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LINKAGE AND THE BASIC PART OF WITT RINGS

Abstract. We prove that the Yucas quaternionic mapping (considered in [6]) satisfies $B_G = X_3(a)$ for any $a \in B_G$. We also give examples of a quaternionic mapping which satisfies $B_G = X_2(a)$ for any $a \in B_G$ but not (L) and a quaternionic mapping which satisfies $B_G = X_1(a)$ for any $a \in B_G$ but not (L).

1. Introduction. Let $q: G \times G \to Q$ be a *quaternionic mapping* in the terminology of [4]. Recall that this means q is a symmetric bilinear mapping, G, Q are Abelian groups, G has exponent two and contains a distinguished element -1 such that $q(a, -a) = 1$ for every $a \in G$.

If a quaternionic mapping also satisfies

(L) $q(a,b) = q(c,d) \Rightarrow \exists\ x \in G$ with $q(a,b) = q(a,x)$ and $q(c,d) = q(c,x)$

then q is said to be a *linked quaternionic mapping*.

For $a \in G$, let $D\langle 1, a\rangle = \{b \in G\colon q(-a,b) = 1\}$. An element $a \in G$ is said to be *rigid* if $D\langle 1,a\rangle = \{1, a\}$ and a is said to be *birigid* (or 2 – *sided rigid*) if both a and $-a$ are rigid. The set

$$B_G = \{\pm 1\} \cup \{a \in G\colon a \text{ is not birigid}\}$$

is said to be the *basic part of G*.

Carson and Marshall [1] proved that if q is a quaternionic mapping with $|G| < \infty$ then (L) implies

$$(*) \qquad B_G = \pm X_1(a)X_3(a) \cup X_1(a)X_2(a)^2$$

for every $a \in B_G \setminus \{1\}$ where $X_1(a) = D\langle 1, a\rangle$ and

$$X_i(a) = \bigcup \{D\langle 1, -x\rangle\colon 1 \neq x \in X_{i-1}(a)\}, \qquad i \geqslant 2.$$

This result is quite strong and it was thought that perhaps (*) was strong enough to imply (L) when $|G| < \infty$ (see [6]). J. L. Yucas [6] gave an example of quaternionic mapping which satisfies (*) but not (L).

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Observing the known examples, one realizes that the structure of B_G is probably much simpler than Carson-Marshall's result (*). K. Szymiczek [5] considered the structure of the basic part of typical fields with an infinite group of square classes including global fields, all purely transcendential extension fields and subfields of real numbers. In all these cases, except for subfields of $\mathbf{R}$, he has got simply $B_G = X_2$ and for subfields of $\mathbf{R}$, he has proved $B_G = X_2 \cup -X_2$ or $X_3 \cup -X_3$ depending on the sign of the number a we start with.

Iwan and Wowk [2] proved that the basic part of Witt rings R of elementary type is $B_R = X_3(a) \cup -X_3(a)$ for any $a \in B_R$. (We refer the reader to [3] for the definition of Witt ring).

Motivated by this we study the problem whether $B_G = X_3(a) \cup -X_3(a)$ implies (L).

In answer to this question we prove that the Yucas quaternionic mapping (considered in [6]) satisfies $B_G = X_3(a)$ for any $a \in B_G$ (J.L. Yucas proved that $B_G = X_1(a)X_3(a)$ for any $a \in B_G$). We also give examples of a quaternionic mapping which satisfies $B_G = X_2(a)$ for any $a \in B_G$ but not (L) and a quaternionic mapping which satisfies $B_G = X_1(a)$ for any $a \in B_G$ but not (L).

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2. Structure of the Yucas' quaternionic mapping. Let $\hat{q}: G \times G \to Q$ be the Yucas quaternionic mapping considered in [6]. Recall that in his construction G is a finite group of exponent two and of dimension $n \geqslant 4$ over $\mathbf{F}_2$ with basis $A = \{a_1, a_2, \ldots, a_n\}$ and Q is any other group of exponent two and of dimension $n-2$ over $\mathbf{F}_2$ with fixed basis $\{q_2, q_3, \ldots, q_{n-1}\}$. The quaternionic mapping $\hat{q}$ is defined on $A \times A$ by

$$\hat{q}(a_i, a_j) = \begin{cases} q_j & \text{if } i = 1 \text{ and } 2 \leqslant j \leqslant n-1, \\ q_i & \text{if } j = 1 \text{ and } 2 \leqslant i \leqslant n-1, \\ q_{n-j+1} & \text{if } i = n \text{ and } 2 \leqslant j \leqslant n-1, \\ q_{n-i+1} & \text{if } j = n \text{ and } 2 \leqslant i \leqslant n-1, \\ 1 & \text{otherwise} \end{cases}$$

and is extended to $\hat{q}: G \times G \to Q$ by bilinearity. $\hat{q}$ is a quaternionic mapping with $-1 = 1$.

We begin by establishing the following

LEMMA. (i) *If $a \in G \backslash D\langle 1, a_2 a_3 \ldots a_{n-1}\rangle$ then there exists $\hat{a} \in G$ such that $a\hat{a} \in D\langle 1, a\rangle \cap D\langle 1, a_2 a_3 \ldots a_{n-1}\rangle$.*

(ii) *If $a, b \in G \backslash D\langle 1, a_2 a_3 \ldots a_n\rangle$ then there exist $\hat{a}, \hat{b} \in G$ such that $b\hat{b} \in D\langle 1, a\hat{a}\rangle \cap D\langle 1, b\rangle$ and $a\hat{a} \in D\langle 1, a\rangle$.*

Proof. (i) This follows from Step 2 in [6]. However, we need an explicit expression for $\hat{a}$. Let H be the span of $a_2, \ldots, a_{n-1}$; $D\langle 1, a_2 \ldots a_{n-1}\rangle = \{1, a_1 a_n\}H$ by [6]. For $h = a_{i_1} \ldots a_{i_k} \in H$ set $\hat{h} = a_{n-i_1+1} \ldots a_{n-i_k+1}$. We note that $\hat{q}(h, \hat{h}) = 1$, $\hat{q}(a_1, h) = \Pi q_{i_j} = \hat{q}(a_n, \hat{h})$ and similarly $\hat{q}(a_n, h) = \hat{q}(a_1, \hat{h})$. As a result, $\hat{q}(a_1 a_n, h\hat{h}) = \hat{q}(a_1, h)\hat{q}(a_n, \hat{h})\hat{q}(a_n, h)\hat{q}(a_1, \hat{h}) = 1$.

Now $a \in a_1 H \cup a_n H$. Suppose $a = a_1 h$ for some $h \in H$. Set $\hat{a} = a_n \hat{h}$ and note $a\hat{a} = a_1 a_n h\hat{h} \in D\langle 1, a_2 \ldots a_{n-1}\rangle$. Further $\hat{q}(a, a\hat{a}) = \hat{q}(a, \hat{a}) = \hat{q}(a_1, a_n)\hat{q}(h, \hat{h})\hat{q}(a_1, \hat{h})\hat{q}(a_n, h) = 1$ and $a\hat{a} \in D\langle 1, a\rangle$. If $a = a_n h$ then take $\hat{a} = a_1 \hat{h}$.

Note that in either case $a\hat{a} = a_1 a_n h\hat{h}$.

(ii) Choose $\hat{a}$ as in (i) and repeat the construction for b, yielding $b\hat{b} = a_1 a_n k\hat{k}$ for some $k \in H$. We need only check $b\hat{b} \in D\langle 1, a\hat{a}\rangle$. We have shown $\hat{q}(a_1 a_n, h\hat{h}) = 1$. Thus

$$\hat{q}(b\hat{b}, a\hat{a}) = \hat{q}(a_1 a_n, a_1 a_n)\hat{q}(a_1 a_n, h\hat{h})\hat{q}(k\hat{k}, a_1 a_n)\hat{q}(k\hat{k}, h\hat{h}) = 1,$$

as desired.

Now we are ready to prove the following

THEOREM. *$\hat{q}$ is a quaternionic mapping which satisfies $G = X_3(a)$ for every $a \in G$.*

Proof. J. L. Yucas [6] proved that if $a \in D\langle 1, a_2 \ldots a_{n-1}\rangle$ then $G = X_3(a)$. Suppose that $a \notin D\langle 1, a_2 a_3 \ldots a_{n-1}\rangle$. By Lemma (i) there exists $a\hat{a} \in D\langle 1, a\rangle \cap D\langle 1, a_2 \ldots a_{n-1}\rangle$.

Consequently, $a_2 a_3 \ldots a_{n-1} \in D\langle 1, a\hat{a}\rangle$ and $a\hat{a} \in D\langle 1, a\rangle = X_1(a)$ so $a_2 a_3 \ldots a_{n-1} \in X_2(a)$ and $D\langle 1, a_2 a_3 \ldots a_{n-1}\rangle \subset X_3(a)$. Thus it suffices to show that if $b \in G \backslash D\langle 1, a_2 a_3 \ldots a_{n-1}\rangle$ then $b \in X_3(a)$.

By the Lemma there exist $\hat{a}, \hat{b}$ such that $b\hat{b} \in D\langle 1, a\hat{a}\rangle \cap D\langle 1, b\rangle$ and $a\hat{a} \in D\langle 1, a\rangle$, which leads to $b \in D\langle 1, b\hat{b}\rangle$, $b\hat{b} \in D\langle 1, a\hat{a}\rangle$. Since $a\hat{a} \in D\langle 1, a\rangle = X_1(a)$ and $b\hat{b} \in D\langle 1, aa\rangle \subset X_2(a)$ we get $b \in D\langle 1, b\hat{b}\rangle \subset X_3(a)$.

3. Some examples. In this section we give examples showing that

(a) $\qquad B_G = X_2(a)$, for every $a \in B_G$ does not imply (L)

and

(b) $\qquad B_G = X_1(a)$, for every $a \in B_G$ does not imply (L).

EXAMPLE (a). Let G be a group of exponent two of dimension $n \geq 6$ over $\mathbf{F}_2$ with a basis $A = \{a_1, a_2, \ldots, a_n\}$ and let Q be a group of exponent two of dimension $n-4$ over $\mathbf{F}_2$ with basis $\{q_2, q_4, q_5, \ldots, q_{n-3}, q_{n-1}\}$. Define $\bar{q}$ on $A \times A$ by the matrix $[\bar{q}(a_i, a_j)]$:

$$\begin{bmatrix}
1 & q_2 & 1 & q_4 & \cdots & q_{n-3} & 1 & q_{n-1} & 1 \\
q_2 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & q_{n-1} \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
q_4 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & q_{n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
q_{n-3} & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & q_4 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
q_{n-1} & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & q_2 \\
1 & q_{n-1} & 1 & q_{n-3} & \cdots & q_4 & 1 & q_2 & 1
\end{bmatrix}$$

Let us observe that for any $a \in B_G$, $X_2(a) = B_G$. Indeed, in this case we get $D\langle 1, a_3\rangle = D\langle 1, a_{n-2}\rangle = B_G = G$. Thus if $a \in G$ we have $a \in D\langle 1, a_3\rangle$ and $a_3 \in D\langle 1, a\rangle = X_1(a)$. This implies $G = D\langle 1, a_3\rangle = X_2(a)$. Of course, this quaternionic mapping does not satisfy (L) $(\bar{q}(a_1, a_2) = \bar{q}(a_n, a_{n-1}) = q_2$ and there is no $x \in G$ with $\bar{q}(a_1, a_2) = \bar{q}(a_1, x) = \bar{q}(a_n, x) = \bar{q}(a_n, a_{n-1}))$.

EXAMPLE (b). Let G be a group of exponent two of dimension 4 over $\mathbf{F}_2$ with a basis $A = \{a_1, a_2, a_3, a_4\}$ and let Q be a group of exponent two of dimension 4 over $\mathbf{F}_2$ with basis $\{q_2, q_3, q_4, q_5\}$. Define $\hat{q}$ on $A \times A$ by the matrix $[\hat{q}(a_i, a_j)]$:

$$\begin{bmatrix} 1 & 1 & q_2 & q_3 \\ 1 & 1 & q_4 & q_2 \\ q_2 & q_4 & 1 & q_5 \\ q_3 & q_2 & q_5 & 1 \end{bmatrix}$$

and extend $\hat{q}$ to $\hat{q}$: $G \times G \to Q$ by bilinearity.

It is easy to verify that $B_G = \{1, a_1, a_2, a_1 a_2\}$ and $D\langle 1, a_1\rangle = D\langle 1, a_2\rangle = D\langle 1, a_1 a_2\rangle = \{1, a_1, a_2, a_1 a_2\}$. This proves $B_G = X_1(a)$ for every $a \in B_G$. Notice that $\hat{q}(a_1, a_3) = \hat{q}(a_2, a_4) = q_2$ but there is no $x \in G$ with $\hat{q}(a_1, a_3) = \hat{q}(a_1, x) = \hat{q}(a_2, x) = \hat{q}(a_2, a_4)$. Thus (L) is not satisfied.

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