ITERATION OF CONTINUOUS FUNCTIONS AND DYNAMICS OF SOLUTIONS FOR SOME BOUNDARY VALUE PROBLEMS

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To the memory of Professor György Targonski

Abstract. It is known there are boundary value problems investigation of which is directly reduced to study of iteration of functions. This allows to carry on detailed and deep analysis of original problems. We consider such a class of simple (in form) problems, solutions of which demonstrate, nevertheless, very complicated behavior, make possible the simulation of self-birthing structures, including self-similar ones, and self-stochasticity phenomenon.

1. At present time, the iteration theory of continuous functions on the real line as a part of the general dynamical system theory takes up a peculiar and very important place in the contemporary theory of dynamical systems and, especially, in its applications to investigation of the surroundings, laws governing the nature.

While on the subject of iteration theory, it is customary to bear in mind a great variety of problems, which occur or might occur in studies on the iterations of functions (each therefore has its range of values to be contained in its domain of definition) depending for the most part on one real or complex variable, although it is not unusual to deal with more general functional spaces. This is how iteration theory is treated in Gy.Targonski's book [1].

Very popular now is the branch of iteration theory, which is referred to as the theory of one-dimensional dynamical systems. As in the case of arbi-

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trary dynamical systems, the main subject of investigation for this theory is trajectories and their properties. Several reasons for this can be pointed out.

First, one-dimensinal dynamical systems (1-DS) have a broad range of dynamical behavior – from stable stationary points to the case where the completely deterministic behavior of trajectories does not differ in practice from random processes; from the viewpoint of the descriptive theory of sets, they are as much complicated as dynamical systems on arbitrary compact sets [2].

Second, due to a relative simplicity of the "phase space" and "control laws", the one-dimensional dynamics allowed in sufficiently short time to create a developed theory, which is rich in deep results and efficient criteria illustrated by simple and visual computer experiments. Here only one example: the statement "topological entropy is positive" is equivalent to each of almost fifty (!) different statements [3].

Third, for wide circles of investigators 1-DS are the matter of a certain phenomenological interest. Namely, they help to understand general rules of initiation and development of real dynamic processes in the range from the simplest ones to chaotic and even turbulent processes. The theory of 1-DS is one of the main elements of chaotic dynamics, and now it is an obligatory part of textbooks on the theory of dynamical systems.

It should be noted that solely 25-30 years ago, we had completely another relation to the iteration theory: the majority of mathematicians dealt with the theory of dynamical systems did not consider iteration of functions on the real line as a matter of their attention. However, the last twenty five years were the period of extensive development of various directions in theory of dynamical systems and, in particular, the iteration theory of different classes of functions. Probably, significant merit in these achievements belongs to the European Conferences on the Iteration Theory, among the main initiators in the organization of which was Professor György Targonski and in which the almost all of mathematicians participating actively in the formation of the theory took part at different times.

2. Very promising for the extension of one-dimensional dynamics advances is the application to research on nonlinear boundary value problems of mathematical physics, in particular, to a simulation of self-arising chaotic evolutions in deterministic systems. It is known there are boundary value problems (BVP) investigation of which is directly reduced to study of iteration of functions.

Already more than 15 years, I and my colleagues study such kind of BVP (see [4, 5, 6]), in particular, our report on ECIT'91 [7] was devoted to these problems.

The situation that today takes place with regard to this avenue of investigation closely resembles the above-mentioned situation prevailed in iteration theory 30 years ago. There persists much speculation that when describing intricate phenomena it is impossible to do without very complicated systems of equations. In this connection, I reason the following statement to be true:

We have *much information* about properties of complicated equations, but we have *not sufficiently much information* about properties of simple equations!

It is quite pertinent to cite herein the words of the book "The Feynman Lectures on Physics" by R.P.Feynman, R.B.Leighton and M.Sands, which has already been invoked in our work [5]:

"...the complexities of things can so easy and dramatically escape the simplicity of the equations which describe them. Unaware of the scope of simple equations, man has often concluded that nothing short of God, not mere equations, is required to explain the complexities of the world."

It is the elucidation of what is ahead in research of very simple (in form) nonlinear BVP that our explorations are directed towards. Below, we represent certain typical assertions concerning the simplest class of such nonlinear problems.

3. We consider such a class of BVP, namely,

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + bu, \qquad x \in [0,1], \quad t \in R^+, \quad a > 0, \tag{1}$$

$$u|_{x=1} = f(u)|_{x=0}, \tag{2}$$

where f is given C^1-smooth function.

BVP of the form (1),(2) are simplest nonlinear problems that one can envision. It takes sense to use them as a model for representation of self-structuring and self-stochasticity phenomena because in this case explanations are, probably, most simple.

The problem (1),(2) is reduced to a difference equation: the general solution of (1) is

$$u(x,t) = \mathrm{e}^{-\frac{b}{a}x} w(x+at), \tag{3}$$

where w is an arbitrary C^1-smooth function, and on substituting (3) into the boundary condition (2), we obtain the difference equation with continuous argument

$$w(\tau+1) = \lambda f(w(\tau)), \qquad \lambda = \exp(b/a), \ \tau \in R^+. \tag{4}$$

The initial condition for Probem (1), (2)

(5) $$u|_{t=0} = \varphi(x)$$

implies the initial condition for the equation (4)

(6) $$w(\tau) = \lambda^{\tau}\varphi(\tau) \qquad \text{for} \quad \tau \in [0,1].$$

It is obvious that equation (4) with the condition (6) has a unique C^1-smooth solution, which we denote by w_φ, if and only if

(7) $$\varphi(1) = f(\varphi(0)) \quad \text{and} \quad \varphi'(1) = f'(\varphi(0))\varphi'(0).$$

A failure of the above assumptions leads to that the solution u_φ of Problem (1), (2) and (5) is not C^1-smooth (and is even not continuous if the first of the assumptions (7) breaks down) only along a countable number of characteristics $x + t = m, \ m = 1, 2, \ldots$

Every solution u_φ can be written in the form

(8) $$u_\varphi(x,t) = \mathrm{e}^{(-b/a)x} f_\lambda^n(\tilde{\varphi}(x+t-n)) \quad \text{for } n \le x+t < n+1, n = 0, 1, \ldots,$$

where

(9) $$\tilde{\varphi}(\tau) = \mathrm{e}^{(b/a)\tau}\varphi(\tau)$$

and f_λ^n is for the n-th iteration of function

(10) $$f_\lambda : w \mapsto \lambda f(w).$$

Thus, the behavior of solutions for Problem (1), (2) is determinated by the 1-D map f_λ. It takes sense to call f_λ *the resolvent map* for Problem (1), (2).

4. What in 1-D maps is most useful for BVP? What properties of 1-D maps result in the "chaotization" of solutions for BVP, namely, in self-structuring and self-stochasticity phenomena for such solutions?

Keeping in mind piecewise monotone 1-D maps, which are typical for applications, one can note the following.

Self-structuring in solutions of a BVP is due to the complex structure of the basin of an attracting cycle or cycle of intervals of its resolvent 1-D map. Such a basin, as a rule, is a union of countably many intervals which are sequential preimages (for the resolvent map) of the domain of so-called immediate attraction of the cycle. This explains why time-evolving cascade processes of appearance of structures arise in solutions of BVP and how they carry on. There can be observed two much different situations during a cascade process: either there exist structures that are conserved for "arbitrarily long", or each of the structures breaks down into smaller ones and the structures scales tend to zero with time. The first situation is the case if the resolvent map of BVP has an attracting cycle, the latter does if the map has a cycle intervals.

Structures that are brought about in solutions of a BVP (during cascade processes) are coherent and, moreover, may be found to be self-similar. These properties have their origin in the geometry of the graphs of iterations of 1-D maps. Namely, when n is large enough, the graph of f^n is self-similar at Misiurewicz's points – that is, at repelling periodic points and their preimages. A map f has a countable number of such points if f possesses a cycle of period > 2.

These two properties of the graphs of resolvent map iteration – self-similarity and fractality – are inherited by the graphs of solutions of BVP.

Self-stochasticity occurs in solutions of a BVP when its resolvent map has absolutely continuous invariant measure (a.c.i.m.). If for a map $w \mapsto f(w)$, there exists a.c.i.m. then one can find only the probability of a trajectory point $f^n(w)$ lying in one or another of regions of the phase space when n is large enough (using well-known G. D. Birkhoff's theorem). Such temporal stochasticity of trajectories $w_n = f(w_{n-1}), n = 1, 2, ...$, transforms into a *spatial-temporal stochasticity* of solutions of BVP with resolvent map f. This is because these BVP are reduced to that of the infinite dimensional dynamical system given by the map

$$w(\tau) \mapsto f(w(\tau)), \quad w \in C^1([0,1], R).$$

Thus, we get a continual family of "oscillators": for every fixed $\tau_* \in [0,1]$ there is an "oscillator" of its own, namely, $w(\tau_*) \mapsto f(w(\tau_*))$. All these "oscillators" act independently of one another (although following the same law), and, in case f has a.c.i.m., the deterministic function $w_n(\tau), \tau \in [0,1]$, behave like random functions of τ, given n large enough. Moreover, such deterministic functions tend in special metric as $n \to \infty$ to a certain random process, whose distributions can be described in terms of the invariant measure of the map f.

The well-known Jakobson's theorem states that among maps closed to quadratic ones, maps that have a.c.i.m. are no exception, namely, for a one-parameter family of quadratic-like maps, Lebesgue measure of the set of those values of the parameter such that their corresponding maps possess a.c.i.m. is positive. This fact suggests at once that the same is true for BVP reducible to maps of this type and the phenomenon of spatial-temporal chaotization of their solution is no exception too.

5. Let us continue analysis of Problem (1), (2).

Among the solutions of Problem (1), (2), it is expedient to separate out those independent of t, namely, functions of the form $u(x,t) = \gamma e^{(b/a)x}$ with an appropriate constant γ. We will call such solutions as *trivial*. For Problem (1), (2) the values of γ are roots of the equation $\gamma = e^{b/a} f(\gamma)$.

We will consider properties of Problem (1), (2) depending on parameter b (assuming that parameter a in the equation (1) is fixed).

5.1. For the sake of simplicity assume that

• *f is a C^3-smooth function with $f'' < 0$ and with negative Schwarzian derivative,*

• $f(0) = f(1) = 0$.

Then f is a unimodal function – i.e. there is the only point, say, w_* such that f has a finite extremum at $w = w_*$ – and it attains a maximum at $w = w_*$.

A class of maps of the form (10) is in such conditions typified by the family of quadratic maps $w \mapsto \lambda w(1-w)$. As the parameter b ranges from $-\infty$ to ∞, the map f_λ exhibits all conceivable types of dynamical behavior that are realizable for quadratic maps (as well-known, these last may appear both very simple – similar to linear ones – and as intricate in a certain sense as dynamical systems on locally compact spaces may appear). In particular, there are bifurcation values $b_1 < b_2 < b_\infty < b_*$ known respectively as

– the largest of those b such that (10) has no fixed points on the interval $(0,1)$,

– the largest of those b such that (10) has no cycles of period 2,

– the largest of those b such that (10) has no cycles of period different from $2^i, i = 0, 1, 2, ...$,

– the largest of those b such that (10) maps the interval $[0,1]$ into itself.

It is easily seen that

• For any $b \le b_*$ the map f_λ possesses a bounded invariant interval I_b with ends being the repelling fixed point of f_λ and its preimage; for $b_1 \le b \le b_*$, $I_b = [0,1]$ and for $b \to -\infty$, the length of I_b increases infinitely.

• For $b \le b_2$, the behavior of trajectories $f_\lambda^n(w)$ is very simple, namely, when $n \to \infty$, $f_\lambda^n(w)$ tends to the (unique) attracting fixed point if $w \in int I_b$ and to $-\infty$ if $w \notin I_b$.

• If $b > b_*$, then the map f_λ has no bounded invariant intervals and for all points $w \in R$ outside of a Cantor-like set of measure zero (which belongs to $[0,1]$), the trajectories $f_\lambda^n(w)$ tend to $-\infty$.

Thus the dynamics of the map f_λ can be found to be intricate only on the interval $[0,1]$ and only for $b \in (b_2, b_*]$. Let us put

$$A(f) = (b_2, b_*] \quad \text{and} \quad A_\infty(f) = (b_\infty, b_*].$$

Remind some facts. In the interval $A(f)$, there exists an open dense subset A_{attr} such that for $b \in A_{attr}$, the map f_λ has an attracting cycle of period ≥ 2 and almost every bounded trajectory $f_\lambda^n(u)$ is asymptotically periodic. In the interval $A_\infty(f)$ there exists a subset A_{acim} of positive measure which consists of those values of b such that f_λ has a smooth invariant measure. As a consequence almost every bounded trajectory is everywhere dense on a set that consists of a finite number of intervals (which make up a cycle of intervals) and its behaviour can be described in terms of probability theory.

The topological entropy of the dynamical system specified by the map f_λ is positive if and only if $b \in A_\infty$; similarly, the closure of the set of the points whose trajectories are Lyapunov unstable is of positive fractal dimension if and only if $b \in A_\infty$ (by the fractal dimension is herein meant the so-called "box-counting" dimension, for example).

5.2. Now we can turn back to solutions of Problem (1), (2).

Whatever b, the problem has (in general, two) trivial solutions, namely, $u^1(x,t) = 0$ and $u^2(x,t) = \gamma \mathrm{e}^{-(b/a)x}$, where γ is the nonzero root of the equation $\gamma = f_\lambda(\gamma)$;

For $b \leq b_2$, a solution u_φ is bounded if and only if $\mathrm{e}^{(b/a)x}\varphi(x) \in I_b$ when $x \in [0,1]$. Therewith, if $\mathrm{e}^{(b/a)x}\varphi(x) \in \int I_b$ for all x, then $u_\varphi(x,t)$ tends as $t \to \infty$ uniformly to one of the trivial solutions and thus $u_\varphi(x,t)$ is asymptotically stable.

Inasmuch as the solutions of Problem (1), (2) are continuous functions, it is evident that bounded when $b > b_*$ are trivial solutions only.

So, we have a criterion for the boundedness of the solutions of Problem (1), (2).

THEOREM 1 (on boundedness of solutions). (i) *Problem* (1), (2) *has bounded solutions different from the trivial ones if and only if* $b \leq b_*$.

(ii) *When* $b_1 \leq b \leq b_*$, *a solution* u_φ *is bounded if and only if* $\varphi \in \Phi(b)$, *where*

$$\Phi(b) = \left\{\varphi \in C^1([0,1], R^+) : \quad 0 \leq \varphi(x) \leq \mathrm{e}^{-(b/a)x} \qquad \textit{for} \quad x \in [0,1]\right\}.$$

Hereinafter we will consider the long-time behavior of the bounded solutions of Problem (1), (2) when $b \in A(f)$. If $b \in A(f)$, then all bounded solutions different from the trivial ones oscillate as $t \to \infty$ with nonvanishing amplitude. Besides, therewith all these solutions, along with both the trivial solutions, are Lyapunov unstable with respect to C^1- and C^0-metrics alike.

5.3. Problem (1), (2) induces on the space of initial functions a dynamical system of translations along solutions, namely,

$$\left\{\Phi(b), R^+, S^t\right\}, \tag{11}$$

where $S^t[\varphi](x) = u_\varphi(x,t)$, $\varphi \in \Phi(b)$. Since $u_\varphi(x,t)$ can be written as in (8), $S^t[\varphi](x)$ takes the form

$$S^t[\varphi](x) = \mathrm{e}^{-(b/a)x} f_\lambda^{[x+t]}(\tilde{\varphi}(\{x+t\})) \tag{12}$$

with $[\cdot]$ and $\{\cdot\}$ on the right of (12) being respectively for the integral and fractional parts of a number. *If $b \in A(f)$, then every solution $u(x,t)$ of Problem* (1), (2) *such that $u(x,0)$ belongs to $\Phi(b)$ matches the trajectory that starts with the "point" $\varphi(x) = u(x,0)$ and every trajectory $S^t[\varphi]$ matches the solution that is generated by the initial condition $u(x,0) = \varphi(x)$.*

What are the ω-limit sets of trajectories of Syst.(11)? The space $\Phi(b)$ equipped *a priori* with the C^1-metric is not compact. This causes the trajectories $S^t[\varphi]$ to be noncompact for almost all $\varphi \in \Phi(b)$. As a consequence, their ω-limit sets either are at all empty or, if not so, are noncompact and therefore they do not characterize completely the asymptotic behavior of $S^t[\varphi]$. Thus, we have to complete the phase space $\Phi(b)$ with help of a metric such that the trajectories of Syst.(11) to be compact in a new extended space.

To carry out this approach, we employ two metrics denoted by ϱ^Δ and $\varrho^\#$. The former effects completing the phase space $\Phi(b)$ with upper semicontinuous functions and the latter does with random functions.

The metric ϱ^Δ is defined on the space of upper semicontinuous (in general, multivalent) functions $\zeta : [0,1] \to 2^I$, with I being a bounded interval, in the following way:

$$\varrho^\Delta(\zeta_1, \zeta_2) = \Delta\left(\operatorname{gr}\zeta_1, \operatorname{gr}\zeta_2\right), \tag{13}$$

where $\operatorname{gr}\zeta$ stands for the graph of a function ζ and $\Delta(\cdot,\cdot)$ stands for Hausdorff distance between sets.

The metric $\varrho^\#$ is so constructed that it evaluates the distance between two functions through the distances between all their finite-dimensional distributions locally ensemble-averraged. This makes possible to employ $\varrho^\#$ both for deterministic and random functions. We do not use the metric $\varrho^\#$ in this work and refer interested readers to other our works (see, for example, [7, 5, 6]).

5.4. How can the ω-limit sets $\omega_b[\varphi]$ of trajectories of Syst.(11) be constructed in the spaces $\Phi^\Delta(b)$ that are arrived at by completing $\Phi(b)$ via the metrics ϱ^Δ?

The space $\Phi^\Delta(b)$ is compact and, consequently, every trajectory of Syst.(11) has a compact ω-limit set in $\Phi^\Delta(b)$. Syst.(11) induces on $\Phi^\Delta(b)$ by continuity the dynamical system

$$\{\Phi^\Delta(b), R^+, S^t\}, \qquad \text{where } S^t[\zeta](x) = \mathrm{e}^{-(b/a)x} f_\lambda^{[x+t]}(\tilde{\zeta}(\{x+t\})), \tag{14}$$

which determinates a motion on ω-limit sets of the original system (11).

THEOREM 2 (on long-time behavior of solutions). *Let* $b \in A(f)$. *For every* $\varphi \in \Phi(b)$ *(excepting* $\varphi(x) = \mathrm{e}^{-(b/a)x} \cdot \mathrm{const}$*), the trajectory* $S^t[\varphi]$ *of Syst.*(11) *has in the space* $\Phi^\Delta(b)$ *an* ω*-limit set* $\omega_b[\varphi]$*, which consists of discontinuous upper semicontinuous functions.*

Thus, for all $b \in A(f)$, Syst.(14) has in $\Phi^\Delta(b)$ a global attractor $\mathcal{AT}(b)$ (by which is meant the smallest closed subset of $\Phi^\Delta(b)$, which contains the ω-limit set of almost every trajectory starting in $\Phi(b)$) and this attractor consists of discontinuous functions. It is worthy of note that the set of discontinuity points of each of these functions might appear to be nowhere dense or, conversely, everywhere dense.

THEOREM 3 (on self-similarity and fractal dimension). *Let* $b \in A(f)$ *and* $\varphi \in \Phi(b)$. *The graph of each of the (upper semicontinuous) functions* ζ_t *that belong to* $\omega_b[\varphi] \subset \Phi^\Delta(b)$ *is self-similar at every point* $x = x_*$ *such that* $w_* = \mathrm{e}^{(b/a)x_*}\varphi(x_*)$ *is a Misiurewicz's point of the resolvent map* f_λ. *Therewith the fractal dimension of the graph* $\mathrm{gr}\,\zeta_t$ *is more than* 1 *if and only if* $b \in A_\infty(f)$.

It should be noted that for every self-similarity point which corresponds, by Theorem 3, to a certain cycle of the resolvent map f_λ, the scaling factor of self-similarity at such a point equals the Lyapunov multiplier of the corresponding cycle of f_λ.

To give a more detail description of the solutions behavior, let us restrict ourselves to those functions $\varphi \in \Phi(b)$ such that

the function $e^{(b/a)x}\varphi(x)$ *is different from a constant on any interval from* $[0, 1]$.

The subset of these functions is denoted by $\Phi_*(b)$. It is clear that $\Phi_*(b)$ is everywhere dense in $\Phi(b)$ with respect to the C^1-topology.

THEOREM 4 (on asymptotic periodicity and stability of solutions). *For almost every* $b \in A(f)$ *there exists an integer* $p = p(b) \geq 2$ *such that whatever* $\varphi \in \Phi_*(b)$, *the* ω*-limit set* $\omega_b[\varphi]$ *of the trajectory* $S^t[\varphi]$ *of Syst.* (11) *consists of upper semicontinuous functions that combine in* $\Phi^\Delta(b)$ *into a cycle of period* p/a *of the extended system* (14); *this cycle is stable with respect to the metric* ϱ^Δ.

In particular, if $b \in A_{attr}$ *then every function* $\zeta_t \in \omega_b[\varphi]$, $t \in [0, p/a]$, *is single-valued (and hence, continuous) on the open dense set*

$$D_t(\varphi) = \big\{x \in [0,1]: \text{ the point } w = \tilde{\varphi}(\{x+t\}) \text{ is stable under the map } f_\lambda\big\},$$

and on every subinterval of $D_t(\varphi)$*, the function* $\zeta_t(x)$ *coincides with one of the functions* $\gamma_i e^{-(b/a)x}$, $i = 1, 2, \ldots, p$*, with* $\{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ *being an attracting cycle of the resolvent map* f_λ*.*

This theorem, in particular, implies that for almost all $b \in A(f)$ the attractor $\mathcal{AT}(b)$ of Syst.(11) consists of an uncountable number of periodic trajectories of Syst.(14) with the same period p/a.

If $p = 1$, we arrive at different situation: whatever initial function $\varphi \in \Phi_*(b)$ we take, its corresponding ω-limit set $\omega_b[\varphi]$ consists of a single point – a certain discontinuous function $\zeta^*(x)$. As a result, the attractor $\mathcal{AT}(b)$ is nothing but the only point $\{\zeta^*\}$, which is clearly a fixed point of Syst.(14). We get a classic example of such a situation in the case where the resolvent map turns out to be the chaotic parabola $w \mapsto 4w(1-w)$, $w \in [0,1]$. In this case, $\zeta^*(x)$, as a function from $[0,1]$ in R, is discontinuous at every point and the value of $\zeta^*(x)$ at given $x \in [0,1]$ is the interval $[0, e^{-(b/a)x}]$.

According to Theorem 4, almost all solutions u_φ of almost every boundary value problem of the form (1), (2) with $b \in A(f)$ are asymptotically periodic in t with the same period. If $P_\varphi(x,t)$ is the p/a-periodic in t function given by

$$P_\varphi(x,t) = \xi_{t \bmod p/a}(x) \quad \text{and} \quad \xi_{t \bmod p/a} \in \omega_b[\varphi], \tag{15}$$

then

$$\varrho^\Delta(u_\varphi(x,T), P_\varphi(x,T)) \to 0 \quad \text{as} \quad T \to \infty. \tag{16}$$

Therewith *the gradient catastrophe* happens to almost every solution u_φ sooner or later, namely, for any $K > 0$ there is a $\tau = \tau(\varphi) > 0$ such that when $T > \tau$, in the domain $[0,1] \times [T, T + p/a]$ there exist characteristics – lines of the form $x + t = \text{const}$ – along which the modulus of gradient for a solution u_φ becames greater than K. If for some $\varphi \in \Phi(b)$, the value of a function $\zeta_t \in \omega_b[\varphi]$ is a nontrivial interval at every point $x \in [0,1]$, then the corresponding solution u_φ behaves extremely irregularly and there is no way of telling certainly its values when t is large enough (note that the set of values of b such that almost every $\varphi \in \Phi(b)$ has this property is of positive Lebesgue measure, specifically, this set contains A_{acim}).

To make a description of such "unpredictable" solutions, the metric $\varrho^\#$ is appropriate. The situation that a deterministic system possesses trajectories whose ω-limit sets consist of random functions was given the name *self-stochasticity* [7, 5] and such a trajectory was called self-stochastic. In the previous example, the resolvent map f_λ, being the chaotic parabola, has a smooth invariant measure with the density $\frac{1}{\pi\sqrt{w(1-w)}}$ and, consequently, almost all the trajectories are attracted to the same fixed point of the extended dynamical system, namely, to the point $\{f_\lambda^\#(e^{(b/a)x} \cdot x)\}$ of the extended phase space, where $f_\lambda^\#$ is the purely random process with the uniform (in z) distribution $F_{f_\lambda^\#}(w; z) = \frac{2}{\pi}\, arcsin\, \sqrt{w}$.

For arbitrary dynamical system, the diversity of the long-time behavior of its trajectories is characterized by the topological entropy of the system. If the attractor of the system consists of periodic trajectories with their periods being bounded in common (for Syst.(11) this is so), then the topological entropy of the system on the attractor is equal to zero. Nevertheless, for a finite-dimensional dynamical system with such attractor, its topological entropy is very often found to be positive – to "settle down" on the unstable trajectories of the system (for 1-D maps, such is the case almost always (in the topological sense)), which means that the greatest diversity of the behavior of trajectories is observed not on the attractor of the system (which is responsible for the long-time behavior of almost all trajectories of the system) but on its repellers. As for Syst.(11), its topological entropy $\mathrm{ent}\,(b)$ can be equal to 0 or ∞ only (whilst on the attractor $\mathcal{AT}(b)$, it is always equal to 0). Namely, $\mathrm{ent}\,(b) = 0$ if $b \notin A(f)$ and $\mathrm{ent}\,(b) = \infty$ if $b \in A(f)$. In particular, in case f_λ is the chaotic parabola (then $b \in A(f)$), $\mathrm{ent}\,(b) = \infty$ whereas the attractor $\mathcal{AT}(b)$ consists of a single fixed point. From the aforesaid it may be inferred that it is desirable to use, along with the notion of topological entropy, another (but closely allied) notions, for example, those used in estimating of the capacity of functional spaces (for instance, "ε-capacity" and "ε-entropy" [8]).

5.5. Further features of BVP can be obtained by reference to the so-called "universal" properties of 1-D maps, that involve, in particular, Feigenbaum's constants $\delta = 4.6992\ldots$ and $\alpha = 2.5029\ldots$

THEOREM 5 (on ordering of bifurcations). *If the resolvent map f_λ for Problem* (1), (2) *has no cycles of period n_1 for $b = b'$ and has a cycle of period n_2 for $b = b''$ and $n_1 \prec n_2$, then for any n such that $n_1 \prec n \prec n_2$, there exists an interval $B_n \subset (b', b'')$ such that for $b \in B_n$ and almost every $\varphi \in \Phi(b)$ the ω-limit set $\omega[\varphi]$ is a cycle of period n/a in the space $\Phi^\Delta(b)$.*

Here, the symbol "$\prec$" is used in the sense of the ordering

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \ldots \prec 5 \cdot 2^k \prec 3 \cdot 2^k \prec \ldots$$
$$\prec 7 \cdot 2 \prec 5 \cdot 2 \prec 3 \cdot 2 \prec \ldots \prec 9 \prec 7 \prec 5 \prec 3.$$

It should be noted here that U. Burkart who was just a Ph.D. student of Prof. Gy. Targonski, was among the first to suggest a new proof of the statement that this ordering of natural number characterizes coexistence of the periods of periodical points for iteration of continuous functions [9, 10].

For any $\varphi \in \Phi = \cap_{b\in(-\infty,b_*]}\Phi(b)$, we put

$$\beta_m[\varphi] = \inf\{b : \omega[\varphi] \text{ is a cycle of period } m/a \text{ in } \Phi^\Delta(b)\},$$

$$\beta^*_m[\varphi] = \inf\{b : \omega[\varphi] \text{ is a cycle of period } m/a \text{ in } \Phi^\Delta(b)$$

and every function from $\omega[\varphi]$ is a multi-valued at each $x \in [0,1]\}$.

For almost all $\varphi \in \Phi$, the values $\beta_m[\varphi]$ and $\beta^*_m[\varphi]$ are independent on φ, and we will write β_m and β^*_m instead of $\beta_m[\varphi]$ and $\beta^*_m[\varphi]$. We have, as a consequence of Theorem 5, $\beta_{m_1} < \beta_{m_2}$ if $m_1 \prec m_2$. In addition, it can be showed that $\beta_{6\cdot 2^n} < \beta^*_{2^n} < \beta_{2^n m}$ for any odd $m > 1$.

THEOREM 6 (on rate of bifurcations).

$$\lim_{n\to\infty} \frac{\beta_{2^n} - \beta_{2^{n-1}}}{\beta_{2^{n+1}} - \beta_{2^n}} = \lim_{n\to\infty} \frac{\beta^*_{2^{n-1}} - \beta^*_{2^n}}{\beta^*_{2^n} - \beta^*_{2^{n+1}}} = \lim_{n\to\infty} \frac{\beta_{2^n m} - \beta_{2^{n-1}m}}{\beta_{2^{n+1}m} - \beta_{2^n m}} = \delta$$

for any odd $m > 1$.

We can characterize the long time behaviour of solutions of problem (1), (2) through the use of Feigenbaum's constant α. We refer to a discontinuous T-periodic in t function $P(x,t)\colon [0,1]\times R^+ \to R$ as *piecewise-exponential*, if $Z_T = [0,1]\times[0,T]$ falls into a finite number of subsets where $P(x,t)$ equals $\gamma e^{(b/a)x}$ with its own constant γ on each subset.

THEOREM 7 (on approximation of solutions). *Let* $\varphi \in \Phi$. *Whatever* $\varepsilon > 0$, *for every* $b \in (\beta_{2^n}, \beta^*_{2^n})$, *there exist* $2^n/a$*-periodic in* t *piecewise-exponential function* $P^\varepsilon_\varphi(x,t)$ *and a set* $B_\varepsilon \subset Z_{2^n/a}$ *with* $\operatorname{mes} B_\varepsilon < \varepsilon$ *such that for solutions of problem* (1), (2), *the relation* $d_n/d_{n+1} \to \alpha (= 2.502...)$ *holds if* $n \to \infty$ *where*

$$d_n = \sup_{b\in(\beta_{2^n},\beta^*_{2^n})} \limsup_{i\to\infty} \sup_{(x,t)\notin B_\varepsilon} |u_\varphi(x, t + i\, 2^n/a) - P^\varepsilon_\varphi(x,t)|.$$

All the theorems presented, owing to the reduction of Problem (1), (2) to the difference equation (4), result directly from the properties of dynamical systems induced by difference equations of the form $w(t+1) = h(w(t))$, $t \in R^+$, whose properties are in turn determined by the dynamics of the 1-D map $w \mapsto h(w)$.

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