Annales Mathematicae Silesianae 9. Katowice 1995, 43–45

Prace Naukowe Uniwersytetu Śląskiego nr 1523

# A NOTE ON THE FRÉCHET THEOREM

ANDRZEJ NOWAK

**Abstract.** We give conditions under which every measurable function is the limit almost everywhere of a sequence of continuous functions.

The following classical result is well known (see e.g. [7, p.110]):

FRÉCHET THEOREM. *Let $E$ be a Lebesgue subset of $\mathbb{R}^m$, and $f$ a Lebesgue measurable extended real-valued function on $E$. Then there exists a sequence of finite continuous functions which converges to $f$ almost everywhere in $E$.*

Recently the problem of the approximation of measurable functions by continuous ones was studied in [8]. In this note we present some more general results.

Let $T$ be a topological space and let $\mu$ be a measure defined on a $\sigma$–algebra $\mathcal{T}$ of subsets of $T$ such that $\mathcal{T}$ contains $\mathcal{B}(T)$, the Borel sets of $T$. We say that $\mu$ is *regular* if for each $A \in \mathcal{T}$ and each $\epsilon > 0$ there exist closed $F$ and open $G$ such that $F \subset A \subset G$ and $\mu(G \setminus F) < \epsilon$ (cf. [2, p.7]).

We shall use a general version of the Lusin theorem given in [9].

LUSIN THEOREM. *Let $(T, \mathcal{T}, \mu)$ be a space of a regular measure, and $X$ a topological space with a countable base. If $f : T \to X$ is measurable, then for each $\epsilon > 0$ there exists a closed set $T_\epsilon \subset T$ such that $\mu(T \setminus T_\epsilon) < \epsilon$ and $f \mid T_\epsilon$ is continuous.*

The following theorem is the main result of the paper. Its proof is rather standard (cf. the proof of the Fréchet theorem in [7]).

THEOREM 1. *Let $(T, \mathcal{T}, \mu)$ be a space of a regular measure with $T$ metrizable; let $X$ be a locally convex, separable and metrizable space. Then for*

AMS (1991) subject classification: Primary 28C15; Secondary 28A20.

*each measurable function $f : T \to X$ there exists a sequence of continuous functions $f_n : T \to X$, $n \in \mathbb{N}$, convergent to $f$ $\mu$–a.e.*

PROOF. By the Lusin theorem, for each $n \in \mathbb{N}$ there is closed $F_n \subset T$ such that $\mu(T \setminus F_n) < 1/n$ and $f \mid F_n$ is continuous. Let $A_n = F_1 \cup \ldots \cup F_n$. Then $A_n \subset A_{n+1}$ and $f \mid A_n$ is continuous. By the Dugundji extension theorem (see e.g. [4, p.92]), $f \mid A_n$ has a continuous extension $f_n : T \to X$. Let $A = \bigcup\{A_n : n \in \mathbb{N}\}$. It is immediate that $\mu(T \setminus A) = 0$, and $f_n(t)$ converges to $f(t)$ for each $t \in A$. It completes the proof. □

For real functions we can relax the assumption of the metrizability of $T$.

THEOREM 2. *Let $(T, \mathcal{T}, \mu)$ be a space of a regular measure, where $T$ is normal. Each measurable extended real-valued function $f$ on $T$ is the limit $\mu$ -a.e. of a sequence of continuous functions $f_n : T \to \mathbb{R}$, $n \in \mathbb{N}$.*

PROOF. Let $h$ be a homeomorphism of the extended real line and the interval $[-1, 1]$, and let $g = h \circ f$. There exists a sequence of continuous functions $g_n : T \to [-1, 1]$, $n \in \mathbb{N}$, which converges to $g$ $\mu$–a.e. It can be constructed in the same way as in the previous proof, but with the use of the Tietze–Urysohn extension theorem. Now we define $f_n(t) = \max\{-n, \min\{h^{-1}(g_n(t)), n\}\}$, $n \in \mathbb{N}$, $t \in T$. It is obvious that the functions $f_n$ are finite, continuous and converge $\mu$–a.e. to $f$. □

REMARKS. **1.** If $\mu$ is a regular measure then its completion is also regular. Hence, in both theorems it suffices to assume that $f$ is measurable with respect to $\mathcal{T}_\mu$, the completion of $\mathcal{T}$. Note that if $f$ is the $\mu$–a.e. limit of a sequence of continuous functions, then $f$ is $\mathcal{T}_\mu$–measurable.

**2.** If $T$ is a Polish space and $\mathcal{T} = \mathcal{B}(T)$, then the assumption of the separability of $X$ in Theorem 1 is superfluous. In fact, for every Borel function $f$ from such $T$ into a metric space the range $f(T)$ is separable (see [3, p.164]).

**3.** There is a less known extension theorem which says that a continuous function from a closed subset of a normal space $T$ into a separable Banach space can be extended to a continuous function on $T$. It follows from the result of Hanner [6] and the Dugundji extension theorem. By an application of this result we obtain a variant of Theorem 1 with $T$ normal and $X$ separable Banach.

**4.** It is well known that a finite Borel measure on a metric space is regular (see e.g. [2, Th.1.1]). More general, if $\mu$ is a measure on $\mathcal{B}(T)$, where $T$ is metrizable and can be represented as the union of countably many open sets of finite measure, then $\mu$ is regular (see [5, p.61]). It implies that the $m$-th dimensional Lebesgue measure is regular (cf. also Remark 1).

We conclude this paper with a variant of Theorem 1 for Baire measures. The smallest $\sigma$-algebra in a topological space $T$ with respect to which all continuous real-valued functions are measurable is called the $\sigma$–algebra of *Baire sets in* $T$, and denoted by $\mathcal{B}_o(T)$. Always $\mathcal{B}_o(T) \subset \mathcal{B}(T)$; if $T$ is metrizable then these two $\sigma$-algebras are equal.

THEOREM 3. *Let $T$ be a normal topological space, $\mu$ a finite measure on $\mathcal{B}_o(T)$, and $X$ a separable Banach space. Each $\mathcal{B}_o(T)$–measurable function $f : T \to X$ is the limit $\mu$–a.e. of a sequence of continuous functions.*

PROOF. A finite measure $\mu$ on $\mathcal{B}_o(T)$ is regular in this sense, that for every Baire set $A \subset T$ and every $\epsilon > 0$ there exist closed $F$ and open $G$ such that $F, G \in \mathcal{B}_o(T)$, $F \subset A \subset G$ and $\mu(G \setminus F) < \epsilon$ (see [1, Cor.7.2.1]). Thus we have a version of the Lusin theorem with $\mathcal{T} = \mathcal{B}_o(T)$ and such $\mu$. Now we argue in the same way as in the proof of Theorem 1, using the extension theorem from Remark 3.

**Acknowledgement.** This research was supported by the Silesian University Mathematics Department (Iterative Functional Equations program).

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MATHEMATICS DEPARTMENT
SILESIAN UNIVERSITY
40-007 KATOWICE
POLAND