ON DIMENSION OF ATTRACTORS OF REACTION-DIFFUSION EQUATIONS WITH PERIODIC RIGHT-HAND SIDE

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To the memory of Professor György Targonski

Abstract. In this paper we study the finite-dimensionality of the global attractor of a discrete dynamical system generated by a reaction–diffusion equation with non-differentiable nonlinear term and periodic right-hand side. The existence of an exponential attractor is also proved. Explicit estimates of the fractal dimension are given.

1. Introduction

One of the main problems in the theory of global attractors in infinite-dimensional dynamical systems is the estimation of the fractal dimension of the attractor. If such an estimate exists, it implies that the observed permanent regime depends only on a finite number of degrees of freedom. In some cases the flow on the attractor is equivalent to the flow defined by a system of ordinary differential equations in a finite-dimensional manifold. This can be obtained using inertial manifolds, that is, smooth finite-dimensional and positively invariant manifolds attracting exponentially all orbits. In [13] the concept of exponential attractor was introduced. An exponential attractor

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is an exponentially attracting compact set containing the global attractor, which is positively invariant with respect to the flow and has finite fractal dimension. It is an intermediate object between global attractors and inertial manifolds. For general results concerning exponential attractors see [1], [12], [13]-[15].

Estimates of the fractal dimension of the global attractor and existence theorems of exponential attractors for autonomous and nonautonomous parabolic equations of reaction–diffusion type have been obtained by several authors (see [1], [2], [3], [4], [5]-[7], [11], [13]-[15], [16], [17], [18], [21], [22], [24], [25]).

In this paper we study the finite-dimensionality of the global attractor and the existence of an exponential attractor for a discrete infinite-dimensional dynamical system generated by the following nonautonomous reaction–diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + f(u) = h + \omega u, \text{ in } \Omega \times (0, +\infty), \\ u \mid_{\partial\Omega} = 0, \\ u \mid_{t=0} = u_0, \end{cases}$$

where h is periodic with respect to the time-variable t. We note that in the same way as in [5]-[7] the function f is not assumed to be differentiable, but instead a Lipschitz condition is imposed. For other conditions avoiding differentiability see [17].

2. Some results on dimension of compact sets

Let H be a Hilbert space and $V : H \to H$ be a continuous mapping.

Let $\mathcal{A} \subset H$ be a compact set such that $V(\mathcal{A}) = \mathcal{A}$. The fractal dimension of $\mathcal{A}$ is defined by

$$d_f(\mathcal{A}) = \inf \{d > 0 : \mu_f(\mathcal{A}, d) = 0\},$$

where

$$\mu_f(\mathcal{A}, d) = \limsup_{\varepsilon \to 0} \varepsilon^d n_\varepsilon,$$

and n_ε is the minimum number of balls of radius less than or equal to ε which are necessary to cover $\mathcal{A}$.

THEOREM 1 (see [6], [7]). *Let us suppose that there exist* $l \in [1, +\infty)$, $\delta \in \left(0, \frac{1}{\sqrt{2}}\right)$ *such that for any* $u, v \in \mathcal{A}$

$$\|V(u) - V(v)\| \leq l \|u - v\|, \tag{1}$$

$$(2) \qquad \|Q_N V(u) - Q_N V(v)\| \leq \delta \|u - v\|,$$

where Q_N *is the projector in* H *into some subspace* $H_N^{\perp}$ *of codimension* $N \in \mathbb{N}$. *Then for any* $\eta > 0$ *such that* $(\sqrt{2}6l)^N (\sqrt{2}\delta)^\eta = \sigma < 1$ *the inequality*

$$(3) \qquad d_f(\mathcal{A}) \leq N + \eta$$

holds.

REMARK 1 [24, p.24]. If $l < 1$ then $\mathcal{A}$ consists of one point, so that $d_f(\mathcal{A}) = 0$.

REMARK 2. Theorem 1 is a modification of a theorem of O.A.Ladyzhenskaya [20].

The map V generates the discrete semigroup $S : \overline{\mathbb{N}} \times H \to H$, $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$, defined by

$$S(n, x) = V^n(x),$$

where V^n denotes the n-th iterate of V. This dynamical system will be denoted by (V, H).

For any $A, B \subset H$, $d(A, B) = \sup_{y \in A} \inf_{x \in B} \|y - x\|$. The compact set $\mathfrak{R}$ is said to be a global attractor of S if $d(S(n, B), \mathfrak{R}) \to 0$, as $n \to \infty$, for any bounded set $B \subset H$ and $S(n, \mathfrak{R}) = \mathfrak{R}$ for each $n \in \mathbb{N}$.

Let $X \subset H$ be a compact set and $V(X) \subset X$. We shall consider the semigroup S restricted to X. The general theory of attractors provides in this case the existence of the global attractor $\mathfrak{R}$ (see [19], [20]).

DEFINITION 1. The compact set $\mathcal{M}$ is said to be an exponential attractor of the dynamical system (V, X) if $\mathfrak{R} \subset \mathcal{M} \subset X$ and

1. $V(\mathcal{M}) \subset \mathcal{M}$;
2. $\mathcal{M}$ has finite fractal dimension;
3. there exist positive constants c_0, c_1 such that

$$d(S(n, X), \mathcal{M}) \leq c_0 \exp(-c_1 n) \quad \text{for} \quad n \geq 1.$$

THEOREM 2 (see [13], [14], [15]). *Let the map* V *be Lipschitz. Suppose that it satisfies the squeezing property, i.e., for some* $\delta \in \left(0, \frac{1}{8}\right)$ *there exists an orthogonal projection* $Q_N(\delta)$ $(P_N = I - Q_N)$ *onto a subspace of codimension* N *such that for any* $u, v \in X$ *either*

$$\|V(u) - V(v)\| \leq \delta \|u - v\|,$$

or

$$\|Q_N(V(u)-V(v))\| \leq \|P_N(u-v)\|.$$

Then (S,X) has an exponential attractor $\mathcal{M}$.

The set $X \subset H$ is called an absorbing set for S if $V(X) \subset X$ and for any bounded set $B \subset H$ there exists n_0 such that $S(n,B) \subset X$ for $n \geq n_0$.

It is clear that if X is a compact absorbing set having an exponential attractor $\mathcal{M}$ then $\mathcal{M}$ attracts exponentially each bounded set B. In that case it is called an exponential attractor for (V,H).

3. Main results

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial\Omega$. Let $H = L_2(\Omega)$ with the norm $\|u\| = \sqrt{\int_\Omega |u|^2 \, dx}$. We consider the following reaction-diffusion equation

$$(4) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + f(u) = h + \omega u, \text{ in } \Omega \times (0,+\infty), \\ u \mid_{\partial\Omega} = 0, \\ u \mid_{t=0} = u_0, \end{cases}$$

where $u = u(x,t)$, $x \in \Omega$, $t \in [0,+\infty)$, $\omega \geq 0$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $h(x,t)$ is a periodic function in t (with period T_0) such that $h \in L_2(\Omega \times [0,T_0])$ and $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz function (with Lipschitz constant ξ). Let us denote

$$C = \int_0^{T_0} \int_\Omega |h(t,x)|^2 \, \mathrm{d}\,x \, \mathrm{d}\,t.$$

It is well known from the theory of maximal monotone operators (see [8], [9]) that for each $u_0 \in L_2(\Omega)$ and $T > 0$ there exists an unique solution of (4), $u(\cdot) \in C([0,T], L_2(\Omega))$, such that

$$u \in W^{1,2}(\delta, T; L_2(\Omega)) \qquad \text{for any} \quad 0 < \delta < T,$$

$$u \text{ is a.e. differentiable on } (0,T),$$

$$u(t) \in H^2(\Omega) \cap H_0^1(\Omega) \quad \text{a.e. on } (0,T),$$

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = h + \omega u \text{ a.e. on } (0,T),$$

$$u(0) = u_0.$$

If $u_0 \in H_0^1(\Omega)$, then $u \in W^{1,2}(0,T;L_2(\Omega))$. Moreover, for any $u_0, v_0 \in L_2(\Omega)$, $t \geq 0$,

$$\|u(t) - v(t)\| \leq \exp(\omega t)\,\|u_0 - v_0\|.$$

We construct a discrete semigroup of operators $S : \overline{\mathbb{N}} \times H \to H$ in the following way:

$$S(n, u_0) = u(nT_0) \qquad \text{for} \quad n \in \overline{\mathbb{N}} \text{ and } u_0 \in H,$$

where $u(\cdot)$ is the unique solution of (4) corresponding to u_0. We note that in this case $V = S(1,\cdot)$.

THEOREM 3. *Let there exist* $\varepsilon > 0, M \geq 0$ *such that*

$$f(s)\,s \geq (-\lambda_1 + \omega + \varepsilon)\,s^2 - M, \tag{5}$$

where λ_1 *is the first eigenvalue of* $-\Delta$ *in* $H_0^1(\Omega)$. *Then the system* (V,H) *has an exponential attractor* $\mathcal{M}$. *The following estimate of the fractal dimension holds*

$$d_f(\Re) \leq K\left((\omega + \xi)^{\frac{n}{2}} \exp(2\omega T_0 n) + (T_0)^{-\frac{n}{2}}\right),$$

where K *depends on* n *and* Ω *and* $\Re$ *is the global attractor of* (V,H).

PROOF. As usual, multiplying (4) by $u(t)$ and using condition (5) we obtain the inequalities

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \varepsilon\,\|u(t)\|^2 \\ &\leq \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + (-\lambda_1 + \varepsilon)\,\|u(t)\|^2 + \|\nabla u(t)\|^2 \\ &\leq M + \frac{1}{2\varepsilon}\|h(t)\|^2 + \frac{\varepsilon}{2}\|u(t)\|^2. \end{aligned} \tag{6}$$

Hence, multiplying inequalities (6) by $\exp(\varepsilon t)$ and integrating on $(0,T)$ we have

$$\|u(T)\|^2 \exp(\varepsilon T) - \|u(0)\|^2 \leq \frac{2M}{\varepsilon}(\exp(\varepsilon T) - 1) + \frac{1}{\varepsilon}\int_0^T \|h(t)\|^2 \exp(\varepsilon t)\,\mathrm{d}\,t.$$

Let $k \geq 0$, $k \in \mathbb{Z}$, be such that $(k-1)\,T_0 < T \leq kT_0$. Being h periodic with period T_0, we can estimate the last term as follows:

$$\begin{aligned}
\frac{1}{\varepsilon}\int_0^T \|h(t)\|^2 \exp(\varepsilon t)\,\mathrm{d}\,t &\le \frac{1}{\varepsilon}\left(\int_{T-T_0}^T \|h(t)\|^2 \exp(\varepsilon t)\,\mathrm{d}\,t\right.\\
&\left.+\int_{T-2T_0}^{T-T_0} \|h(t)\|^2 \exp(\varepsilon t)\,\mathrm{d}\,t+\ldots+\int_0^{T-(k-1)T_0} \|h(t)\|^2 \exp(\varepsilon t)\,\mathrm{d}\,t\right)\\
&\le \frac{1}{\varepsilon}\left(\exp(\varepsilon T)+\ldots+\exp(\varepsilon(T-(k-1)T_0))\right)\int_0^{T_0}\|h(t)\|^2\\
&= \frac{C\exp(\varepsilon T)}{\varepsilon}\left(1+\exp(-\varepsilon T_0)+\ldots+\exp(-\varepsilon(k-1)T_0)\right)\\
&\le \frac{C\exp(\varepsilon T)}{\varepsilon}\left(1-\exp(-\varepsilon T_0)\right)^{-1}.
\end{aligned}$$

Therefore, the following inequality holds

$$\|u(T)\|^2 \le \|u(0)\|^2 \exp(-\varepsilon T)+\frac{2M}{\varepsilon}\left(1-\exp(-\varepsilon T)\right)+\frac{C}{\varepsilon}\left(1-\exp(-\varepsilon T_0)\right)^{-1}.$$

Let $\rho^2 = \frac{C}{\varepsilon}\left(1-\exp(-\varepsilon T_0)\right)^{-1}+\frac{2M}{\varepsilon}+\eta$, $\eta > 0$. It follows from the last inequality that for any $R > 0$ there exists $T(R)$ such that $\|u(t)\| < \rho$, if $t > T(R)$, $\|u_0\| < R$. Let

$$X_\rho = \{y \in H : \ \exists u_0 \in H, \|u_0\| < \rho,\, t \ge 0, \text{ such that } u(t) = y\}.$$

It is clear that if $u_0 \in X_\rho$, then $u(t) \in X_\rho$ for $t \ge 0$. Thus, the set X_ρ is absorbing for V. It is also evident that X_ρ is a bounded set. Hence, there exists $\rho_1 > 0$ such that $\|u\| < \rho_1$ for $u \in X_\rho$.

Further, we have to obtain an absorbing ball in $H_0^1(\Omega)$. Let $R > 0$ and $N(R)$ be such that $S(N, u_0) \in X_\rho$, if $\|u_0\| < R$. Let $t \ge NT_0$ and $0 < r \le T_0$. Integrating (6) on $(t, t+r)$ and using the fact that $u(\tau) \in X_\rho$ for $\tau \ge NT_0$, we get

$$\begin{aligned}
\int_t^{t+r} \|\nabla u(\tau)\|^2\,\mathrm{d}\,\tau &\le \left(\frac{1}{2}+\lambda_1 r\right)\rho_1^2 + Mr + \frac{1}{2\varepsilon}\int_t^{t+r}\|h(\tau)\|^2\,\mathrm{d}\,\tau\\
&\le \left(\frac{1}{2}+\lambda_1 r\right)\rho_1^2 + Mr + \frac{C}{2\varepsilon}.
\end{aligned}$$

It is clear from the Lipschitz condition for f that there exist constants K_1, K_2 such that for any $u \in L_2(\Omega)$

$$\|f(u)\| \le K_1 + K_2\|u\|.$$

Hence, multiplying (4) by $\frac{du}{dt}$ we obtain

$$\left\|\frac{du}{dt}\right\|^2 - \left(\Delta u, \frac{du}{dt}\right) = \left(-f(u) + \omega u + h, \frac{du}{dt}\right)$$
$$\leq \frac{3}{4}\left\|\frac{du}{dt}\right\|^2 + \|h\|^2 + \omega^2\|u\|^2 + 2K_1^2 + 2K_2^2\|u\|^2.$$

Then the equality $\left(-\Delta u, \frac{du}{dt}\right) = \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2$ implies

$$\frac{1}{2}\left\|\frac{du}{dt}\right\|^2 + \frac{d}{dt}\|\nabla u\|^2 \leq 2\|h\|^2 + 4K_1^2 + \left(2\omega^2 + 4K_2^2\right)\|u\|^2$$
$$\leq 2\|h(t)\|^2 + 4K_1^2 + \left(2\omega^2 + 4K_2^2\right)\rho_1^2,$$

if $t \geq NT_0$.

Let us recall the uniform Gronwall Lemma (see [24, p. 89]):

LEMMA 1. *Let g, z, y be three positive locally integrable functions on $(t_0, +\infty)$ such that $\frac{dy}{dt}$ is also locally integrable on $(t_0, +\infty)$ and for $t \geq t_0$*

$$\frac{dy}{dt} \leq gy + z,$$

$$\int_t^{t+r} g(s)\,\mathrm{d}s \leq a_1, \quad \int_t^{t+r} z(s)\,\mathrm{d}s \leq a_2, \quad \int_t^{t+r} y(s)\,\mathrm{d}s \leq a_3,$$

where $r, a_1, a_2, a_3 > 0$. Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right)\exp(a_1) \qquad \text{for} \quad t \geq t_0.$$

Applying Lemma 1 with $g(t) \equiv 0$, $y(t) = \|\nabla u\|^2$ and $z(t) = 2\|h(t)\|^2 + 4K_1^2 + \left(2\omega^2 + 4K_2^2\right)\rho_1^2$, $t_0 = NT_0$, we have

$$\begin{aligned} &\|\nabla u(t)\|^2 \\ (7) \qquad &\leq \left(\frac{\left(\frac{1}{2} + \lambda_1 r\right)\rho_1^2 + Mr + \frac{C}{2\varepsilon}}{r} + 2C + \left(4K_1^2 + \left(2\omega^2 + 4K_2^2\right)\rho_1^2\right)r\right) \\ &= \rho_2^2 \end{aligned}$$

for $t \geq r + NT_0$. Let us define the ball $B_{\rho_2}^V = \left\{u \in H_0^1(\Omega) : \|u\|_{H_0^1(\Omega)} < \rho_3\right\}$, $\rho_3 = \rho_2 + \eta$, $\eta > 0$. We set $r = \frac{T_0}{2}$, $X = \overline{X_\rho \cap B_{\rho_2}^V}$. We claim that X is a compact absorbing set. Since the injection $H_0^1(\Omega) \subset L_2(\Omega)$ is compact, X

is compact . It is clear from (7) that $S(n, B_R) \subset X$, if $n \geq N(R)+1$, where $B_R = \{u \in H : \|u\| < R\}$. Finally, we must prove that $V(X) \subset X$. First let $u_0 \in X_\rho \cap B^V_{\rho_2}$. In this case $N(\rho) = 0$, so that $S(1, u_0) = V(u_0) \in B_\rho \cap B^V_{\rho_2}$. Being V continuous, $V(X) \subset X$.

It follows that S has the global attractor $\mathfrak{R}$.

In order to obtain the estimate of the fractal dimension we have to check that (1)-(2) hold for the map V.

Condition (1) is always satisfied, since

$$\|S(1, u_0) - S(1, v_0)\| \leq \exp(\omega T_0) \|u_0 - v_0\| \qquad \text{for} \quad u_0, v_0 \in H.$$

Hence $l = \exp(\omega T_0)$.

Further, let us prove that (2) is satisfied. For arbitrary solutions $u(t)$, $v(t)$, corresponding to $u_0, v_0 \in L_2(\Omega)$, respectively, we have

$$(8) \qquad \begin{cases} \frac{d(u-v)}{dt} - \Delta(u-v) + f(u) - f(v) - \omega(u-v) = 0 \quad \text{a.e.,} \\ u(0) - v(0) = u_0 - v_0. \end{cases}$$

Denote $m(t) = u(t) - v(t)$. Multiplying the last equality by $Q_N m(t)$ and integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|Q_N m(t)\|^2 + \|\nabla Q_N m(t)\|^2 \\ &\quad + \int_\Omega (f(u(t)) - f(v(t)) - \omega m(t)) Q_N m(t) \mathrm{d}x = 0. \end{aligned}$$

Now the inequalities $\|\nabla Q_N m(t)\| \geq \lambda_{N+1} \|Q_N m(t)\|^2$, $\|m(t)\| \leq \exp(\omega t)$ $\|m_0\|$ and the Lipschitz condition for f imply

$$\begin{aligned} \frac{d}{dt} \|Q_N m(t)\|^2 &\leq -2\lambda_{N+1} \|Q_N m(t)\|^2 + 2(\xi + \omega) \|m(t)\|^2 \\ &\leq -2\lambda_{N+1} \|Q_N m(t)\|^2 + 2(\xi + \omega) \exp(2\omega t) \|m_0\|^2, \end{aligned}$$

where λ_{N+1} is the $N+1$-th eigenvalue of $-\Delta$ in $H^1_0(\Omega)$. Multiplying both sides by $\exp(2\lambda_{N+1} t)$, we have

$$\frac{d}{dt}\left(\|Q_N m(t)\|^2 \exp(2\lambda_{N+1} t)\right) \leq 2(\xi + \omega) \exp(2(\omega + \lambda_{N+1})t) \|m_0\|^2.$$

Integrating on $(0, T_0)$ we get

$$\begin{aligned} &\|Q_N m(T_0)\|^2 \exp(2\lambda_{N+1} T_0) \\ &\quad \leq \|Q_N m_0\|^2 + \|m_0\|^2 \frac{\xi + \omega}{\omega + \lambda_{N+1}} (\exp(2(\omega + \lambda_{N+1})T_0) - 1). \end{aligned}$$

Hence,

$$\begin{aligned}\|Q_N m(T_0)\|^2 &\le \|m_0\|^2 \left(\frac{\lambda_{N+1}-\xi}{\omega+\lambda_{N+1}}\exp\left(-2\lambda_{N+1}T_0\right)+\frac{\xi+\omega}{\omega+\lambda_{N+1}}\exp\left(2\omega T_0\right)\right)\\ &\le \|m_0\|^2\left(\exp\left(-2\lambda_{N+1}T_0\right)+\frac{\xi+\omega}{\omega+\lambda_{N+1}}\exp\left(2\omega T_0\right)\right)\\ &= \delta^2(N)\,\|m_0\|^2 .\end{aligned}$$

Choosing an appropriate N we obtain $\delta(N) < \frac{1}{\sqrt{2}}$. Then (2) is satisfied on $\Re$ for the map V.

Let now $\omega < \lambda_1$. Multiplying (8) by $m(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\|\,m(t)\|^2+\|\nabla\,m(t)\|^2+\int_\Omega \left(f(u(t))-f(v(t))\,m(t)\mathrm{d}\,x-\omega\,\|m(t)\|^2=0.\right.$$

Since $\|\nabla m(t)\| \ge \lambda_1 \|m(t)\|^2$ and being f non-decreasing, we obtain

$$\frac{d}{dt}\|m(t)\|^2 \le 2(\omega-\lambda_1)\,\|m(t)\|^2 .$$

Hence,

$$\|m(T_0)\|^2 \le \|m_0\|^2\exp\left(2(\omega-\lambda_1)T_0\right).$$

Therefore, it follows from Remark 1 that $d_f(\Re)=0$.

It is well known (see [10, p. 201], [23, p. 136]) that $\lambda_N = O(N^{\frac{2}{n}})$, as $N \to \infty$. Hence, there exists $D > 0$ such that $\frac{\lambda_N}{N^{\frac{2}{n}}} \ge D$ for $N \in \mathbb{N}$. Let $\gamma = 12$. We take $(\gamma\delta(N)l)^2 = \gamma^2\left(\exp\left(-2\lambda_{N+1}T_0+2\omega T_0\right)+\frac{\omega+\xi}{\omega+\lambda_{N+1}}\exp\left(4\omega T_0\right)\right)$ and choose λ_{N+1} in such a way that for some $0<\alpha<1$

$$\exp\left(-2(\lambda_{N+1}-\omega)T_0\right) \le \frac{1}{2\gamma^2}, \tag{9}$$

$$\frac{(\omega+\xi)}{\omega+\lambda_{N+1}}\exp\left(4\omega T_0\right) \le \frac{1}{2\gamma^2+\alpha}. \tag{10}$$

Hence, conditions (9), (10) will be satisfied if the next inequalities holds

$$\lambda_{N+1} \ge \omega + \frac{\log\left(\sqrt{2}\gamma\right)}{T_0},$$

$$\lambda_{N+1} \ge \left(2\gamma^2+\alpha\right)(\omega+\xi)\exp\left(4\omega T_0\right)-\omega.$$

It is sufficient to find N such that

$$\lambda_{N+1} \ge \left(2\gamma^2+\alpha\right)(\omega+\xi)\exp\left(4\omega T_0\right)+\frac{\log\left(\sqrt{2}\gamma\right)}{T_0}.$$

Using $\lambda_{N+1} \geq D(N+1)^{\frac{2}{n}}$ we obtain that the last inequality holds as soon as

$$N+1 \geq \left(\frac{(2\gamma^2+\alpha)\,(\omega+\xi)}{D} \exp\,(4\omega T_0) + \frac{\log\,(\sqrt{2}\gamma)}{DT_0} \right)^{\frac{n}{2}} = \beta.$$

We choose $N = [\beta]$. It is clear that there exist constants D_1, D_2 (depending on Ω and n) for which $N \leq D_1\,(\omega+\xi)^{\frac{n}{2}} \exp\,(2\omega T_0 n) + D_2\,(T_0)^{-\frac{n}{2}}$.

We can assume that $N \geq 1$. If $N = 0$ it is clear that $\omega < \lambda_1$ and then $d_f(\Re) = 0$. We have obtained that for such N, $(\gamma\delta(N)l)^N < 1$ and then all conditions of Theorem 1 hold for $\eta = N$. Hence,

$$d_f(\Re) \leq 2N \leq K\left((\omega+\xi)^{\frac{n}{2}} \exp\,(2\omega T_0 n) + (T_0)^{-\frac{n}{2}} \right).$$

Finally, it is clear that for N great enough $\delta\,(N) < \frac{1}{16}$. This implies that the squeezing property is satisfied and then from Theorem 2 the existence of an exponential attractor follows.

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