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STABILITY OF A SYSTEM OF GENERALIZED TRIGONOMETRIC EQUATIONS

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Abstract. Addition formulas for generalized trigonometric functions corresponding to a given symmetric bounded and convex planar set containing the origin as an inner point are derived. Connections with the theory of characters on (semi) groups are considered. Hyers–Ulam stability of a suitable system of functional equations is investigated. It is also shown that superstability phenomenon fails to hold for that system.

Let S be the boundary of a planar convex bounded set F, symmetric with respect to zero and such that $0 \in \operatorname{Int} F$. By Minkowski Theorem there exists a norm in $\mathbb{R}^2$ such that S is the unit sphere corresponding to this norm. We denote that norm by $\|\cdot\|$. We define "new" trigonometric functions Cos and Sin in a way analogous to that used to define the usual functions cos and sin with the aid of the unit circle. Namely, we proceed as follows: since arbitrary half–line having the beginning in zero cuts the sphere S in exactly one point p, the first and second coordinate of p will be called the Cosinus and Sinus of the argument x of the point p, respectively. Now, we can find addition formulas for functions Cosinus and Sinus, which coincide with the well–known formulas in the case where S is the unit circle at the Euclidean plane.

In the sequel, we denote by $|\cdot|$ the usual Euclidean norm in $\mathbb{R}^2$. Obviously, both $\|\cdot\|$ and $|\cdot|$ norms are understood as norms in the linear space $\mathbb{C}$ of all complex numbers over the field $\mathbb{R}$ of reals. Moreover, $(T, \cdot)$ will stand for the multiplicative group $\{z \in \mathbb{C} : z = 1\}$.

By a character on a groupoid $(X, +)$ we mean any homomorphism between X and T.

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1. Let $(X,+)$ be a groupoid with zero. In what follows, at first we shall consider a pair of real-valued functions f,g defined on X in place of the generalized Cos and Sin functions.

THEOREM 1. *Suppose that functions $f,g : X \to \mathbb{R}$ do not vanish simultaneously and $m : X \to \mathbb{C}$ is a function defined by the formula*

$$m(x) = f(x) + ig(x) \tag{1}$$

for all $x \in X$. Then the following conditions are equivalent:
(I) $m(x) \in S$ *and* $\arg m(x+y) = \arg m(x) + \arg m(y)$ *for all* $x,y \in X$,
(II) $m(x) \in S$ *for all* $x \in X$ *and* f,g *satisfy the following system of functional equations:*

$$\begin{cases} f(x+y) = \dfrac{|m(x+y)|\mathrm{Re}(m(x)m(y))}{|m(x)||m(y)|} = \dfrac{|m(x+y)|(f(x)f(y)-g(x)g(y))}{|m(x)||m(y)|} \\ g(x+y) = \dfrac{|m(x+y)|\mathrm{Im}(m(x)m(y))}{|m(x)||m(y)|} = \dfrac{|m(x+y)|(f(x)g(y)+f(y)g(x))}{|m(x)||m(y)|} \end{cases}$$

for all $x,y \in X$;
(III) *the pair (f,g) yields a solution to the system*

$$\begin{cases} f(x+y) = \dfrac{\mathrm{Re}(m(x)m(y))}{\|m(x)m(y)\|} \\ g(x+y) = \dfrac{\mathrm{Im}\ (m(x)m(y))}{\|m(x)m(y)\|} \end{cases}$$

for all $x,y \in X$.

PROOF. First we prove that condition (I) implies (II). From (I) it follows that

$$\begin{aligned} m(x+y) =& |m(x+y)| \exp(i \arg m(x+y)) = \\ & |m(x+y)| \exp(i \arg m(x)) \exp(i \arg m(y)) = \\ & |m(x+y)| \frac{m(x)}{|m(x)|} \frac{m(y)}{|m(y)|} \end{aligned}$$

for all $x,y \in X$. Since $f(x+y) = \mathrm{Re}\ m(x+y)$ and $g(x+y) = \mathrm{Im}\ m(x+y)$ for all $x,y \in X$, we get condition (II).

Now assume that f,g satisfy condition (II). Since $m(x) \in S$ for all $x \in X$, we have $\|m(x)\| = 1$ for all $x \in X$. System (II) implies that

$$\begin{aligned}
1 =& \|m(x+y)\| = \|(f(x+y), g(x+y))\| = \\
& \left\| \left(\frac{|m(x+y)|\mathrm{Re}(m(x)m(y))}{|m(x)||m(y)|}, \frac{|m(x+y)|\mathrm{Im}\ (m(x)m(y))}{|m(x)||m(y)|} \right) \right\| = \\
& \frac{|m(x+y)|}{|m(x)||m(y)|} \|(\mathrm{Re}(m(x)m(y)), \mathrm{Im}\ (m(x)m(y)))\| = \\
& \frac{|m(x+y)|}{|m(x)||m(y)|} \|m(x)m(y)\|
\end{aligned}$$

for all $x, y \in X$. Hence

$$\frac{|m(x+y)|}{|m(x)||m(y)|} = \frac{1}{\|m(x)m(y)\|}$$

for all $x, y \in X$. Therefore f, g satisfy (III).

To prove implication (III) $\Rightarrow$ (I), note that

$$\begin{aligned}
\|m(x+y)\| =& \|(f(x+y), g(x+y))\| = \\
& \left\| \left(\frac{\mathrm{Re}(m(x)m(y))}{\|m(x)m(y)\|}, \frac{\mathrm{Im}\ (m(x)m(y))}{\|m(x)m(y)\|} \right) \right\| = \\
& \frac{1}{\|m(x)m(y)\|} \|(\mathrm{Re}(m(x)m(y)), \mathrm{Im}\ (m(x)m(y)))\| = \\
& \frac{1}{\|m(x)m(y)\|} \|m(x)m(y)\| = 1
\end{aligned}$$

for all $x, y \in X$. Putting $y = 0$ we get $\|m(x)\| = 1$ for all $x \in X$, whence $m(x) \in S$ for all $x \in X$. Moreover, system (III) implies that

$$m(x+y) = \frac{1}{\|m(x)m(y)\|} m(x)m(y)$$

for all $x, y \in X$. Hence

$$\arg m(x+y) = \arg(m(x)m(y)) = \arg m(x) + \arg m(y)$$

for all $x, y \in X$.

This completes the proof. □

THEOREM 2. *Suppose that functions $f, g : X \to \mathbf{R}$ do not vanish simultaneously and $m : X \to \mathbf{C}$ is a function defined by formula (1). Then* *functions f, g satisfy system (III) on X if and only if there exists a character $h : X \to T$ such that*

$$f(x) = \frac{\operatorname{Re} h(x)}{\|h(x)\|}, \quad g(x) = \frac{\operatorname{Im} h(x)}{\|h(x)\|} \tag{2}$$

for all $x \in X$.

PROOF. Let f, g satisfy (III). By Theorem 1 we infer that f, g satisfy condition (II) of Theorem 1.

Let $h : X \to T$ be a function defined by the formula

$$h(x) = \frac{m(x)}{|m(x)|} \tag{3}$$

for all $x \in X$, where $m : X \to \mathbf{C}$ is the function defined by formula (1). From system (II) it follows that h is a character on X. Moreover $m(x) \in S$ for all $x \in X$ and $m = |m|h$. Therefore

$$1 = \|m(x)\| = \||m(x)|h(x)\| = |m(x)|\|h(x)\|$$

for all $x \in X$, whence

$$|m(x)| = \frac{1}{\|h(x)\|}$$

for all $x \in X$. Consequently

$$m(x) = \frac{h(x)}{\|h(x)\|}$$

for all $x \in X$ and, therefore,

$$f(x) = \frac{\operatorname{Re} h(x)}{\|h(x)\|}, \quad g(x) = \frac{\operatorname{Im} h(x)}{\|h(x)\|}$$

for all $x \in X$.

Now, assume that h is a character on X and f, g satisfy condition (2).

Let $m : X \to \mathbb{C}$ be defined by formula (1). Then

$$\begin{aligned}m(x)m(y) =&(f(x)g(y) - g(x)g(y)) + i(f(x)g(y) + f(y)g(x)) = \\ &\frac{\text{Re } h(x)\text{Re } h(y) - \text{Im } h(x)\text{Im } h(y)}{\|h(x)\| \, \|h(y)\|} + \\ &i\frac{\text{Re } h(x)\text{Im } h(y) + \text{Im } h(x)\text{Re } h(y)}{\|h(x)\| \, \|h(y)\|} = \\ &\frac{\text{Re } h(x+y)}{\|h(x)\| \, \|h(y)\|} + i\frac{\text{Im } h(x+y)}{\|h(x)\| \, \|h(y)\|} = \\ &\frac{\|h(x+y)\|}{\|h(x)\| \, \|h(y)\|}\left(\frac{\text{Re } h(x+y)}{\|h(x+y)\|} + i\frac{\text{Im } h(x+y)}{\|h(x+y)\|}\right) = \\ &\frac{\|h(x+y)\|}{\|h(x)\| \, \|h(y)\|}(f(x+y) + ig(x+y)) = \\ &\frac{\|h(x+y)\|}{\|h(x)\| \, \|h(y)\|}m(x+y)\end{aligned}$$

for all $x, y \in X$. Moreover

$$\begin{aligned}\|m(x)\| = \|(f(x), g(x))\| = &\left\|\left(\frac{\text{Re } h(x)}{\|h(x)\|}, \frac{\text{Im } h(x)}{\|h(x)\|}\right)\right\| = \\ &\frac{1}{\|h(x)\|}\|h(x)\| = 1\end{aligned}$$

for all $x \in X$, which implies that

$$\|m(x)m(y)\| = \frac{\|h(x+y)\|}{\|h(x)\| \, \|h(y)\|}\|m(x+y)\| = \frac{\|h(x+y)\|}{\|h(x)\| \, \|h(y)\|}$$

for all $x, y \in X$. Hence

$$m(x+y) = \frac{m(x)m(y)}{\|m(x)m(y)\|}$$

for all $x, y \in X$ and, consequently, f, g satisfy system (III).

This finishes the proof. □

Observe that in the case where X is the additive group of all real numbers and $f = \text{Cos}$, $g = \text{Sin}$, then f, g do not vanish simultaneously and satisfy condition (I) of Theorem 1. In fact, $m(x) = \text{Cos}\, x + i\ \text{Sin}\, x$, $x \in S$ and $\arg m(x) = \{x + 2k\pi : k \in \mathbb{Z}\}$ for all $x \in X$, where $\mathbb{Z}$ stands for the set of all integers. Hence $\arg m(x+y) = \arg m(x) + \arg m(y)$ for all $x, y \in X$. Consequently, Cos and Sin satisfy systems (II) and (III). Moreover, if $S = T$

then $\|\cdot\| = |\cdot|$, Cos and Sin are the usual cos and sin functions, and system (III) reduces to the usual system of trigonometric equations. Therefore, in the sequel, real functions f, g defined on a groupid X with zero and satisfying system (III) will be called the generalized sine and cosine functions.

For example, we can consider the curve $S = \{(a,b) \in \mathbf{R}^2 : \ a^n + b^n = 1\}$, where n is an even positive integer. Then $\|(a,b)\| = \sqrt[n]{a^n + b^n}$; we have considered that case in [1].

2. In this section we shall consider the stability of system (III) of functional equations in the sense of Hyers and Ulam.

In what follows, λ, μ will denote two positive real numbers such that

$$\lambda|p| \leq \|p\| \leq \mu|p| \tag{4}$$

for all $p \in \mathbf{R}^2$. Such numbers do exist since, obviously, the norms $\|\cdot\|$ and $|\cdot|$ are equivalent.

Let $\varepsilon > 0$ be arbitrarily fixed. Suppose that functions $f, g : X \to \mathbf{R}$ do not vanish simultaneously and $m : X \to \mathbf{C}$ is the function defined by formula (1). We shall consider the following system of functional inequalities:

$$\left\{ \begin{array}{l} \left| f(x+y) - \dfrac{\text{Re } (m(x)m(y))}{\|m(x)m(y)\|} \right| < \varepsilon \\ \left| g(x+y) - \dfrac{\text{Im } (m(x)m(y))}{\|m(x)m(y)\|} \right| < \varepsilon \end{array} \right. \tag{(III)$_\varepsilon$}$$

for all $x, y \in X$.

LEMMA 1. *If functions $f, g : X \to \mathbf{R}$ do not vanish simultaneously and satisfy system (III)$_\varepsilon$ of functional inequalities and $h : X \to T$ is the function defined by formula (3), then*

$$|h(x+y) - h(x)h(y)| < 2\sqrt{2}\varepsilon\mu \tag{5}$$

for all $x, y \in X$.

PROOF. System (III)$_\varepsilon$ implies that

$$\begin{aligned} &\big|\|m(x)m(y)\|m(x+y) - m(x)m(y)\big| = \\ &\quad \left(\big|\|m(x)m(y)\|f(x+y) - \text{Re } (m(x)m(y))\big|^2 + \right. \\ &\quad \left. \big|\|m(x)m(y)\|g(x+y) - \text{Im } (m(x)m(y))\big|^2\right)^{\frac{1}{2}} < \\ &\quad (2\|m(x)m(y)\|^2\varepsilon^2)^{\frac{1}{2}} = \sqrt{2}\varepsilon\|m(x)m(y)\| \end{aligned}$$

for all $x, y \in X$. Thus, on account of condition (4), we get

$$|h(x+y)-h(x)h(y)| = \left|\frac{m(x+y)}{|m(x+y)|} - \frac{m(x)}{|m(x)|}\frac{m(y)}{|m(y)|}\right| \leq$$

$$\left|\frac{m(x+y)}{|m(x+y)|} - \frac{\|m(x)m(y)\|m(x+y)}{|m(x)||m(y)|}\right| +$$

$$\left|\frac{\|m(x)m(y)\|m(x+y)}{|m(x)||m(y)|} - \frac{m(x)m(y)}{|m(x)||m(y)|}\right| <$$

$$|m(x+y)|\left|\frac{|m(x)||m(y)| - |m(x+y)|\|m(x)m(y)\|}{|m(x+y)||m(x)||m(y)|}\right| +$$

$$\frac{\sqrt{2}\varepsilon\|m(x)m(y)\|}{|m(x)||m(y)|} \leq \frac{||\|m(x)m(y)\|m(x+y) - m(x)m(y)|}{|m(x)||m(y)|} +$$

$$\frac{\sqrt{2}\varepsilon\|m(x)m(y)\|}{|m(x)||m(y)|} < \frac{2\sqrt{2}\varepsilon\|m(x)m(y)\|}{|m(x)||m(y)|} \leq 2\sqrt{2}\mu\varepsilon$$

for all $x, y \in X$. □

LEMMA 2. *Let H_1, H_2 be two characters on X. If*

$$|H_1(x) - H_2(x)| < \sqrt{3}$$

for all $x \in X$, then $H_1 = H_2$.

PROOF. Let $r : X \to T$ be a function defined by the formula:

$$r(x) = \frac{H_2(x)}{H_1(x)}$$

for all $x \in X$. Then r is a character of X, as well. Moreover

$$|r(x) - 1| = \left|\frac{H_2(x)}{H_1(x)} - 1\right| = \frac{|H_2(x) - H_1(x)|}{|H_1(x)|} = |H_2(x) - H_1(x)| < \sqrt{3}$$

for all $x \in X$. On the other hand

$$|r(x) - 1|^2 = 2 - 2\cos \operatorname{Arg} r(x)$$

for all $x \in X$. In this case $2 - 2\cos \operatorname{Arg} r(x) < 3$ for all $x \in X$ and therefore $\cos \operatorname{Arg} r(x) > -\frac{1}{2}$ for all $x \in X$. Hence

$$\operatorname{Arg} r(x) \in \left(-\frac{2}{3}\pi, \frac{2}{3}\pi\right)$$

for all $x \in X$.

Assume, that there exists an $x_0 \in X$ such that Arg $r(x_0) \neq 0$. If Arg $r(x_0) > 0$, then there exists a positive integer k such that

$$\text{Arg } r(x_0) \in \left\langle \frac{1}{k+1}\frac{2}{3}\pi, \frac{1}{k}\frac{2}{3}\pi \right).$$

Since $r((k+1)x_0) = (r(x_0))^{k+1}$, we have

$$(k+1)\text{Arg } r(x_0) \in \arg r((k+1)x_0) \subset \left(-\frac{2}{3}\pi, \frac{2}{3}\pi\right) + 2\pi\mathbb{Z}.$$

On the other hand one has

$$(k+1)\text{Arg } r(x_0) \in \left\langle \frac{2}{3}\pi, \frac{k+1}{k}\frac{2}{3}\pi \right) \subset \left\langle \frac{2}{3}\pi, \frac{4}{3}\pi \right),$$

which is a contradiction.

Analogously, the assumption Arg $r(x_0) < 0$ leads to a contradiction.

Hence Arg $r(x) = 0$ for all $x \in X$ and therefore $r(x) = 1$ for all $x \in X$. Consequently $H_1 = H_2$ and the proof has been completed. □

REMARK 1. If $H_1, H_2 : X \to T$ are two characters such that $|H_1(x) - H_2(x)| \leq \sqrt{3} + \varepsilon$ for all $x \in X$, where $\varepsilon \geq 0$, then H_1, H_2 may happen to be different as can be seen from the following

EXAMPLE 1. Assume that $(X, +) = \mathbb{Z}_3$. Put $H_1 = 1$, $H_2(0) = 1$, $H_2(1) = \exp(i\frac{2}{3}\pi)$, $H_2(2) = \exp(-i\frac{2}{3}\pi)$. Then H_1, H_2 are characters and $|H_1(x) - H_2(x)| \leq \sqrt{3}$ for all $x \in X$; clearly, $H_1 \neq H_2$.

In the sequel we shall assume that $(X, +)$ is an Abelian group.

LEMMA 3. *Let $\varepsilon \in (0, \sqrt{2})$ be arbitrarily fixed. If a function $k : X \to T$ satisfies inequality:*

$$|k(x+y) - k(x)k(y)| < \varepsilon \tag{6}$$

for all $x, y \in X$, then there exists a pair of functions $H, r : X \to T$ such that

$$k(x) = H(x)r(x) \quad \text{for all} \quad x \in X; \tag{7}$$

$$H \quad \text{is a character of } X; \tag{8}$$

$$\text{(9)} \qquad \text{Arg } r(x) \in \left\langle -\arccos\left(1-\frac{\varepsilon^2}{2}\right), \arccos\left(1-\frac{\varepsilon^2}{2}\right)\right\rangle \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \text{for every } x \in X.$$

Moreover, if $\varepsilon \in (0,1)$, *then such a pair is unique and Arg* $r(x) \in (-\frac{\pi}{3}, \frac{\pi}{3})$ *for all* $x \in X$.

PROOF. Assumption (6) implies that

$$\text{(10)} \qquad \begin{aligned} &\varepsilon^2 > |k(x+y) - k(x)k(y)|^2 = \\ &(k(x+y) - k(x)k(y)(\overline{k(x+y)} - \overline{k(x)}\ \overline{k(y)}) = \\ &2 - 2\text{Re } (k(x+y)\overline{k(x)}\ \overline{k(y)}) \end{aligned}$$

for all $x, y \in X$.

Let $t : X \to \mathbb{R}$ be a function such that

$$t(x) \in \arg k(x)$$

for all $x \in X$. Then

$$k(x) = \exp(it(x))$$

for all $x \in X$ whence

$$\text{Re } (k(x+y)\overline{k(x)}\ \overline{k(y)}) = \text{Re } \exp(i(t(x+y) - t(x) - t(y))) - \cos(t(x+y) - t(x) - t(y))$$

for all $x, y \in X$. This jointly with condition (10) implies that

$$\varepsilon^2 > 2 - 2\cos(t(x+y) - t(x) - t(y))$$

for all $x, y \in X$. Consequently

$$\text{(11)} \qquad \cos(t(x+y) - t(x) - t(y)) > 1 - \frac{\varepsilon^2}{2}$$

for all $x, y \in X$.

Put

$$\delta = \arccos\left(1 - \frac{\varepsilon^2}{2}\right).$$

Condition (11) implies that

$$\text{(12)} \qquad t(x+y) - t(x) - t(y) \in (-\delta, \delta) + 2\pi\mathbf{Z}$$

for all $x, y \in X$.

Let $s : X \to \mathbb{R}$ be a function defined by the formula:

$$s(x) = \frac{t(x)}{2\pi}$$

for all $x \in X$. Putting

$$\eta = \frac{\delta}{2\pi}$$

we observe that $0 < \eta < \frac{1}{4}$. Moreover, by (12)

$$s(x+y) - s(x) - s(y) \in \mathbb{Z} + (-\eta, \eta)$$

for all $x, y \in X$. By Corollary 3 in [2] it follows that there exists a function $p : X \to \mathbb{R}$ such that

(13) $$p(x+y) - p(x) - p(y) \in \mathbb{Z} \quad \text{for all} \quad x, y \in X$$

and

(14) $$|s(x) - p(x)| \leq \eta \quad \text{for all} \quad x \in X.$$

Let $q : X \to \mathbb{R}$, $H, r : X \to T$ be functions defined by the formulas:

$$q(x) = s(x) - p(x)$$
$$H(x) = \exp(i2\pi p(x))$$
$$r(x) = \exp(i2\pi q(x))$$

for all $x \in X$. By (13) we get the equality

$$\frac{H(x+y)}{H(x)H(y)} = \exp(i2\pi(p(x+y) - p(x) - p(y))) = 1$$

for all $x, y \in X$, which says that H is a character of X. However, condition (14) implies that

$$|2\pi q(x)| \leq 2\pi\eta = \delta = \arccos\left(1 - \frac{\varepsilon^2}{2}\right) < \frac{\pi}{2}$$

for all $x \in X$. Consequently

$$\operatorname{Arg} r(x) \in \left\langle -\arccos\left(1 - \frac{\varepsilon^2}{2}\right), \arccos\left(1 - \frac{\varepsilon^2}{2}\right)\right\rangle \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

for all $x \in X$. Note that $\operatorname{Arg} r(x) \in (-\frac{\pi}{3}, \frac{\pi}{3})$ for all $x \in X$, whereas $\varepsilon \in (0,1)$. Since

$$t(x) = 2\pi s(x) = 2\pi p(x) + 2\pi q(x)$$

for all $x \in X$, we have

$$k(x) = \exp(it(x)) = \exp(i2\pi p(x)) \exp(i2\pi q(x)) = H(x)r(x)$$

for all $x \in X$.

Assume that $\varepsilon \in (0,1)$. Let $H_1, H_2, r_1, r_2 : X \to T$ be functions such that both pairs $(H_1, r_1), (H_2, r_2)$ satisfy conditions (7), (8) and (9). Then $\operatorname{Arg} r_i(x) \in (-\frac{\pi}{3}, \frac{\pi}{3})$ for $i = 1, 2$ and for all $x \in X$. Therefore

$$\operatorname{Arg} r_1(x) - \operatorname{Arg} r_2(x) \in \left(-\frac{2}{3}\pi, \frac{2}{3}\pi\right)$$

for all $x \in X$. Consequently

$$\begin{aligned} |H_1(x) - H_2(x)|^2 = \left|\frac{k(x)}{r_1(x)} - \frac{k(x)}{r_2(x)}\right|^2 = \left|\frac{k(x)}{r_1(x)}\right|^2 \left|1 - \frac{r_1(x)}{r_2(x)}\right|^2 = \\ |1 - \exp(i(\operatorname{Arg} r_1(x) - \operatorname{Arg} r_2(x)))|^2 = \\ 2 - 2\cos(\operatorname{Arg} r_1(x) - \operatorname{Arg} r_2(x)) < 3 \end{aligned}$$

for all $x \in X$, whence

$$|H_1(x) - H_2(x)| < \sqrt{3}$$

for all $x \in X$. By Lemma 2 we have $H_1 = H_2$ and, consequently, $r_1 = r_2$. This finishes the proof of the uniqueness of the pair (H, r) satisfying conditions (7), (8), (9) and completes the proof. □

In the sequel, if functions $f, g \in X \to \mathbf{R}$ do not vanish simultaneously and $m : X \to \mathbf{C}$ is the function defined by formula (1), then functions $f_1, g_1 : X \to \mathbf{R}$, $m_1 : X \to \mathbf{C}$ are defined by the formulas:

$$(15)\qquad \begin{cases} f_1(x) = \dfrac{\operatorname{Re}\,(m(x)m(0))}{\|m(x)m(0)\|} \\ g_1(x) = \dfrac{\operatorname{Im}\,(m(x)m(0))}{\|m(x)m(0)\|} \end{cases}$$

$$(16)\qquad m_1(x) = f_1(x) + ig_1(x)$$

for all $x \in X$. Definitions (15) and (16) imply that

$$(17)\qquad \|m_1(x)\| = 1$$

for all $x \in X$. Consequently $m_1(x) \in S$ for all $x \in X$ and functions f_1, g_1 are bounded.

REMARK 2. If functions $f, g : X \to \mathbb{R}$ do not vanish simultaneously and satisfy system $(\mathrm{III})_\varepsilon$ on X, then

$$|f(x) - f_1(x)| < \varepsilon \quad \text{and} \quad |g(x) - g_1(x)| < \varepsilon \tag{18}$$

for all $x \in X$, f, g are bounded and

$$|m(x)| < \sqrt{2}\varepsilon + \frac{1}{\lambda}, \tag{19}$$

$$\|m(x)\| < \sqrt{2}\mu\varepsilon + 1 \tag{20}$$

for all $x \in X$.

PROOF. Setting $y = 0$ in system $(\mathrm{III})_\varepsilon$ we obtain (18). In that case f, g are bounded because so are f_1, g_1. However (17) and (18) imply that

$$|m(x)| \le |m(x) - m_1(x)| + |m_1(x)| \le$$
$$(|f(x) - f_1(x)|^2 + |g(x) - g_1(x)|^2)^{\frac{1}{2}} + \frac{1}{\lambda}\|m_1(x)\| < \sqrt{2}\varepsilon + \frac{1}{\lambda}$$

for all $x \in X$. On the other hand

$$\|m(x)\| \le \|m(x) - m_1(x)\| + \|m_1(x)\| \le \mu|m(x) - m_1(x)| + 1 <$$
$$\sqrt{2}\mu\varepsilon + 1$$

for all $x \in X$.

THEOREM 3. *Let $(X, +)$ be an Abelian group and let $\varepsilon \in (0, \frac{1}{2\mu})$ be arbitrarily fixed. If functions $f, g : X \to \mathbb{R}$ do not vanish simultaneously and satisfy system $(III)_\varepsilon$ on X, then there exists a pair of functions $F, G : X \to \mathbb{R}$ not vanishing simultaneously and satisfying system of functional equations (III) on X with $M = F + iG$ on X instead of m and such that*

$$\|(F(x), G(x)) - (f(x), g(x))\| < \sqrt{2}\mu(1 + 4\delta)\varepsilon \tag{21}$$

and

$$|(F(x), G(x)) - (f(x), g(x))| < \sqrt{2}\delta(3 + 2\delta)\varepsilon \tag{22}$$

for all $x \in X$, where $\delta := \frac{\mu}{\lambda}$. Moreover, if $\varepsilon < \sqrt{6}(4\mu\delta(3+2\delta)(1+\frac{\sqrt{2}}{2}+\delta))^{-1}$, then such a pair (F, G) is unique.

Moreover, if $S = T$, and $\varepsilon \in (0, \frac{1}{2})$ then

$$|(F(x), G(x)) - (f(x), g(x))| < 3\sqrt{2}\varepsilon \tag{23}$$

for all $x \in X$ and the pair (F, G) is unique provided that $\varepsilon < \frac{1}{2\sqrt{6}}$.

PROOF. Let $m : X \to \mathbb{C}$, $h : X \to T$ be the functions defined by formulas (1) and (3). By Lemma 1

$$|h(x+y) - h(x)h(y)| < 2\sqrt{2}\mu\varepsilon$$

for all $x, y \in X$. However Lemma 3 implies that there exist functions $H, r : X \to T$ such that

$$h(x) = H(x)r(x) \qquad \text{for all} \quad x \in X; \tag{24}$$

$$H \quad \text{is a character of} \quad X; \tag{25}$$

$$\begin{gathered}\operatorname{Arg} r(x) \in \Big\langle -\arccos(1 - 4\mu^2\varepsilon^2), \arccos(1 - 4\mu^2\varepsilon^2)\Big\rangle \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \text{for all} \quad x \in X.\end{gathered} \tag{26}$$

Let $F, G : X \to \mathbb{R}$, $M : X \to \mathbb{C}$ be functions defined by the formulas:

$$F(x) = \frac{\operatorname{Re} H(x)}{\|H(x)\|}, \quad G(x) = \frac{\operatorname{Im} H(x)}{\|H(x)\|} \tag{27}$$

$$M(x) = F(x) + iG(x) \tag{28}$$

for all $x \in X$. Obviously

$$M(x) = \frac{H(x)}{\|H(x)\|} \tag{29}$$

for all $x \in X$. Theorem 2 implies that F, G satisfy system (III) on X with m replaced by M. From condition (24) it follows that

$$\arg h(x) = \arg H(x) + \operatorname{Arg} r(x)$$

for all $x \in X$. This jointly with (26) implies that

$$|h(x) - H(x)|^2 = 2 - 2\cos \operatorname{Arg} r(x) \le 2 - 2(1 - 4\mu^2\varepsilon^2) = 8\mu^2\varepsilon^2$$

for all $x \in X$, whence

$$|h(x) - H(x)| \le 2\sqrt{2}\mu\varepsilon \tag{30}$$

for all $x \in X$.

Let $m_0 : X \to \mathbb{C}$ be a function defined by the formula:

$$m_0(x) = \frac{h(x)}{\|h(x)\|} \tag{31}$$

for all $x \in X$. Obviously $\|m_0(x)\| = 1$ for all $x \in X$ and, consequently, $m_0(x) \in S$ for all $x \in X$. Moreover

$$\arg m_0(x) = \arg h(x) = \arg m(x) \tag{32}$$

for all $x \in X$. Moreover, conditions (29), (30) and (31) imply that

$$|m_0(x) - M(x)| = \left| \frac{h(x)}{\|h(x)\|} - \frac{H(x)}{\|H(x)\|} \right| =$$
$$\frac{\big|\|H(x)\|h(x) - \|h(x)\|H(x)\big|}{\|h(x)\|\,\|H(x)\|} \le$$
$$\frac{\big|\|H(x)\|h(x) - \|H(x)\|H(x)\big| + \big|\|H(x)\|H(x) - \|h(x)\|H(x)\big|}{\|h(x)\|\,\|H(x)\|} =$$
$$\frac{\|H(x)\|\,|h(x) - H(x)| + |H(x)|\,\big|\|H(x)\| - \|h(x)\|\big|}{\|h(x)\|\,\|H(x)\|} \le$$
$$\frac{\|H(x)\|\,|h(x) - H(x)| + \frac{1}{\lambda}\|H(x)\|\,\|H(x) - h(x)\|}{\|h(x)\|\,\|H(x)\|} \le$$
$$\frac{|h(x) - H(x)| + \frac{1}{\lambda}\mu|h(x) - H(x)|}{\lambda|h(x)|} \le (1 + \frac{\mu}{\lambda})2\sqrt{2}\mu\varepsilon\lambda^{-1}$$

for all $x \in X$. Putting

$$\delta = \frac{\mu}{\lambda} \tag{33}$$

we obtain

$$|m_0(x) - M(x)| \le 2\sqrt{2}(1 + \delta)\delta\varepsilon \tag{34}$$

for all $x \in X$. On the other hand, conditions (29), (30), (31) and (33) imply that

$$\|m_0(x) - M(x)\| = \left\| \frac{h(x)}{\|h(x)\|} - \frac{H(x)}{\|H(x)\|} \right\| =$$
$$\frac{\big\| \|H(x)\|h(x) - \|h(x)\|H(x) \big\|}{\|h(x)\| \, \|H(x)\|} \le$$
$$\frac{\big\| \|H(x)\|h(x) - \|H(x)\|H(x) \big\| + \big\| \|H(x)\|H(x) - \|h(x)\|H(x) \big\|}{\|h(x)\| \, \|H(x)\|} \le$$
$$\frac{\|H(x)\| \, \|h(x) - H(x)\| + \big| \|H(x)\| - \|h(x)\| \big| \, \|H(x)\|}{\|h(x)\| \, \|H(x)\|} \le$$
$$\frac{2\|h(x) - H(x)\|}{\|h(x)\|} \le \frac{2\mu|h(x) - H(x)|}{\lambda|h(x)|} \le 4\sqrt{2}\delta\mu\varepsilon$$

for all $x \in X$. We have

$$\|m_0(x) - M(x)\| \le 4\sqrt{2}\delta\mu\varepsilon \tag{35}$$

for all $x \in X$.

Let $f_1, g_1 : X \to \mathbb{R}$, $m_1 : X \to \mathbb{C}$ be functions defined by formulas (15) and (16). In view of (18) we have

$$|m(x) - m_1(x)| < \sqrt{2}\varepsilon$$

for all $x \in X$ whence

$$\|m(x) - m_1(x)\| < \sqrt{2}\mu\varepsilon \tag{36}$$

for all $x \in X$.

Now, we shall prove that

$$\|m(x) - m_0(x)\| < \sqrt{2}\mu\varepsilon \tag{37}$$

for all $x \in X$.

Note that the equality $\|m_0(x)\| = 1$ for all $x \in X$ and (32) imply that

$$\|m(x) - m_0(x)\| = \big|\|m(x)\| - 1\big| \tag{38}$$

for all $x \in X$.

Suppose that there exists a $y \in X$ such that

$$\|m(y) - m_0(y)\| \ge \sqrt{2}\mu\varepsilon. \tag{39}$$

If $\|m(y)\| < 1$, then conditions (36), (38) and (39) imply that

$$\|m_1(y)\| \leq \|m_1(y) - m(y)\| + \|m(y)\| < \sqrt{2}\mu\varepsilon + 1 - \|m(y) - m_0(y)\| \leq 1,$$

i.e. $\|m_1(y)\| < 1$. However, by (17), we have $\|m_1(y)\| = 1$, a contradiction.

Assume now that $\|m(y)\| > 1$; then by (36), (38) and (39) we obtain

$$\|m_1(y)\| \geq \|m(y)\| - \|m_1(y) - m(y)\| > \|m(y) - m_0(y)\| + 1 - \sqrt{2}\mu\varepsilon \geq 1,$$

a contradiction, again.

The assumption $\|m(y)\| = 1$ jointly with (38) implies that $\|m(y) - m_0(y)\| = 0$ which contradicts (39).

This finishes the proof of inequality (37).

Now, from (33) and (37) it follows that

$$|m(x) - m_0(x)| < \sqrt{2}\delta\varepsilon \tag{40}$$

for all $x \in X$. Finally, conditions (34) and (40) imply that

$$|m(x) - M(x)| < \sqrt{2}\delta\varepsilon + 2\sqrt{2}(1+\delta)\delta\varepsilon = \sqrt{2}\delta(3+2\delta)\varepsilon$$

for all $x \in X$. Moreover, conditions (35) and (37) imply that

$$\|m(x) - M(x)\| < \sqrt{2}\mu\varepsilon + 4\sqrt{2}\delta\mu\varepsilon = \sqrt{2}\mu(1+4\delta)\varepsilon$$

for all $x \in X$. This proves that the functions F, G satisfy conditions (21) and (22).

Let $\varepsilon \in (0, \sqrt{6}(4\mu\delta(3+2\delta)(1+\frac{\sqrt{2}}{2}+\delta))^{-1})$. We shall show that there exists exactly one pair $F, G : X \to \mathbb{R}$ of functions satisfying system (III) on X and conditions (21) and (22).

Let (F_1, G_1), (F_2, G_2) be two pairs of real functions on X satisfying (III) on X along with (21) and (22).

Let

$$M_j(x) = F_j(x) + iG_j(x)$$

for all $x \in X$ and $j = 1, 2$. On account of Theorem 2, there exist characters $H_1, H_2 : X \to T$ such that

$$M_j(x) = \frac{H_j(x)}{\|H_j(x)\|}$$

for all $x \in X$, $j = 1, 2$. Then

$$\|M_j(x)\| = 1 \quad \text{and} \quad |M_j(x)| = \frac{1}{\|H_j(x)\|}$$

for all $x \in X$, $j = 1,2$. Hence

$$H_j(x) = \frac{M_j(x)}{|M_j(x)|}$$

for all $x \in X$, $j = 1,2$. Moreover, by (22),

$$|M_j(x) - m(x)| < \sqrt{2}\delta(3+2\delta)\varepsilon$$

for all $x \in X$, $j = 1,2$, and therefore,

$$\begin{aligned}
&|H_1(x) - H_2(x)| = \left|\frac{M_1(x)}{|M_1(x)|} - \frac{M_2(x)}{|M_2(x)|}\right| \leq \\
&\left|\frac{M_1(x)}{|M_1(x)|} - \frac{m(x)}{|M_1(x)|}\right| + \left|\frac{m(x)}{|M_1(x)|} - \frac{m(x)}{|M_2(x)|}\right| + \\
&\left|\frac{m(x)}{|M_2(x)|} - \frac{M_2(x)}{|M_2(x)|}\right| \leq \frac{\mu|M_1(x) - m(x)|}{\|M_1(x)\|} + \\
(41) \qquad &|m(x)|\frac{||M_2(x)| - M_1(x)||\mu^2}{\|M_1(x)\|\ \|M_2(x)\|} + \frac{\mu|M_2(x) - m(x)|}{\|M_2(x)\|} < \\
&2\sqrt{2}\delta(3+2\delta)\mu\varepsilon + \mu^2|m(x)||M_2(x) - M_1(x)| \leq \\
&2\sqrt{2}\delta(3+2\delta)\mu\varepsilon + \mu^2|m(x)|(|M_2(x) - m(x)| + |m(x) - M_1(x)|) < \\
&2\sqrt{2}\delta(3+2\delta)\mu\varepsilon + 2\sqrt{2}\delta(3+2\delta)\mu^2\varepsilon|m(x)| = \\
&2\sqrt{2}\delta(3+2\delta)\mu\varepsilon(1 + \mu|m(x)|)
\end{aligned}$$

for all $x \in X$. Since $\varepsilon < \frac{1}{2\mu}$, condition (19) implies that

$$1 + \mu|m(x)| < 1 + \mu(\sqrt{2}\varepsilon + \frac{1}{\lambda}) < 1 + \frac{\sqrt{2}}{2} + \delta$$

for all $x \in X$. From here and from (41) we deduce that

$$|H_1(x) - H_2(x)| < 2\sqrt{2}\delta(3+2\delta)(1 + \frac{\sqrt{2}}{2} + \delta)\mu\varepsilon < \sqrt{3}$$

for all $x \in X$. By Lemma 2 one obtains the equality $H_1 = H_2$ which implies that $(F_1, G_1) = (F_2, G_2)$.

Now, assume that $S = T$. Then $\|\cdot\| = |\cdot|$ as well as $\mu = \lambda = \delta = 1$ and $\varepsilon \in (0, \frac{1}{2})$. By (29) and (31) one has $M = H$ and $m_0 = h$. Hence conditions (30) and (40) imply that

$$\begin{aligned}
|M(x) - m(x)| = &|H(x) - m(x)| \leq |H(x) - h(x)| + |m_0(x) - m(x)| < \\
&2\sqrt{2}\varepsilon + \sqrt{2}\varepsilon = 3\sqrt{2}\varepsilon
\end{aligned}$$

for all $x \in X$.

Let (F_1, G_1), (F_2, G_2) be two pairs of real functions on X satisfying (III) on X and such that

$$|(F_j(x), G_j(x)) - (f(x), g(x))| < 3\sqrt{2}\varepsilon$$

for all $x \in X$, $j = 1, 2$. By Theorem 2, there exist characters H_1, H_2 of X such that $H_j(x) = F_j(x) + iG_j(x)$ for all $x \in X$, $j = 1, 2$. Then

$$|H_1(x) - H_2(x)| < 6\sqrt{2}\varepsilon$$

for all $x \in X$. If $\varepsilon < \frac{1}{2\sqrt{6}}$, then $|H_1(x) - H_2(x)| < \sqrt{3}$ for all $x \in X$. By means of Lemma 2 we get $H_1 = H_2$. Consequently $(F_1, G_1) = (F_2, G_2)$ which ends the proof. □

Now, we shall show that system (III) is not superstable, i.e. there exists a solution of system $(\mathrm{III})_\varepsilon$ which deos not satisfy system (III). This is exhibited in the following.

EXAMPLE 2. Suppose that functions $F, G : X \to \mathbb{R}$ do not vanish simultaneously and satisfy system of functional equations (III) and $\epsilon > 0$ is fixed. If $c : X \to \mathbb{R}$ is a function satisfying the following conditions:

(42) $$c(x) > 0 \quad \text{for all} \quad x \in X \quad \text{or} \quad c(x) < 0 \quad \text{for all} \quad x \in X;$$

(43) $$0 < |c(x) - 1| < \lambda\varepsilon \quad \text{for all} \quad x \in X;$$

then the functions $f, g : X \to \mathbb{R}$ such that:

(44) $$f(x) = c(x)F(x), \quad g(x) = c(x)G(x) \quad \text{for all} \quad x \in X$$

satisfy the system of functional inequalities considered but fail to be solution of system (III). Moreover

(45) $$|f(x) - F(x)| < \varepsilon \quad \text{and} \quad |g(x) - G(x)| < \varepsilon$$

for all $x, y \in X$.

PROOF. Let $m, M : X \to \mathbb{C}$ be functions defined by formulas (1) and (28), respectively. From (1), (28) and (44) it follows that

(46) $$m(x) = c(x)M(x)$$

for all $x \in X$. By Theorem 1 we get

$$\|M(x)\| = 1 \tag{47}$$

for all $x \in X$. Conditions (43), (46) and (47) imply that

$$|f(x) - F(x)| = |m(x) - M(x)| = |c(x)M(x) - M(x)| =$$
$$|M(x)||c(x) - 1| < \frac{1}{\lambda}\|M(x)\|\lambda\varepsilon = \varepsilon$$

for all $x \in X$. Analogously,

$$|g(x) - G(x)| < \varepsilon$$

for all $x \in X$ which proves the validity of (45).

Now, we shall show that the functions f, g satisfy system $(\mathrm{III})_\varepsilon$.

Obviously, f, g do not vanish simultaneously. Since F, G satisfy system (III), then conditions (42), (43), (44), (46) and (47) imply that

$$\begin{aligned}
&\left|f(x+y) - \frac{\mathrm{Re}\ (m(x)m(y))}{\|m(x)m(y)\|}\right| = \\
&\left|c(x+y)F(x+y) - \frac{\mathrm{Re}\ (c(x)c(y)M(x)M(y))}{\|c(x)c(y)M(x)M(y)\|}\right| = \\
&\left|c(x+y)F(x+y) - \frac{c(x)c(y)\mathrm{Re}\ (M(x)M(y))}{c(x)c(y)\|M(x)M(y)\|}\right| = \\
&|c(x+y)F(x+y) - F(x+y)| = |F(x+y)||c(x+y) - 1| \leq \\
&|M(x+y)||c(x+y) - 1| < \frac{1}{\lambda}\|M(x+y)\|\lambda\varepsilon = \varepsilon
\end{aligned} \tag{48}$$

for all $x \in X$. Analogously,

$$\left|g(x+y) - \frac{\mathrm{Im}\ (m(x)m(y))}{\|m(x)m(y)\|}\right| = |G(x+y)||c(x+y) - 1| < \varepsilon \tag{49}$$

for all $x, y \in X$. Hence f, g satisfy system $(\mathrm{III})_\varepsilon$.

Finally, we shall prove that the equalities (III) fail to hold for every $(x, y) \in X^2$.

Assume the contrary, i.e. there exists a pair $(u, v) \in X^2$ such that system (III) holds true. From conditions (48) and (49) it follows that

$$|F(u+v)||c(u+v) - 1| = 0 \quad \text{and} \quad |G(u+v)||c(u+v) - 1| = 0.$$

By (43) we get $F(u+v) = 0$ and $G(u+v) = 0$ which is a contradiction because F, G do not vanish simultaneously.

This finishes the proof. □

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