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DUALITY PRINCIPLE OF W. SIERPIŃSKI IN THE ABSTRACT BAIRE-CATHEGORY THEORY

Abstract. Let $\mathscr{C}$ be an $\mathfrak{M}$-family of subsets of X and $\mathscr{C}_I$ — the family of its "first category" sets. It is proven that one and only one of the following conditions is satisfied: (*) each $\mathscr{C}_I$-set is at most countable; (**) X is the union of $\mathscr{C}_I$-set and a set having property (L), which are disjoint; (***) each $\mathscr{C}$-residual set contains an uncountable $\mathscr{C}_I$-set.

Moreover, if $\mathscr{C} \subset 2^X$ and $\mathscr{D} \subset 2^Y$ are two $\mathfrak{M}$-families, the "duality principle" holds (i.e. there exists a bijection $f: X \to Y$ transforming $\mathscr{C}_I$-sets onto $\mathscr{D}_I$-sets) iff $\mathscr{C}$ and $\mathscr{D}$ satisfy the same of the conditions above.

Also, some considerations are added, concerning the coincidence between the properties of the family $\mathscr{C}_I$ and a σ — ideal.

1. Introduction. In 1934 W. Sierpiński ([11]) pointed out at some analogies between the sets of first category of Baire and the sets of Lebesgue measure zero, appearing in different contexts before. These analogies were collected in the most elegant way by J. C. Oxtoby in [7] (see also [8], [9]). The cycle of papers of J. C. Morgan II (see [2], [3], [4], [5], [6]), in which the concept of $\mathfrak{M}$-family and, more general, $\mathfrak{K}$-family has been introduced, was an important step on in this direction. This concept has been initially used in some game — theoretical investigations (see [2]), and later as a generalization of the concepts of measure and topology, Baire property ([5]) and the absolute Baire property ([4], [6]), This paper contains the generalization of duality principle of W. Sierpiński and some considerations on relations between concepts of a $\mathfrak{M}$-family and a σ-ideal.

2. Some definitions and theorems. Let X be a nonempty set. The members of any family $\mathscr{A}$ of subsets of X will be called *$\mathscr{A}$-sets.*

DEFINITION 1. A family $\mathscr{C}$ of subsets of X is called an *$\mathfrak{M}$-family* if the following axioms are satisfied:

1. $X = \bigcup \mathscr{C}$;

2. Let A be a $\mathscr{C}$-set and let $\mathscr{D}$ be a nonempty family of disjoint $\mathscr{C}$-sets, which has power less than the power of $\mathscr{C}$;

(a) if $A \cap \bigcup \mathscr{D}$ contains a $\mathscr{C}$-set, then there is a $\mathscr{D}$-set D, such that $A \cap D$ contains $\mathscr{C}$-set,

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(b) if $A \cap \bigcup \mathscr{D}$ contains no $\mathscr{C}$-set, then there is a $\mathscr{C}$-set $B \subset A \setminus \bigcup \mathscr{D}$;

3. The intersection of any descending sequence of $\mathscr{C}$-sets is nonempty;

4. If $x \in X$, then each $\mathscr{C}$-set A contains a $\mathscr{C}$-set B, such that $x \notin B$.

DEFINITION 2. A set $S \subset X$ is ***$\mathscr{C}$-singular*** if each $\mathscr{C}$-set A contains a $\mathscr{C}$-set B disjoint from S. The family of all countable unions of $\mathscr{C}$-singular sets is denoted by $\mathscr{C}_{\mathrm{I}}$. The family of all subsets of X, which are not $\mathscr{C}_{\mathrm{I}}$-sets is denoted by $\mathscr{C}_{\mathrm{II}}$. If X is $\mathscr{C}_{\mathrm{II}}$-set, then the complement of $\mathscr{C}_{\mathrm{I}}$-set is called ***$\mathscr{C}$-residual set***.

Clearly, $\mathscr{C}$-singular sets form an ideal. Hence, $\mathscr{C}_{\mathrm{I}}$ is a σ-ideal.

Below, we present some examples of $\mathfrak{M}$-families.

EXAMPLE 1 (see [5, p. 20]). Let X be an uncountable set and let $\mathscr{C}$ be the family of all sets, whose complement is finite. The $\mathscr{C}$-singular sets, $\mathscr{C}_{\mathrm{I}}$-sets and $\mathscr{C}_{\mathrm{II}}$-sets coincide here with the finite sets, countable sets and uncountable sets, respectively.

EXAMPLE 2 (see [5, p. 20]). Let (X, d) be a complete, separable metric space, with no isolated points, let Q be a countable set, dense in X, and let $\mathscr{C}$ be the family of all closures of open balls $\left\{x \in X : d(x, r) < \frac{1}{n}\right\}$, $r \in Q$, $n = 1, 2, \ldots$ $\mathscr{C}$-singular sets coincide here with the nowhere dense sets, whereas $\mathscr{C}_{\mathrm{I}}$-sets and $\mathscr{C}_{\mathrm{II}}$-sets coincide with the sets of first and second Baire category, respectively.

EXAMPLE 3 (see [5, p. 20]). Let (X, d) be a complete, separable metric space, let μ be a σ-finite, regular measure, defined on σ-field of Borel sets and let $\mathscr{C}$ be the family of all compact sets having positive measure μ. $\mathscr{C}$-singular sets and $\mathscr{C}_{\mathrm{I}}$-sets coincide here with the sets of $\bar{\mu}$-measure zero ($\bar{\mu}$ denotes the completion of μ).

EXAMPLE 4 (see [2, p. 12]). Let $X \subset [0, 1]$ be a perfect set and let $\mathscr{C}$ be the family of all perfect sets of the form $X \cap I$, where I is a closed subinterval of $[0, 1]$. $\mathscr{C}$-singular sets, $\mathscr{C}_{\mathrm{I}}$-sets and $\mathscr{C}_{\mathrm{II}}$-sets coincide here with nowhere dense sets with respect to X, sets of first category with respect to X and the sets of second category with respect to X, respectively.

EXAMPLE 5. Let X be the union of two disjoint, uncountable sets X_1 and X_2. Define

$$\mathscr{C} := \{A \subset X : X_2 \setminus A \text{ is finite}\}.$$

It is easy to see, that $\mathscr{C}$ is the $\mathfrak{M}$-family and X_1 is $\mathscr{C}$-singular set.

EXAMPLE 6 (see [2, p. 13—14]). Let X be the unit interval. The family $\mathscr{C}$ of all perfect subsets of X is the $\mathfrak{M}$-family.

Some other examples of $\mathfrak{M}$-families may be found in [2, § 4].

Unless otherwise specified, the symbol $\mathscr{C}$ will denote a $\mathfrak{M}$-family.

The following lemma is an easy consequence of definitions.

The following facts are simple corollaries of definitions and lemma.

I. All countable sets are $\mathscr{C}_{\mathrm{I}}$-sets.

II. Each $\mathscr{C}$-set is $\mathscr{C}_{\mathrm{II}}$-set.

III. X is $\mathscr{C}_{\mathrm{II}}$-set.

IV. The intersection of a $\mathscr{C}_{\mathrm{II}}$-set and a $\mathscr{C}$-residual set is a $\mathscr{C}_{\mathrm{II}}$-set.

In what follows, $\mathscr{E}$ will denote the family of all sets being complements of $\mathscr{C}$-sets.

THEOREM 1 (see [5, p. 15]). *If* $\mathscr{C}$ *satisfies* CCC (the countable chain condition; see, for instance, [1, § 14]), *then each $\mathscr{C}$-singular set is contained in a $\mathscr{C}$-singular $\mathscr{E}_\delta$-set, and each $\mathscr{C}_{\mathrm{I}}$-set is contained in an $(\mathscr{E}_{\delta\sigma}\cap\mathscr{C}_{\mathrm{I}})$-set.*

3. Duality principle. In 1934 W. Sierpiński in [11] proved the existence of the bijective function $f\colon \boldsymbol{R}\to\boldsymbol{R}$ ($\boldsymbol{R}$ denotes the real line) such that $S\subset\boldsymbol{R}$ is the set of first Baire category if and only if $f(S)$ is the set of Lebesgue measure zero. This result has next been strengthened in 1943 by P. Erdös (see, e.g. [7, p. 76]). Hence, a natural problem arises, what conditions should be satisfied by a $\mathfrak{M}$-family $\mathscr{C}$ of ƒubsets of X and $\mathfrak{M}$-family $\mathscr{D}$ of subsets of Y, in order that a bijective function s: $X\to Y$ exists, such that $S\subset X$ is a $\mathscr{C}_{\mathrm{I}}$-set if and only if $f(S)$ is a $\mathscr{D}_{\mathrm{I}}$-set. Following considerations give a solution of this problem, under the following hypotheses. Assume the continuum hypothesis: $\omega_1=\mathfrak{c}$. Let X and Y be nonempty sets, such that $|X|=|Y|=\mathfrak{c}$; let $\mathscr{C}$ be a $\mathfrak{M}$-family of subsets of X, let $\mathscr{D}$ be a $\mathfrak{M}$-family of subsets of Y; $|\mathscr{C}|\leqslant\mathfrak{c}$, $|\mathscr{D}|\leqslant\mathfrak{c}$; lastly, let $\mathscr{C}$ and $\mathscr{D}$ satisfy CCC.

If $\mathscr{C}$ is defined as a family of closures of open balls (see Ex. 2) and $\mathscr{D}$ is defined as a family of all compact sets of positive Lebesgue measure (see Ex. 3), the classical duality principle of W. Sierpiński is obtained. Indeed, $\mathscr{C}_{\mathrm{I}}$ coincides with the family of all sets of first Baire category, whereas $\mathscr{D}_{\mathrm{I}}$ coincides with the family of all sets of Lebesgue measure zero.

DEFINITION 3. A set $N\subset X$ is said to have *property* (L) *with respect to* $\mathscr{C}$ if and only if N is uncountable and has at most countably many common points with any $\mathscr{C}_{\mathrm{I}}$-set; equivalently, if and only if N is uncountable and every uncountable subset of N is a $\mathscr{C}_{\mathrm{II}}$-set.

The following considerations yields, in particular, the existence of a set having property (L).

THEOREM 2. *$\mathscr{C}$ satisfies one and only one of the following three conditions*:

(*) *Each $\mathscr{C}_{\mathrm{I}}$-set is at most countable.*

(**) *X is the union of two uncountable, disjoint sets T and N, where T is a $\mathscr{C}_{\mathrm{I}}$-set and N has property* (L) *with respect to $\mathscr{C}$.*

(***) *Each $\mathscr{C}$-residual set contains an uncountable $\mathscr{C}_{\mathrm{I}}$-set.*

Proof. Assume, that (*) and (***) are not satisfied. Each $\mathscr{C}$-residual set is uncountable (see I, III). Since (***) is not satisfied, there is a $\mathscr{C}$-residual set M, having property (L). Furthermore, (*) is not satisfied; hence, there exists an uncountable $\mathscr{C}_{\mathrm{I}}$-set P. The set $R:=X\setminus M$ is a $\mathscr{C}_{\mathrm{I}}$-set; hence $T:=P\cup R$ is a $\mathscr{C}_{\mathrm{I}}$-set. It is easy to see that the set $N:=X\setminus T$ has property (L) with respect to $\mathscr{C}$. Thus

$$X=T\cup N$$

is the desired partition of X.

Obviously, any two of the above three conditions cannot be satisfied simultaneously.

The condition (*) is satisfied by the $\mathfrak{M}$-family defined in Example 1, the condition (**) is satisfied by the $\mathfrak{M}$-family defined in Example 5, and the condition (***) is satisfied by the $\mathfrak{M}$-families defined in Examples 2, 3 and 4.

LEMMA 2. *$\mathscr{C}$ satisfies (***) if and only if there exists an uncountable family $\mathscr{A}$ of uncountable, disjoint $\mathscr{C}_I$-sets, such that $X = \bigcup \mathscr{A}$ and each $\mathscr{C}_I$-set is contained in at most countable union of $\mathscr{A}$-sets.*

The sufficiency follows from the properties of the family $\mathscr{A}$. The necessity is an immediate consequence of Theorem 19.5 in [7].

DEFINITION 4. The bijective function $f: X \to Y$ such that $S \subset X$ is a $\mathscr{C}_I$-set if and only if $f(S)$ is a $\mathscr{D}_I$-set is called a *$\mathscr{C}$—$\mathscr{D}$ category function.*

THEOREM 3. *A $\mathscr{C}$—$\mathscr{D}$ category function exists if and only if $\mathscr{C}$ and $\mathscr{D}$ satisfy simultaneously one of the conditions*: (*), (**), (***).

Proof. The necessity is obvious, since $\mathscr{C}$—$\mathscr{D}$ category function transforms $\mathscr{C}_I$-sets into $\mathscr{D}_I$-sets and preserves property (L) as well as the power of the set. To prove the sufficiency consider the following three possibilities:

1. $\mathscr{C}$ and $\mathscr{D}$ satisfy (*), simultaneously; then, any bijective function $f: X \to Y$ is a $\mathscr{C}$—$\mathscr{D}$ category function.

2. $\mathscr{C}$ and $\mathscr{D}$ satisfy (**), simultaneously. Then, there exist four uncountable sets: T_1, T_2, N_1, N_2 such that

$$X = T_1 \cup N_1, \qquad Y = T_2 \cup N_2$$

and

$$T_1 \cap N_1 = \emptyset, \qquad T_2 \cap N_2 = \emptyset.$$

Moreover, T_1 is a $\mathscr{C}_I$-set, T_2 is a $\mathscr{D}_I$-set, N_1 and N_2 have property (L) with respect to $\mathscr{C}$ and $\mathscr{D}$, respectively. Let $f_1 : T_1 \to T_2$ and $f_2 : N_1 \to N_2$ be the bijective functions (they have to exist, since T_1 and T_2 as well as N_1 and N_2 are of the same power). Define

$$f(x) := \begin{cases} f_1(x) & \text{for } x \in T_1, \\ f_2(x) & \text{for } x \in N_1. \end{cases}$$

It is not hard to check that f is a bijection.

Let $S \subset X$ be a $\mathscr{C}_I$-set. Then $S = (S \cap T_1) \cup (S \cap N_1)$, where the set $S \cap N_1$ is at most countable. Hence,

$$f(S) = f((S \cap T_1) \cup (S \cap N_1)) = f(S \cap T_1) \cup f(S \cap N_1).$$

Since $f(S \cap T_1) \subset T_2$, $f(S \cap T_1)$ is a $\mathscr{D}_I$-set, and $f(S \cap N_1)$ is at most countable. Thus $f(S)$ is a $\mathscr{D}_I$-set.

Now, let $Q \subset Y$ be a $\mathscr{D}_I$-set. It is easy to prove in the same way, that $f^{-1}(Q)$ is a $\mathscr{C}_I$-set.

3. $\mathscr{C}$ and $\mathscr{D}$ satisfy (***), simultaneously. Then, from Lemma 2, there exist two uncountable families $\mathscr{A}$ and $\mathscr{B}$ of uncountable, disjoint $\mathscr{C}_I$-sets and $\mathscr{D}_I$-sets

respectively, such that $X=\bigcup \mathscr{A}$, $Y=\bigcup \mathscr{B}$, and each $\mathscr{C}_{\mathrm{I}}$-set is contained in at most countable union of $\mathscr{A}$-sets as well as each $\mathscr{D}_{\mathrm{I}}$-set is contained in at most countable union of $\mathscr{B}$-sets. Let

$$\mathscr{A}=\{P_\alpha : \alpha<\omega_1\} \text{ and } \mathscr{B}=\{R_\alpha : \alpha<\omega_1\}.$$

For each $\alpha<\omega_1$ there exists a bijective function $f_\alpha : P_\alpha \to R_\alpha$, since P_α and R_α are of the same power. Define

$$f(x):=f_\alpha(x) \text{ for } x\in P_\alpha.$$

It is easily seen that f is a bijection.

Let $S\subset X$ be a $\mathscr{C}_{\mathrm{I}}$-set. There exists a sequence of ordinal numbers $\{\alpha_k : k=1,2,\ldots\}$ such that $S\subset \bigcup\{P_{\alpha_k} : k=1,2,\ldots\}$. Thus,

$f(S)\subset f(\bigcup\{P_{\alpha_k} : k=1,2,\ldots\})=\bigcup\{f(P_{\alpha_k}) : k=1,2,\ldots\}=\bigcup\{R_{\alpha_k} : k=1,2,\ldots\}$.

Since R_{α_k} are $\mathscr{D}_{\mathrm{I}}$-sets, $f(S)$ is a $\mathscr{D}_{\mathrm{I}}$-set.

In the same way we can prove that if $Q\subset Y$ is a $\mathscr{D}_{\mathrm{I}}$-set, then $f^{-1}(Q)$ is a $\mathscr{C}_{\mathrm{I}}$-set.

The existence of a set having property (L) is an immediate consequence of Theorem 2.

THEOREM 4. *Each $\mathscr{C}_{\mathrm{II}}$-set contains a set having property* (L) *with respect to $\mathscr{C}$.*

Proof. Let $K\subset X$ be a $\mathscr{C}_{\mathrm{II}}$-set. Consider the following three possibilities:

1. $\mathscr{C}$ satisfies (*); then K has property (L) with respect to $\mathscr{C}$.

2. $\mathscr{C}$ satisfies (**): then $X=T\cup N$, T is a $\mathscr{C}_{\mathrm{I}}$-set, N has property (L) with respect to $\mathscr{C}$. The intersection $K\cap N$ is an uncountable set; hence, it has property (L) with respect to $\mathscr{C}$.

3. $\mathscr{C}$ satisfies (***); then there exists an uncountable family $\mathscr{A}$ of uncountable, disjoint $\mathscr{C}_{\mathrm{I}}$-sets, such that each $\mathscr{C}_{\mathrm{I}}$-set is contained in at most countable union of $\mathscr{A}$-sets and $X=\bigcup \mathscr{A}$. Hence, there exists an uncountable family $\mathscr{A}'$, being a subfamily of $\mathscr{A}$, such that $S\in\mathscr{A}'$ implies that $K\cap S\neq\varnothing$. Let $x_s\in K\cap S$ for $S\in\mathscr{A}'$. Define

$$N:=\{x_s : S\in\mathscr{A}'\}.$$

Obviously, $N\subset K$ and N has property (L) with respect to $\mathscr{C}$.

The following corollaries are simple consequences of Theorem 4.

COROLLARY 1. *There exists an injective function f: $X\to Y$ such that $f(M)$ is a $\mathscr{D}_{\mathrm{II}}$-set provided that M is uncountable set.*

COROLLARY 2. *Each $\mathscr{C}_{\mathrm{II}}$-set contains uncountably many disjoint $\mathscr{C}_{\mathrm{II}}$-sets.*

4. Some remarks on σ-ideals. Assume the continuum hypothesis: $\omega_1=\mathfrak{c}$. It is an immediate consequence of Definition 1 and Theorem 1, that if $\mathscr{C}$ is a $\mathfrak{M}$-family of subsets of a set X of power $\mathfrak{c}$, the power of $\mathscr{C}$ is less or equal $\mathfrak{c}$ and $\mathscr{C}$ satisfies CCC, then $\mathscr{C}_{\mathrm{I}}$ has following properties:

(i) it is a proper σ-ideal of subsets of a set of power $\mathfrak{c}$;

(ii) one-point sets belong to this σ-ideal;

(iii) it contains a subfamily of power $\mathfrak{c}$ such that each member of this σ-ideal is a subset of a member of this subfamily.

In what follows, we consider families of sets, having properties (i)—(iii).

DEFINITION 5. Let the family $\mathscr{J} \subset 2^X$ (2^X denotes the set of all subsets of a set X) has properties (i)—(iii). The set $N \subset X$ is said to have *property* (L) *with respect to* $\mathscr{J}$, if and only if N is uncountable and for each $\mathscr{J}$-set S the intersection $S \cap N$ is at most countable.

THEOREM 5. *Let* $\mathscr{J} \subset 2^X$ *have properties* (i)—(iii). *Then* $\mathscr{J}$ *satisfies one and only one of the three following conditions*:

(*) *Each* $\mathscr{J}$*-set is at most countable.*

(**) X *is the union of two uncountable, disjoint sets* T *and* N, *where* T *is an* $\mathscr{J}$*-set nad* N *has property* (L) *with respect to* $\mathscr{J}$.

(***) *For each* $\mathscr{J}$*-set* P *there exists an uncountable* $\mathscr{J}$*-set* R, *disjoint with* P.

The proof of this theorem is identical with the proof of Theorem 2, after suitable change of denotations. Similarly, the following lemma can be proved.

LEMMA 3. *If* $\mathscr{J}$ *satisfies* (***), *then there exists an uncountable family* $\mathscr{A}$ *of uncountable, disjoint* $\mathscr{J}$*-sets such that* $\bigcup \mathscr{A} = X$ *and each* $\mathscr{J}$*-set is contained in at most countable union of* $\mathscr{A}$*-sets.*

REMARK. It is obvious, that if $\mathscr{C}$ in an $\mathfrak{M}$-family satisfying assumptions of part 3 of this paper, then $\mathscr{C}_\mathrm{I}$ has properties (i)—(iii). Furthermore, $\mathscr{C}$ satisfies the conditions (*), (**), (***) from Theorem 2 if and only if $\mathscr{C}_\mathrm{I}$ satisfies the conditions (*), (**), (***) from Theorem 5, respectively. This fact justifies the use of the same denotations.

THEOREM 6. *Let* $\mathscr{I} \subset 2^X$ *and* $\mathscr{J} \subset 2^Y$ *have properties* (i)—(iii). $\mathscr{I}$ *and* $\mathscr{J}$ *satisfy one of the conditions* (*), (**), (***), *simultaneously, if and only if there exists a bijective function* $f\colon X \to Y$ *such that the equivalence* $S \in \mathscr{I} \Leftrightarrow f(S) \in \mathscr{J}$ *holds.*

The proof of this theorem is the same as the proof of Theorem 3 (after suitable change of denotations). As immediate consequences of Theorem 6 the following corollaries may easily be obtained.

COROLLARY 3. *If a* σ*-ideal* $\mathscr{J} \subset 2^Y$ *has properties* (i)—(iii), *then there exists an* $\mathfrak{M}$*-family* $\mathscr{D}$ *of subsets of* Y, *satysfying* CCC, *such that* $\mathscr{J}$ *coincides with* $\mathscr{D}_\mathrm{I}$.

Proof. $\mathscr{J}$ satisfies one of the conditions (*), (**), (***). Consider one of the $\mathfrak{M}$-families defined in Examples 1—5, satisfying the same condition. Then $\mathscr{J}$ and $\mathscr{C}_\mathrm{I}$ satisfy the same condition and, from Theorem 6, there exists a bijection $f\colon X \to Y$ such that the equivalence $S \in \mathscr{C}_\mathrm{I} \Leftrightarrow f(S) \in \mathscr{J}$ holds. Obviously, the $\mathfrak{M}$-family $\mathscr{D}$, defined as follows:

$$\mathscr{D} := \{f(B) : B \in \mathscr{C}\}$$

satisfies our assertion.

With the use of fact, that the family of all sets of Lebesgue measure zero and the family of all sets of first category of Baire satisfy condition (***), the following two corollaries can be proved in the similar way.

COROLLARY 4. *If* $\mathscr{J} \subset 2^X$, *having properties* (i)—(iii), *satisfies the condition* (***), *then there exists a metric topology in* X, *homoemorphic with the topology of the real line and such that* $\mathscr{J}$ *coincides with the family of all sets of first category of Baire with respect to this topology.*

COROLLARY 5. *If* $\mathscr{J} \subset 2^X$, *having properties* (i)—(iii), *satisfies the condition* (***), *then there exists a metric topology in* X, *homoemorphic with the topology of* *the real line and a complete measure μ, defined on the completion of σ-field of Borel sets, regular, σ-finite and non-atomic, such that $\mathscr{J}$ coincides with the family of all sets of μ-measure zero.*

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