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ON THE ESTIMATIONS OF SOLUTION OF DELATED STOCHASTIC DIFFERENTIAL EQUATIONS

Abstract. The paper refers to the subject of the estimation of the difference between the solutions of two delated stochastic equations, which as a consequence gives also some criterions of the uniqueness for these equations. The results are obtained by using some integral inequalities and applying them to the more general class of equations with local integrable martingales.

Introduction. In this paper we shall discuss the problem of estimate of the difference between the solutions of two delated stochastic differential equations what as a consequence gives us some criterions of uniqueness for these equations. These results generalize some known uniqueness criterions for the Itô's differential equations (see [1], [2]) and give us in particular an estimate of the 2-nd order moment of solution. In the present paper for the proofs we use non-linear integral inequalities (see [4]) and many suggestions from [1].

Definitions and notations. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, and let $(\mathscr{F}_t, t \geqslant 0)$ be an increasing family of sub-σ-fields of $\mathscr{F}$. We assume, as usual, that $\mathscr{F}_0$ contains all the null sets of $\mathscr{F}$ and that the family $(\mathscr{F}_t, t \geqslant 0)$ is continuous from the right.

We shall say that the function f belongs to the set $D([0, T], R)$ where $0 \leqslant T < +\infty$, iff f is finite, right continuous and has finite left limits for all $t \in [0, T]$. By D we denote the set $D((-\infty, 0], R)$.

Process $(x_t, t \geqslant 0)$ is *cadlag* if for almost all ω, the function $t \to x_t(\omega)$ belongs to $D([0, \infty), R)$. Let $\mathscr{M}_2$ denotes the set of all martingales μ_t with respect to the family $(\mathscr{F}_t, t \geqslant 0)$, such that μ_t is cadlag and

$$\sup_{t \geqslant 0} E\mu_t^2 < \infty$$

holds true. We shall say that the process μ_t is *continuous* if, for almost all ω the function $t \to x_t(\omega)$ is continuous. Let $\mathscr{M}_2^c$ be the subset of $\mathscr{M}_2$ containing all continuous martingales. For each $\mu_t \in \mathscr{M}_2$, μ_t is a submartingale and from Meyer's theorem there exists only one integrable process $\langle \mu, \mu \rangle_t$ and a martingale v_t such that

$$\mu_t^2 = v_t + \langle \mu, \mu \rangle_t$$

holds. By $\mathscr{M}_2'$ we denote the class of all martingales μ_t such that $\langle \mu, \mu \rangle_t$ is continuous.

We define an operator θ_t mapping $D((-\infty, T], R) \to D$ such that

$$\theta_t \varphi(s) = \varphi(s+t), \qquad s \leqslant 0.$$

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D is a metric space with pseudometric ϱ generated by seminorm $\|\cdot\|_*$ such that

$$\|\varphi\|_* = \{ \int_{-\infty}^{0} |\varphi(s)|^2 K(ds)\}$$

where $K(\cdot)$ is some measure on the σ-field of Borel sets of $(-\infty, 0]$ such that $K((-\infty, 0)) = 1$.

Estimate of the difference between the solution of two stochastic differential equations. Let $\mu_t \in \mathcal{M}_2^r$. We consider the equation

$$\text{(1)} \qquad dx_t = a(t, \theta_t x)dt + b(t, \theta_t x)d\mu_t, \quad t \in (0, T] \qquad x_t = \varphi_t, \quad t \leqslant 0$$

where φ_t is cadlag and the functions a and b satisfy the following conditions:

1° $a(s, \varphi) = a(s, \varphi, \omega)$ and $b(s, \varphi) = b(s, \varphi, \omega)$ are two operators mapping $[0, T] \times D \times \Omega \to \mathbf{R}$ such that for each $t \leqslant T$ the functions $a: [0, t] \times D \times \Omega \to \mathbf{R}$ and $b: [0, t] \times D \times \Omega \to \mathbf{R}$ are $\mathcal{L}_t \times \mathcal{B}_D \times \mathcal{F}_t$ – measurable where $\mathcal{L}_t$ is σ-field of Borel sets on $[0, t]$, $\mathcal{B}_D$ is σ-field in D generated by cylindric sets in D;

2° for each ω the classes $\{a(t, \cdot), t \in [0, T]\}$ and $\{b(t, \cdot), t \in [0, T]\}$ where $a(t, \cdot)$ and $b(t, \cdot)$ are meant as the functions of the argument φ are uniformly continuous in D with respect to ϱ:

3° for each ω the functions $a(\cdot, \varphi)$ and $b(\cdot, \varphi)$ belong to $D([0, T], \mathbf{R})$.

The operator θ defined by $\varphi_t = \theta_t \psi$ where $\psi \in D((-\infty, T], \mathbf{R})$, $t \in [0, T]$ is a Borel function and for each $\mathcal{B}_D \times \mathcal{L}_t$ – measurable function $g(\varphi, t)$ the mapping $g(\theta_t \varphi, t)$ is a Borel function of argument t (see [2]).

Under the above assumptions about $a(t, \varphi)$ and $b(t, \varphi)$ both integrals

$$\int_0^t a(s, \theta_s x)ds \text{ and } \int_0^t b(s, \theta_s x)d\mu_s, \qquad \mu_t \in \mathcal{M}_2^r$$

exist.

DEFINITION. By a solution of the equation (1) we mean a probability space $(\Omega, \mathcal{F}, P)$ with an increasing family of sub-σ-fields $(\mathcal{F}_t)$ and a family of stochastic processes (x_t, μ_t) defined on it such that

(i) with probability one x_t and μ_t belong to D and $\mu_0 = 0$,
(ii) they are adapted to $\mathcal{F}_t$ for each t,
(iii) μ_t is an integrable martingale,
(iv) (x_t, μ_t) satisfies

$$x_t - x_0 = \int_0^t a(s, \theta_s x)ds + \int_0^t b(s, \theta_s x)d\mu_s, \qquad t \in [0, T] \text{ a.e.,}$$

$$x_t = \varphi_t, \qquad t \leqslant 0.$$

Let us consider two stochastic differential equations: equation (1) and

$$\text{(2)} \qquad d\bar{x}(t) = a(t, \theta_t x)\, dt + \bar{b}(t, \theta_t x)d\mu_t, \qquad t \in [0, T], \qquad \bar{x}(t) = \bar{\varphi}(t), \qquad t \leqslant 0,$$

where $\bar{\varphi}(t) \in D$.

THEOREM 1. *Let us assume that*

1) *the functions* $a, \bar{a}, b, \bar{b}: [0, T] \times D \times \Omega \to \mathbf{R}$ *are Borel functions satisfying the conditions* 1°—3°;

2) *there exists a continuous function* $\Phi: [0, T] \times \mathbf{R}_+ \to \mathbf{R}_+$ *non-decreasing with respect to* $t \in [0, T]$ *such that for every* $(t, x), (t, y)$,

$$(a(t,x) - \bar{a}(t,y))^2 + (b(t,x) - \bar{b}(t,y))^2 \leqslant \Phi(t, \|x-y\|_*^2);$$

3) *for every random variable* $\eta: \Omega \to \mathbf{R}_+$ *such that* $E\eta < \infty$ *the inequality*

$$E\Phi(t, \eta) \leqslant V\Phi(t, E\eta)$$

holds with a constant V;

4) $\sup_{t \leqslant 0} E\varphi_t^2 < \infty$ *and* $\sup_{t \leqslant 0} E\bar{\varphi}_t^2 < \infty$;

5) *the right-hand maximum solution* $M(t; 0, \tilde{\eta})$ *of the non-random differential equation*

$$y' = K\Phi(t, y)$$

where $K = 3(T+U)V$ *through* $(0, \tilde{\eta})$ *exists in the interval* $[0, T]$;

6) *there exist solutions* (x_t, μ_t) *and* $(\bar{x}_t, \mu_t)$ *of the equations* (1) *and* (2) *respectively and*

$$\sup_{t \in [0,T]} Ex_t^2 < \infty, \quad \sup_{t \in [0,T]} E\bar{x}_t^2 < \infty;$$

7) $\mu_t \in \mathcal{M}_2^c$ *such that* $\mathrm{d}\langle \mu, \mu \rangle_t / \mathrm{d}t$ *is bounded* in $[0, T]$.

Then $\sup_{0 \leqslant t \leqslant T} E|x_t - \bar{x}_t|^2 \leqslant M(t; 0, \tilde{\eta}) - H, t \in [0, T]$, $H = \sup_{u<0} E|\varphi_u - \bar{\varphi}_u|^2$, $\tilde{\eta} = 3E(\varphi_0 - \bar{\varphi}_0)^2 + H$.

Proof. Let us consider the difference $x_t - \bar{x}_t$. From the inequality $(a+b+c)^2 \leqslant 3a^2 + 3b^2 + 3c^2$ we have for $t \geqslant 0$

$$E|x_t - \bar{x}_t|^2 \leqslant 3E|\varphi_0 - \bar{\varphi}_0|^2 + 3E\Big(\int_0^t [a(s, \theta_s x) - \bar{a}(s, \theta_s \bar{x})] \, \mathrm{d}s\Big)^2 +$$

$$+ 3E\Big(\int_0^t [b(s, \theta_s x) - \bar{b}(s, \theta_s \bar{x})] \, \mathrm{d}\mu_s\Big)^2.$$

From the assumptions 2), 3), 5), 7) we have

$$I_2 = \int_0^t [b(s, \theta_s x) - \bar{b}(s, \theta_s \bar{x})] \, \mathrm{d}\mu_s \in \mathcal{M}_2^c$$

and

$$E\Big(\int_0^t [b(s, \theta_s x) - \bar{b}(s, \theta_s \bar{x})] \, \mathrm{d}\mu_s\Big)^2 = E\Big(\int_0^t [b(s, \theta_s x) - \bar{b}(s, \theta_s \bar{x})]^2 \mathrm{d}\langle \mu, \mu \rangle_s\Big).$$

By Cauchy's inequality we have

$$E|x_t-\bar{x}_t|^2 \leqslant 3E(\varphi_0-\bar{\varphi}_0)^2+3TE\int_0^t [a(s,\theta_s x)-\bar{a}(s,\theta_s\bar{x})]^2 ds+$$

$$+3E\int_0^t [b(s,\theta_s x)-\bar{b}(s,\theta_s\bar{x})]^2 d\langle\mu,\mu\rangle_s,$$

$$E|x_t-\bar{x}_t|^2 \leqslant 3C(T+U)\int_0^t E\Phi(s, \|\theta_s x-\theta_s\bar{x}\|_*^2)ds \qquad (C, T, U = \text{const}),$$

and

$$(3) \qquad E|x_t-\bar{x}_t|^2 \leqslant 3C+K\int_0^t \Phi(s, E\|\theta_s x-\theta_s\bar{x}\|_*^2)ds, \qquad t\in[0,T].$$

Let $z(t) := \sup\limits_{0\leqslant s\leqslant t} E|x_s-\bar{x}_s|^2$. Then

$$E\|\theta_s x-\theta_s\bar{x}\|_*^2 = E\int_{-\infty}^0 [(x-\bar{x})(s+u)]^2 K(du) \leqslant \{\sup_{u<0} E[(x-\bar{x})(u)]^2+$$

$$+\sup_{0\leqslant u\leqslant s} E[(x-\bar{x})(u)]^2\}\, K(-\infty,0] \leqslant H+z(s),$$

where $H = \sup\limits_{u<0} E|\varphi_u-\bar{\varphi}_u|^2$. Therefore we get from (3)

$$E|x_t-\bar{x}_t|^2 \leqslant 3C+K\int_0^t \Phi(s, z(s)+H)ds, \qquad t\in[0,T],$$

and

$$z(t)+H \leqslant 3C+H \leqslant K\int_0^t (s, z(s)+H)ds, \qquad t\in[0,T].$$

From Opial's theorem we have

$$z(t)+H \leqslant M(t;0, 3C+H), \qquad t\in[0,T],$$

$$\sup_{0\leqslant s\leqslant t} K|x_s-\bar{x}_s|^2 \leqslant M(t;0, 3C+H)-H, \qquad t\in[0,T].$$

REMARK. When in particular $a = \bar{a}$ and $b = \bar{b}$ then Theorem 1 gives us the estimation of the difference between two solutions of the same equation with the different initial functions φ and $\bar{\varphi}_t$.

COROLLARY. *Under the assumptions of Theorem 1 for $a = \bar{a}, b = \bar{b}, \varphi_t = \bar{\varphi}_t$ assume that $M(t;0,0) = 0$ for $t\in[0,T]$. Then the equation (1) has a unique solution.*

When in particular $\Phi(t, y) = a(t)q(y)$, where $a(t)$ is a non-negative continuous function in $\mathbf{R}_+$, and $q(y)$ is a continuous, non-decreasing in $\mathbf{R}$, $q(y) \neq 0$ and $\lim_{\varepsilon \to 0^+} \int_\varepsilon^u \frac{1}{q(s)} ds = +\infty$, $u > \varepsilon$, we have the uniqueness criterion for (1).

In a similar way as in Theorem 1 we have

$$z(t) \leqslant K \int_0^t a(s)q(z(s))ds$$

hence

$$z(t) \leqslant \varepsilon + K \int_0^t a(s)\, q(z(s))\, ds$$

and

$$z(t) \leqslant G^{-1}\left[G(\varepsilon) + K \int_0^t a(s)ds\right],$$

where

$$G(u) = \int_{-\infty}^{u} \frac{1}{q(s)} ds;$$

for $\varepsilon \to 0$ the right term of the last inequality tends to 0. When $\Phi(t, y) = Ly, L > 0$ then in the assumption 2) of Theorem 1 we have well known Lipschitz condition.

Estimation of the error of an approximate solution. Let x_t and $\bar{x}_t$ be two unique solutions of the equations (1) and (2) respectively and let $a, \bar{a}, b, \bar{b}$ be the functions satisfying the conditions 1°—3°. Then we have

THEOREM 2. *Assume that there exist functions* $\Phi, \bar{\Phi}, \tilde{\Phi}: [0, T] \times \mathbf{R}_+ \to \mathbf{R}_+$ *such that*

1) $(a(t, x) - \bar{a}(t, x))^2 + (b(t, x) - \bar{b}(t, x))^2 \leqslant \tilde{\Phi}(t, \|x\|_*^2), \quad (t, x) \in [0, T] \times D;$
2) $(\bar{a}(t, x) - \bar{a}(t, y))^2 + (\bar{b}(t, x) - \bar{b}(t, y))^2 \leqslant \bar{\Phi}(t, \|x - y\|_*^2);$
3) $(a(t, x))^2 + (b(t, x))^2 \leqslant \Phi(t, \|x\|_*^2);$
4) *functions* $\Phi, \bar{\Phi}, \tilde{\Phi}$ *are continuous, non-decreasing with respect to t and for every random variable* $\xi: \Omega \to \mathbf{R}_+$ *such that* $E\xi < \infty$ *the inequalities*

$$E\Phi(t, \xi) \leqslant V\Phi(t, \xi),$$

$$E\bar{\Phi}(t, \xi) \leqslant V\bar{\Phi}(t, \xi),$$

$$E\tilde{\Phi}(t, \xi) \leqslant V\tilde{\Phi}(t, \xi),$$

where $V = \text{const.}$, *are true;*

5) *let* $M(t; 0, \eta)$ *be the right-hand maximum solution of non-random differential equation*

$$y' = \lambda\Phi(t, y)$$

through $(0, \eta)$ *and* $M_1(t; 0,0)$ — *the right-hand maximum solution of equation*

$$y' = K\bar{\Phi}(t, y + \tilde{\mu}(t))$$

through (0,0) *for some constans* K, λ.

$$\tilde{\mu}(t) = C + K \int_0^t \tilde{\Phi}(s, M(t; 0, \eta))\mathrm{d}s \qquad (\textit{where } C, K = \textit{const});$$

6) $\mu_t \in \mathcal{M}_2^c$ *such that* $\mathrm{d}\langle \mu, \mu \rangle_t / \mathrm{d}t$ *is bounded in* $[0, T]$.
Then

$$\sup_{0 \leqslant s \leqslant t} E|x_t - \bar{x}_t|^2 = M_1(t; 0,0) + \tilde{\mu}(t) - H, \qquad t \in [0, T],\ H = \sup_{u < 0} E\varphi_u^2.$$

Proof. In a similar way as in Theorem 1 we have for $t \in [0, T]$

$$E|x_t|^2 \leqslant 3\varphi_0^2 + 3E\Big(\int_0^t a(s, \theta_s x)\mathrm{d}s\Big)^2 + 3E\Big(\int_0^t b(s, \theta_s x)\mathrm{d}\mu_s\Big)^2 \leqslant$$

$$\leqslant \tilde{\eta} + 3T \int_0^t Ea^2(s, \theta_s x)\mathrm{d}s + 3 \int_0^t E\big(b(s, \theta_s x)\big)^2 \mathrm{d}\langle \mu, \mu \rangle_s \leqslant$$

$$\leqslant \tilde{\eta} + 3(T + B)\, V \int_0^t \Phi(s, E\|\theta_s x\|_*^2)\mathrm{d}s$$

but

$$E\|\theta_s x\|_*^2 \leqslant \sup_{0 \leqslant u \leqslant s} Ex_u^2 + \sup_{u < 0} E\varphi_u^2 = z(s) + H,$$

hence

$$\sup_{0 \leqslant s \leqslant t} Ex_s^2 + H \leqslant H + \tilde{\eta} + 3(T + B)V \int_0^t \Phi(s, z(s) + H)\mathrm{d}s,$$

and from Opial's theorem

$$H + z(t) \leqslant M(t; 0, \eta), \qquad t \in [0, T], \tag{4}$$

where $\eta = \tilde{\eta} + H$, and from assumption 4)

$$\tilde{\Phi}(t, z(t)) \leqslant \tilde{\Phi}(t, M(t; 0, \eta)), \qquad t \in [0, T].$$

Let us consider the difference $x_t - \bar{x}_t$ for $t \in [0, T]$. In a similar way as in Theorem 1 we have

$$E|x_t - \bar{x}_t|^2 \leqslant 3E|\varphi_0 - \bar{\varphi}_0|^2 + 3T \int_0^t E[a(s, \theta_s x) - \bar{a}(s, \theta_s \bar{x})]^2 \mathrm{d}s +$$

$$+ 3B \int_0^t E[b(s, \theta_s x) - \bar{b}(s, \theta_s \bar{x})]^2 \mathrm{d}s \leqslant$$

$$\leqslant C + 6(T + B) \int_0^t E\{[a(s, \theta_s x) - \bar{a}(s, \theta_s x)]^2 +$$

$$+ [b(s, \theta_s x) - \bar{b}(s, \theta_s x)]^2 + [\bar{a}(s, \theta_s x) - \bar{a}(s, \theta_s \bar{x})]^2 +$$

$$+ [\bar{b}(s, \theta_s x) - \bar{b}(s, \theta_s \bar{x})]^2\}\, \mathrm{d}s,$$

$$E|x_t - \bar{x}_t|^2 \leqslant C + 6(T+B)\int_0^t E\tilde{\Phi}(s, \|\theta_s x\|_*^2)\mathrm{d}s + 6(T+B)\int_0^t E\Phi(s, \|\theta_s x - \theta_s \bar{x}\|_*^2)\mathrm{d}s.$$

Hence, from the assumptions 4) and 3)

$$E|x_t - \bar{x}_t|^2 \leqslant C + K\int_0^t \tilde{\Phi}(s, M(s; 0, \eta))\mathrm{d}s + K\int_0^t \bar{\Phi}(s, E\|\theta_s x - \theta_s \bar{x}\|_*^2)\mathrm{d}s, \ t \in [0, T]$$

and

$$H + z(t) \leqslant \tilde{\mu}(t) + K\int_0^t \bar{\Phi}(s, z(s) + H)\mathrm{d}s$$

where $H = \sup\limits_{t<0} E|\varphi_t - \bar{\varphi}_t|^2$. The above inequality implies (see [4]) the following:

$$H + \sup_{0 \leqslant s \leqslant t} E|x_t - \bar{x}_t|^2 \leqslant M_1(t; 0,0) + \tilde{\mu}(t), \qquad t \in [0, T]$$

which completes the proof.

Theorem 2 gives us the estimation of error if, instead of the solution of a given system which may be "difficult to solve", we take the solution of an approximate one, which is "easier to solve".

REMARK. Let $\bar{a} = \bar{b} = 0$ for $(t, x) \in [0, T] \times D$ and $\bar{\varphi} = 0$. Then we can assume that $\bar{\Phi} = 0$ and $\tilde{\Phi}(t, y) = \Phi(t, y)$. Hence and from the fact that $M_1(t; 0,0) = 0$ for $t \in [0, T]$ we have the estimation

$$\sup_{0 \leqslant t \leqslant T} E|x_t|^2 \leqslant \tilde{\mu}(t), \qquad t \in [0, T].$$

If, in particular $\tilde{\Phi}(t, y) = K(1 + y)$, we get the well known growth condition. We have in this case

$$\sup_{0 \leqslant t \leqslant T} E|x_t|^2 \leqslant A + B\mathrm{e}^{CT}$$

where A, B, C are constants dependent on H, K, V.

We can generalize above theorems by applying some more general stochastic integrals. Let us consider the equation

$$x_t = x_0 + \int_0^t a(s, \theta_s x)\mathrm{d}s + \int_0^t b(s, \theta_s x)\mathrm{d}w + \int_0^t \int_{\mathbf{R}} c(s, \theta_s x, y)\mu(\mathrm{d}t, \mathrm{d}y), \qquad t \in [0, T],$$

$$x_t = \varphi_t, \qquad t < 0,$$

where $\varphi_t \in D$, w_t is a Wiener process and $\mu(t, A)$ is a measur such that

a) for each $A \in \mathscr{B}$, $\mu(t, A)$ is a Poisson process and $\langle \mu, \mu \rangle_t(A) = v(t, A)$, $Ev(t, A) = tq(A)$, where $q(A)$ is a measure on $\mathscr{B}$, and $\mathscr{B}$ is a σ-field of Borel sets in $\mathbf{R}$.

b) if $B_1 \cap B_2 = \emptyset$ then $\nu(t, B_1 \cup B_2) = \nu(t, B_1) + \nu(t, B_2)$,

c) the classes of random variables $\{\nu(s, A), s \in [0, t], A \in \mathscr{B}\}$ and $\{\nu(s', C) - \nu(t, C), s' > t, C \in \mathscr{B}\}$ are independent for each $t > 0$. The process $\tilde{\nu}(t, A) := \nu(t, A) - tq(A)$ is called *Poisson measure* (see [3]). In this case, if the assumptions 1), 3)—6) of Theorem 1 are satisfied and

$$(a(t,x)-\bar{a}(t,y))^2+(b(t,x)-\bar{b}(t,y))^2+(\int_{\mathbf{R}}[c(t,x,u)-\bar{c}(t,y,u)]^2 q(\mathrm{d}u) \leqslant$$

$$\leqslant \Phi(t, \|x-y\|_*^2)$$

then

$$\sup_{t \leqslant T} E|x_t - \bar{x}_t|^2 \leqslant M(t; 0, \eta), \qquad t \in [0, T].$$

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