta_y$ is compact;
- (H6) the mapping $(x,y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space of nonvoid compact subsets of K is continuous, the latter being endowed by the Michael–topology.

If $\delta_x * \delta_y = \delta_y * \delta_x$ for all x, y in K, then we call the hypergroup *commutative.* For instance, if $K = G$ is a locally compact Haussdorff–group, $\delta_x * \delta_y = \delta_{xy}$ for all x, y in K, $x^\vee$ is the inverse of x, and e is the identity of G, then we obviously have a hypergroup $(K, *, \vee, e)$, which is commutative if and only if the group G is commutative. However, not every hypergroup originates in this way.

Let $0 < \theta \leq 1$ be arbitrary and let $K = \{0, 1\}$. We define e as 0 and involution as the identity map. The products $\delta_0 * \delta_0 = \delta_0$, $\delta_0 * \delta_1 = \delta_1 * \delta_0 = \delta_1$ are obvious, and we let

$$\delta_1 * \delta_1 = \theta\delta_0 + (1-\theta)\delta_1 .$$

It is easy to see that we get a hypergroup for any θ in $]0, 1]$. For $\theta = 1$ we get the two–element group of integers modulo 2.

We identify x by δ_x and we define the *translation operator* T_y by the element y in K according to the formula:

$$T_y f(x) = \int_K f d(\delta_x * \delta_y),$$

for any f integrable with respect to $\delta_x * \delta_y$. In particular, T_y is defined for any continuous complex valued function on K.

In other words we have

$$f(x * y) = \int_K f d(\delta_x * \delta_y)$$

for any x, y in K.

Having translation operators we may consider the classical functional equations on hypergroups. On commutative hypergroups one can study

- *exponentials*:
$$m(x * y) = m(x)m(y),$$
with $m(x) \neq 0$ for all x, y in K, which are common eigenfunctions of all translation operators;
- *additive functions*:
$$a(x * y) = a(x) + a(y)$$
for all x, y in K;
- *polynomial functions*:
$$(T_y - I)^{n+1} f(x) = 0$$
for all x in K, where n is a nonnegative integer, y is arbitrary in K and $I = T_e$ is the identity operator;
- *d'Alembert–equation*:
$$f(x * y) + f(x * y^{\vee}) = 2f(x)f(y)$$
for all x, y in K;
- *stability problems* for the classical equations;
- *spectral synthesis problems* for translation invariant function spaces; etc.

The main goal of this work is to call attention to hypergroups and to the possibility of studying functional equations on hypergroups. It seems that some of the classical methods can be adopted to the hypergroup–case but in some cases new ideas are needed.

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SZOSTOK, TOMASZ: *On a generalized orthogonal additivity*

Logical connections between the modified version of orthogonal Cauchy equation and the following unconditional equation

$$f(x+y) = g\left(\frac{\|x-y\|}{\|x+y\|}\right)[f(x)+f(y)]$$

are examined. Namely, it is proved that under some assumptions this equation preserves the solutions of orthogonal Cauchy equation. Further the Cauchy equation with the right-hand side multiplied by some constant is considered. This equation is assumed for all x, y satisfying the equality $\frac{\|x-y\|}{\|x+y\|} = \alpha$. Finally solutions of this conditional equation in the case of odd functions defined on inner product spaces and α lying in some interval are determined.

2. Problems and Remarks

1. Problem and Remarks. Let F be a field. A mapping $|\ |^* : F \to \mathbb{R}$ is called a *valuation* if it is positive definite, multiplicative, and subadditive. If, instead of subadditivity, the stronger inequality

$$|x+y|^* \leq \max\{|x|^*, |y|^*\} \tag{1}$$

holds for all $x, y \in F$, then the valuation $|\ |^*$ is called *non-archimedean*. We say that $(X, \|\ \|)$ is a normed space over the field F with valuation $|\ |^*$ if X is a linear space over F and the mapping $\|\ \| : X \to \mathbb{R}$ is positive definite, subadditive, and we have $\|\lambda x\| = |\lambda|^*\|x\|$ for all $\lambda \in F$, $x \in X$. A normed space is called non-archimedean if the norm satisfies an analogue of the inequality (). If a normed space is complete with respect to the metric generated by the norm, it is called a Banach space. Let us note that every field with a valuation is also a normed space over itself.

PROBLEM. Determine (classes of) pairs (S, X) such that $(S, +)$ is a groupoid, X is a normed space over a field with a valuation, and, for functions mapping S into X, the additive Cauchy equation is stabil in the Hyers–Ulam sense.

As a counterexample, we mention the case $S = X = \mathbb{Q}$. In what follows, we list some cases when stability holds.

EXAMPLE 1. If $(S,+)$ is a commutative semigroup and X is a Banach space over $\mathbb{R}$ or $\mathbb{C}$, then Hyers' classical result (more exactly, its proof) establishes stability.

EXAMPLE 2. If S is a vector space over $\mathbb{Q}$ and X is a non–archimedean Banach space over a p–adic field $\mathbb{Q}_p$, then an analogue of Hyers' method can be applied to prove stability, as it was presented by Z. Kaiser (cf. the abstract in this report).

EXAMPLE 3. Let K be a field and $X = K(x)$ be the set of all rational functions in one variable. For non–zero polynomials $p, q \in K[x]$ define

$$\left|\frac{p}{q}\right|^* = 2^{deg(p)-deg(q)}.$$

One can easily check that this yields a non–archimedean valuation on the field $K(x)$. Moreover, for every $\varepsilon > 0$, the set $\{r \in K(x) : |r|^* < \varepsilon\}$ is a K–linear subspace of K(x). Now we can apply [1, Lemma 1] to prove the stability without any assumptions on S.

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Z. BOROS

2. Problem. Let

$$A(x,y,z,t) := \frac{x+y+z+t}{4}, \quad G(x,y,z,t) := \frac{\sqrt{xy}+\sqrt{zt}}{2} \quad (x,y,z,t) \in \mathbb{R}_+^4 :$$

The following iteration was interpreted by Borchardt in 1876 for every $(x,y,z,t) \in \mathbb{R}_+^4$.

$$\begin{aligned} x_1 &:= x, \ y_1 := y, \qquad z_1 := z, \ t_1 := t; \\ x_{n+1} &:= A(x_n, y_n, z_n, t_n), y_{n+1} := G(y_n, x_n, z_n, t_n), \\ z_{n+1} &:= G(z_n, x_n, y_n, t_n), t_{n+1} := G(t_n, x_n, y_n, z_n), \end{aligned}$$

where $n \geq 1$. It is a consequence of general results, that there exists one and only one function $M : \mathbb{R}_+^4 \to \mathbb{R}_+$ for which

$$M(x,y,z,t) = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} t_n.$$

M has been determinded by Borchardt on the domain

$$T := \left\{((x,y,z,t) \in \mathbb{R}_+^4 \mid x > y > z > t, \ xt > yz\right\}.$$

The question is the following: What kind of (as far as possible elementary) method can be used to determine the explicit form of M on $\mathbb{R}_+^4$. This problem substantially means to solve the functional equation (invariance–equation)

$$M[A(x,y,z,t),G(y,x,z,t),G(z,x,y,t),G(t,x,y,z)] = \\ = M(x,y,z,t) \qquad ((x,y,z,t) \in \mathbb{R}_+^4),$$

where M is a mean with four variables on positive real numbers.

Z. DARÓCZY

3. Remark. Connected to R. Ger's and B. Koclęga–Kulpa's presentations at this meeting and to a problem raised by K. Nikodem during the 38^{th} International Symposium on Functional Equations (Noszvaj, Hungary, 2000), we prove the following statement. If G is an Abelian group divisible by 2, H is a Hilbert space, ε is a nonnegative real number and a function $f : G \to H$ satisfies

$$\|f(x-y) - 2f(x) - 2f(y)\| \leq \|f(x+y)\| + \varepsilon \quad (x, y \in G),$$

then there exists a function $g : G \to H$ fulfilling

$$g(x+y) + g(x-y) - 2g(x) - 2g(y) = 0 \quad (x, y \in G)$$

and

$$\|f(x) - g(x)\| \leq \frac{5}{2}\varepsilon \quad (x \in G).$$

A. GILÁNYI

4. Problem. Given intervals I, $J \in \mathbb{R}$ and a mapping $\alpha : I \to J$ find all functions $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ such that

$$f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} = g\left(\frac{\alpha(x)+\alpha(y)}{2}\right) - \frac{g(\alpha(x))+g(\alpha(y))}{2}$$

for every x, $y \in I$. The case where α, f and g are diffeomorphisms is also of interest.

W. JARCZYK AND J. MATKOWSKI

5. Problem. Motivated by the notion of quasiadditive functions due to Józef Tabor, a function $f : S \to X$ is called *quasimonomial* if, for some $0 \leq \varepsilon < 1$,

$$\left\|\frac{1}{n!}\Delta_y f(x) - f(y)\right\| \leq \varepsilon \min\left(\left\|\frac{1}{n!}\Delta_y f(x)\right\|, \|f(y)\|\right) \qquad (x, y \in S).$$

Does there exist a "nice" function $\varphi : X \to X$ and a monomial function $A_n : S \to X$ of degree n such that f admits the decomposition

$$f = \varphi \circ A_n \ ?$$

Here S is a commutative semigroup and X is a reflexive Banach space. For the case $n = 1$ the answer is affirmative and was obtained by R. Badora, R. Ger and myself (*Additive selections and the stability of the Cauchy functional equation*, Bull. Austr. Math. Soc., accepted.).

ZS. PÁLES

6. Problem. The stability theorem of the Pexider functional equation due to K. Nikodem states that if S is a commutative semigroup, $f, g, h : S \to \mathbb{R}$ satisfy

$$\|f(xy) - g(x) - g(y)\| \leq \varepsilon \qquad (x, y \in S),$$

then there exist functions $F, G, H \to \mathbb{R}$ such that

$$F(xy) = G(x) + H(y) \qquad (x, y \in S)$$

and

$$\|F - f\| \leq c_f\varepsilon, \qquad \|G - g\| \leq c_g\varepsilon, \qquad \|H - h\| \leq c_h\varepsilon$$

with $(c_f, c_g, c_h) = (3, 4, 4)$. Using the invariant mean technique, one can obtain $(c_f, c_g, c_h) = (3, 1, 1)$. Is it possible to reach $(c_f, c_g, c_h) = (1, 1, 1)$?

ZS. PÁLES

7. Remark. J. Matkowski investigated the functional equation

$$F\Big(\frac{2xy}{x+y}, \frac{x+y}{2}\Big) = F(x, y) \qquad (x, y > 0) \tag{1}$$

Assuming continuity of F at diagonal points (x, x), he proved that F is always of the form

$$F(x, y) = f(xy) \qquad (x, y > 0). \tag{2}$$

We show that this statement cannot be obtained without the above continuity assumption. For, observe that (1) implies the symmetry of F. Thus, it is enough to describe F on the cone $K := \{(x, y) : 0 < x \leq y\}$. For fixed $(x, y) \in K$, define the sequence (x_k, y_k) by the iteration

$$x_1 = x, \quad y_1 = y, \qquad x_{k+1} = \frac{2x_ky_k}{x_k + y_k}, \quad y_{k+1} = \frac{x_k + y_k}{2} \quad (k \geq 2).$$

We say that (x, y) is equaivalent to (u, v) if either there exists $k \in \mathbb{N}$ such that $(u, v) = (x_k, y_k)$ or there exists $n \in \mathbb{N}$ such that $(x, y) = (u_n, v_n)$. One can esily verify that this relation is indeed an equivalence relation, all the equivalence classes are subsets of some hyperbola $H_c = \{(x, y) \in K : xy = c^2\}$, and the meaning of equation (1) is that F is constant on the equivalence classes. This way, one can obtain the general solution of (1). Since, for $c > 0$, there are more than two disjoint equaivalence classes in H_c, there are solutions of (1) that are not of the form (2). All the equivalence classes contained in H_c accumulate to the point (c, c), therefore, in order that F be of the form (2) it is necessary and suffiecient that F be continuous at (c, c) when restricted to H_c for all $c > 0$.

Zs. Páles

8. Problem. If M and N are continuous strict means on the interval I, then their *Gauss–convolution* $M \otimes N$ is defined in the following way:

$$M \otimes N(x, y) := \lim_{k \to \infty} x_k = \lim_{k \to \infty} y_k, \qquad (x, y \in I)$$

where the sequences (x_k) and (y_k) are constructed from x, y by the following iteration:

$$x_1 = x, \quad y_1 = y, \quad x_{k+1} = M(x_k, y_k), \quad y_{k+1} = N(x_k, y_k) \quad (k \geq 2).$$

For instance, the Gauss–convolution of the arithmetic and geomtric means is the famous *medium arithmeticum–geometricum*, the arithmetic–geometric mean introduced by Gauss. It is a hard fact that this mean has an explicite representation via elliptic integrals.

Questions:

(1) Are there nice algebraic properties of the operation $\otimes$ (bisymmetry, square symmetry, power associativity, etc.)?

(2) If M and N are given means, is there a one parameter family $\{M_t : t \in [0, 1]\}$ of means which is closed under $\otimes$ and $M_0 = M$, $M_1 = N$. This problem seems to be interesting also for the case when M and N are the geometric and arithmetic means, respectively.

(3) Is there a representation for the Gauss–convolution of power means similar to that of for the arithmetic–geometric mean? Can one also compute the Gauss–convolution of Gini means or Stolarsky means?

Zs. Páles

(compiled by Justyna Sikorska)