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ON WRIGHT–CONVEX STOCHASTIC PROCESSES

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Abstract. Some characterizations of Wright-convex stochastic processes are presented.

In 1974 B. Nagy [1] considered additive stochastic processes and in 1980 K. Nikodem [3] obtained some properties of convex stochastic processes which are a generalization of properties of convex functions. The subject of this paper is a characterization of Wright-convex stochastic processes. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(a,b) \subset \mathbf{R}$ be an interval.

We say that a stochastic process $X : (a,b) \times \Omega \to \mathbf{R}$ is

a) convex if

$$X(\lambda s + (1-\lambda)t, \cdot) \leq \lambda X(s, \cdot) + (1-\lambda)X(t, \cdot) \qquad \text{(a.e.)}$$

for all $s, t \in (a,b)$ and $\lambda \in [0,1]$,

b) λ-convex (where λ is a fixed number from $(0,1)$) if

$$X(\lambda s + (1-\lambda)t, \cdot) \leq \lambda X(s, \cdot) + (1-\lambda)X(t, \cdot) \qquad \text{(a.e.)}$$

for all $s, t \in (a,b)$,

c) Wright-convex (W-convex) if

$$X(\lambda s + (1-\lambda)t, \cdot) + X((1-\lambda)s + \lambda t, \cdot) \leq X(s, \cdot) + X(t, \cdot) \qquad \text{(a.e.)}$$

for all $s, t \in (a,b)$ and $\lambda \in [0,1]$.

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A stochastic process $A: \mathbf{R} \times \Omega \to \mathbf{R}$ is called additive if

$$A(s+t,\cdot) = A(s,\cdot) + A(t,\cdot) \qquad \text{(a.e.)},$$

for all $s, t \in \mathbf{R}$.

Let us denote by
C_λ – the set of all λ-convex stochastic processes,
C – the set of all convex stochastic processes,
W – the set of all W-convex stochastic processes.

PROPOSITION 1.

$$C \subset W \subset C_{1/2}.$$

This fact is obvious.

PROPOSITION 2.

$$C \subset C_\lambda \subset C_{1/2}, \qquad \textit{for all} \quad \lambda \in (0,1).$$

PROOF. The first inclusion is trivial.

To prove the second one assume that $X \in C_\lambda$ and take arbitrary points $s, t \in (a,b)$. Since X is λ-convex and

$$\frac{s+t}{2} = \lambda\left(\lambda\frac{s+t}{2} + (1-\lambda)s\right) + (1-\lambda)\left(\lambda t + (1-\lambda)\frac{s+t}{2}\right),$$

we get

$$X\left(\frac{s+t}{2},\cdot\right) \leq \lambda\left(\lambda X\left(\frac{x+t}{2},\cdot\right) + (1-\lambda)X(s,\cdot)\right)$$
$$+ (1-\lambda)\left(\lambda X(t,\cdot) + (1-\lambda)X\left(\frac{s+t}{2},\cdot\right)\right) \qquad \text{(a.e.)}.$$

Hence

$$X\left(\frac{s+t}{2},\cdot\right) \leq \frac{X(s,\cdot) + X(t,\cdot)}{2} \qquad \text{(a.e.)},$$

which ends the proof. □

The following proposition is due to K. Nikodem (cf. [3], Lemma 1).

PROPOSITION 3.

$$C_{1/2} \subset C_\lambda, \qquad \textit{for all} \quad \lambda \in (0,1) \cap \mathbf{Q}.$$

By Proposition 2 and 3 we obtain

REMARK.
$$C_{1/2} = C_\lambda, \qquad \text{for all} \quad \lambda \in (0,1) \cap \mathbb{Q}.$$

Now we shall prove the main result of this paper.

THEOREM. *Let $X : (a,b) \times \Omega \to \mathbb{R}$ be a stochastic process. The following conditions are equivalent:*

a) *X is W-convex,*

b) *X is $\frac{1}{2}$ - convex and*

(1)
$$X(\lambda s + (1-\lambda)(t,\cdot) + X((1-\lambda)s + \lambda t, \cdot) \leq 2\max\{X(s,\cdot), X(t,\cdot)\} \qquad \text{(a.e.)}$$

for all $s,t \in (a,b)$ and $\lambda \in [0,1]$,

c) *there exist an additive stochastic processes $A : \mathbb{R} \times \Omega \to \mathbb{R}$ and a convex stochastic process $Y : (a,b) \times \Omega \to \mathbb{R}$ such that*

$$X(t,\cdot) = A(t,\cdot) + Y(t,\cdot) \qquad \text{(a.e.)},$$

for all $t \in (a,b)$,

d) *X is $\frac{1}{2}$ - convex and there exists a concave stochastic process Y ($-Y$ is convex) such that $X + Y$ is $\frac{1}{2}$ - concave.*

PROOF. Implication a) $\Rightarrow$ b) is trivial.

For the proof of implication b) $\Rightarrow$ c) let us fix points $p,q \in (a,b)$, $p < q$. Let $t \in [p,q]$. Then there is a number $\lambda \in [0,1]$ such that

$$t = \lambda p + (1-\lambda)q.$$

Since

$$p + q - t = (1-\lambda)p + \lambda q$$

we get, by (1),

$$X(t,\cdot) + X(p+q-t,\cdot) \leq 2\max\{X(p,\cdot), X(q,\cdot)\} \qquad \text{(a.e.)},$$

and hence

$$X(t,\cdot) \leq -X(p+q-t,\cdot) + 2\max\{X(p,\cdot), X(q,\cdot)\} \qquad \text{(a.e.)}.$$

Thus the $\frac{1}{2}$ - convex stochastic process X is bounded from above on the interval $[p,q]$ by a $\frac{1}{2}$ - concave stochastic process. This implies (cf. [5], Theorem 4) that there exist an additive stochastic process $A : \mathbb{R} \times \Omega \to \mathbb{R}$ and a convex stochastic process $U : (p, q) \times \Omega \to \mathbb{R}$ such that

$$X(t, \cdot) = A(t, \cdot) + U(t, \cdot) \qquad \text{(a.e.)},$$

for all $t \in (a, b)$.

Now, let

$$Y(t, \omega) := X(t, \omega) - A(t, \omega), \qquad t \in (a, b),\ \omega \in \Omega.$$

Of course, Y is $\frac{1}{2}$ - convex on (a, b) and it is also convex on the interval (p, q). Therefore, for arbitrary fixed $r, s \in (p, q)$, $r < s$, and every $t \in (r, s)$ we have

$$Y(t, \cdot) \leq |Y(r, \cdot)| + |Y(s, \cdot)| \qquad \text{(a.e.)},$$

which implies that Y is P–upper bounded on (r, s). Hence Y is continuous on (a, b) and, consequently, convex on (a, b) (cf. [3], Theorems 4 and 5). Thus

$$X(t, \cdot) = A(t, \cdot) + Y(t, \cdot), \qquad \text{(a.e.)}$$

for all $t \in (a, b)$, where A is additive and Y is convex.
Implication c) $\Rightarrow$ a) is clear.
Equivalence c) $\Leftrightarrow$ d) is proved in [5, Theorem 3].
This completes the proof. □

REMARK. An analogous characterization of Wright–convex functions was obtained by K. Nikodem [4] (cf. also C. T. Ng [2]).

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