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ON r-JACOBSTHAL AND r-JACOBSTHAL–LUCAS NUMBERS

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Abstract. Recently, Bród introduced a new Jacobsthal-type sequence which is called r-Jacobsthal sequence in current study. After defining the appropriate r-Jacobsthal–Lucas sequence for the r-Jacobsthal sequence, we obtain some properties of these two sequences. For simpler results, we define two new sequences and examine their properties, too. Finally, we generalize some well-known identities.

1. Introduction

The Lucas sequences generalize many famous integer sequences defined by a second order linear recurrence relation such as the Fibonacci numbers, the Lucas numbers, the Pell numbers, the Pell–Lucas numbers, the Jacobsthal numbers and the Jacobsthal–Lucas numbers. Let A and B be integers. The roots of $x^2 - Ax + B = 0$ are $x_1 = \frac{1}{2}(A + \sqrt{A^2 - 4B})$ and $x_2 = \frac{1}{2}(A - \sqrt{A^2 - 4B})$. The Lucas sequences (see [3] for details) are defined by the following Binet-like formula

$$P_n = \frac{x_1^n - x_2^n}{x_1 - x_2} \quad \text{and} \quad Q_n = \frac{x_1^n + x_2^n}{x_1 + x_2}.$$

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For $(A,B)=(1,-1)$, $(2,-1)$ and $(1,-2)$, the sequence $\{P_n\}$ gives the Fibonacci, the Pell and the Jacobsthal numbers, respectively. Similarly, for $(A,B)=(1,-1)$, $(2,-1)$ and $(1,-2)$, the sequence $\{Q_n\}$ gives the Lucas, the Pell–Lucas and the Jacobsthal–Lucas numbers, respectively.

The Jacobsthal sequence $\{J_n\}$ and the Jacobsthal–Lucas sequence $\{j_n\}$ are defined by the same second order recurrence relation, namely

$$T_n = T_{n-1} + 2T_{n-2}$$

except the initial conditions. While the initial conditions of the Jacobsthal sequence are $J_0=0$ and $J_1=1$, the initial conditions of the Jacobsthal–Lucas sequence are $j_0=2$ and $j_1=1$. Binet formulas for the Jacobsthal and the Jacobsthal–Lucas numbers are

$$J_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad j_n = \frac{\gamma^n + \delta^n}{\gamma + \delta},$$

respectively, where $\gamma = 2$ and $\delta = -1$ are roots of the characteristic equation $x^2 - x - 2 = 0$. Generating functions for the sequences $\{J_n\}$ and $\{j_n\}$ are

$$\sum_{n=0}^{\infty} J_n x^n = \frac{x}{1-x-2x^2} \quad \text{and} \quad \sum_{n=0}^{\infty} j_n x^n = \frac{2-x}{1-x-2x^2},$$

respectively. A comprehensive study about the Jacobsthal and the Jacobsthal–Lucas numbers was made by Horadam ([7]). He gave a lot of interesting properties and beautiful identities of these numbers. We recall some of them.

$$J_n + j_n = 2J_{n+1},$$

$$j_n^2 + 9J_n^2 = 2j_{2n}, \tag{1.1}$$

$$j_n^2 - 9J_n^2 = (-1)^n 2^{n+2}, \tag{1.2}$$

$$J_m j_n + J_n j_m = 2J_{n+m}, \tag{1.3}$$

$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}. \tag{1.4}$$

There are some generalizations of the Jacobsthal and the Jacobsthal–Lucas numbers defined in different ways. Falcon ([5]) defined the k-Jacobsthal numbers, Jhala, Sisodiya and Rathore ([8]) gave another definition for the k-Jacobsthal numbers, Dasdemir ([2]) defined the Jacobsthal p-numbers and Uygun ([10]) introduced the (s,t)-Jacobsthal numbers. Similarly, Uygun and Owusu ([11]) defined the bi-periodic Jacobsthal numbers. All of these authors changed the recurrence relation of the Jaconsthal sequence, slightly, while preserving the initial conditions.

Some general sequences also generalize the Jacobsthal and the Jacobsthal--Lucas numbers. We can refer to the Horadam sequence ([6]) and the bi-periodic generalized Fibonacci sequence ([4]) as examples of this approach.

In [1], Bród defined another one parameter Jacobsthal sequence $\{\mathcal{J}_{r,n}\}$ (for integers $n \geq 0$ and $r \geq 0$) which is called r-Jacobsthal sequence, by the recurrence relation

$$\mathcal{J}_{r,n} = 2^r \mathcal{J}_{r,n-1} + (2^r + 4^r)\mathcal{J}_{r,n-2}, \quad n \geq 2 \tag{1.5}$$

with the initial conditions $\mathcal{J}_{r,0} = 1$ and $\mathcal{J}_{r,n} = 1 + 2^{r+1}$. It is clear that $J_n = \mathcal{J}_{0,n-2}$.

The r-Jacobsthal sequence has an application in the theory of graphs (see [1]). Recall this graph interpretation of the r-Jacobsthal numbers.

Let G be a finite, undirected, simple graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subset V(G)$ is an independent set of G if for any two distinct vertices $x, y \in S$ holds $xy \notin E(G)$. A subset of $V(G)$ containing only one vertex and the empty set are independent sets of G, too. The number of independent sets of a graph G is denoted by $NI(G)$. The parameter $NI(G)$ was studied not only in mathematical literature. In 1989, Merrifield and Simmons ([9]) introduced the number of independent sets into the chemical literature as the index σ. They showed a correlation between this index in a molecular graph and some chemical properties. The parameter $NI(G)$ is often represented by the Fibonacci numbers and the Lucas numbers. This fact may be a motivation to ask the following question: Are there any generalizations of the Fibonacci numbers that have graph interpretations due to the number of independent sets in the graph?

Consider a graph $G_{n,r}$ (Figure 1), where $n \geq 1$, $r \geq 0$.

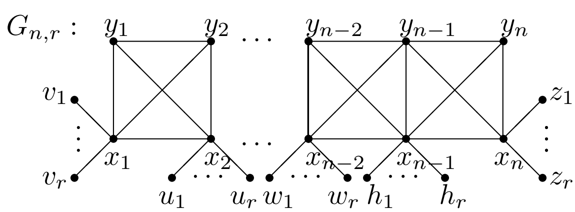

Figure 1. A graph $G_{n,r}$

In [1], the following result was proved.

THEOREM 1.1 ([1]). *Let n, r be integers such that $n \geq 1$, $r \geq 0$. Then*

$$NI(G_{n,r}) = \mathcal{J}_{r,n}.$$

The graph interpretation of the r-Jacobsthal numbers can be used for proving some identities.

THEOREM 1.2 ([1] convolution identity). *Let n, m, r be integers such that $m \geq 3, n \geq 2$, $r \geq 0$. Then*

$$\mathcal{J}_{r,n} = 2^r \mathcal{J}_{r,m-1}\mathcal{J}_{r,n} + (4^r + 8^r)\mathcal{J}_{r,m-2}\mathcal{J}_{r,n-1}.$$

COROLLARY 1.3 ([1]). $J_{m+n} = J_m J_{n+1} + 2J_{m-1}J_n$.

Bród gave the following Binet formula for the r-Jacobsthal numbers

$$\mathcal{J}_{r,n} = \frac{(\sqrt{4\cdot 2^r + 5\cdot 4^r} + 3\cdot 2^r + 2)\lambda_1^n + (\sqrt{4\cdot 2^r + 5\cdot 4^r} - 3\cdot 3^r - 2)\lambda_2^n}{2\sqrt{4\cdot 2^r + 5\cdot 4^r}},$$

where λ_1 and λ_2 are roots of the characteristic equation $\lambda^2 - 2^r\lambda - (2^r + 4^r) = 0$, namely

$$\lambda_1 = \frac{2^r + \sqrt{4\cdot 2^r + 5\cdot 4^r}}{2} \quad \text{and} \quad \lambda_2 = \frac{2^r - \sqrt{4\cdot 2^r + 5\cdot 4^r}}{2}.$$

We can change this Binet formula easily with

$$\mathcal{J}_{r,n} = \frac{\lambda_1^*\lambda_1^n - \lambda_2^*\lambda_2^n}{\lambda_1 - \lambda_2}, \tag{1.6}$$

where $\lambda_1^* = 2^r + 1 + \lambda_1$ and $\lambda_2^* = 2^r + 1 + \lambda_2$.

Now we define the r-Jacobsthal–Lucas sequence $\{\mathcal{K}_{r,n}\}_{n=0}^{\infty}$ with the same recurrence relation

$$\mathcal{K}_{r,n} = 2^r\mathcal{K}_{r,n-1} + (2^r + 4^r)\mathcal{K}_{r,n-2}, \quad n \geq 2,$$

with the initial conditions $\mathcal{K}_{r,0} = 3 + 2^{1-r}$ and $\mathcal{K}_{r,1} = 3 + 4\cdot 2^r$. It is easily seen that $j_n = \mathcal{K}_{0,n-2}$. Some initials terms of the r-Jacobsthal–Lucas sequence are

$$\mathcal{K}_{r,0} = 3 + 2^{1-r},$$

$$\mathcal{K}_{r,1} = 3 + 4\cdot 2^r,$$

$$\mathcal{K}_{r,2} = 2 + 8\cdot 2^r + 7\cdot 4^r,$$

$$\mathcal{K}_{r,3} = 5\cdot 2^r + 15\cdot 4^r + 11\cdot 8^r,$$

$$\mathcal{K}_{r,4} = 2\cdot 2^r + 15\cdot 4^r + 30\cdot 8^r + 18\cdot 16^r.$$

Binet formula for the r-Jacobsthal–Lucas numbers is given in the following theorem.

THEOREM 1.4. *For any nonnegative integer n, the nth r-Jacobsthal–Lucas number is*

$$\mathcal{K}_{r,n} = \frac{\lambda_1^* \lambda_1^n + \lambda_2^* \lambda_2^n}{\lambda_1 + \lambda_2}.$$

PROOF. For the r-Jacobsthal–Lucas sequence $\{\mathcal{K}_{r,n}\}_{n=0}^{\infty}$, we have

$$\mathcal{K}_{r,n} = A_1 \lambda_1^n + A_2 \lambda_2^n. \tag{1.7}$$

By using the initial conditions of r-Jacobsthal–Lucas sequence, we get the following system of equations

$$\begin{cases} A_1 + A_2 = 3 + 2^{1-r}, \\ A_1\lambda_1 + A_2\lambda_2 = 3 + 4 \cdot 2^r. \end{cases}$$

Solutions of the system are

$$A_1 = \frac{1 + 2^r + \lambda_1}{2^r} \quad \text{and} \quad A_2 = \frac{1 + 2^r + \lambda_2}{2^r}.$$

After substitution A_1 and A_2 into (1.7), we obtain the theorem. □

Now we present generating function for the r-Jacobsthal–Lucas sequence $\{\mathcal{K}_{r,n}\}_{n=0}^{\infty}$.

THEOREM 1.5. *The generating function for the r-Jacobsthal–Lucas sequence is*

$$\sum_{n=0}^{\infty} \mathcal{K}_{r,n} x^n = \frac{3 + 2^{1-r} + (2^r + 1)x}{1 - 2^r x - (2^r + 4^r)x^2}.$$

PROOF. Let us define $K(x) = \sum_{n=0}^{\infty} \mathcal{K}_{r,n} x^n$. Then we obtain

$$K(x) = 3 + 2^{1-r} + (3 + 4 \cdot 2^r)x + \sum_{n=2}^{\infty} \mathcal{K}_{r,n} x^n. \tag{1.8}$$

By multiplying both sides of (1.8) by $-2^r x$ and $-(2^r+4^r)x^2$, we have

$$-2^r xK(x) = -(2+3\cdot 2^r)x - \sum_{n=2}^{\infty} 2^r \mathcal{K}_{r,n-1}x^n \tag{1.9}$$

and

$$-(2^r+4^r)x^2K(x) = -\sum_{n=2}^{\infty}(2^r+4^r)\mathcal{K}_{r,n}x^n, \tag{1.10}$$

respectively. Adding side by side equalities (1.8), (1.9) and (1.10) gives

$$\begin{aligned}&\left[1-2^r x-(2^r+4^r)x^2\right]K(x)\\ &\quad = 3+2^{1-r}+(2^r+1)x+\sum_{n=2}^{\infty}\left[\mathcal{K}_{r,n}-2^r\mathcal{K}_{r,n-1}-(2^r+4^r)\mathcal{K}_{r,n-2}\right]x^n.\end{aligned}$$

The last equation and the recurrence relation for the r-Jacobsthal–Lucas numbers complete the proof. □

It should be noted that we need the following equation for later use

$$\lambda_1^* \cdot \lambda_2^* = (2^r+1)^2.$$

2. Second type of r-Jacobsthal and r-Jacobsthal–Lucas numbers

For simpler results, we need a new type of the r-Jacobsthal and the r-Jacobsthal–Lucas numbers. We preserve the recurrence relation but change the initial conditions. Namely, the second type of the r-Jacobsthal numbers satisfies the recurrence relation

$$\mathcal{J}'_{r,n} = 2^r\mathcal{J}'_{r,n-1} + (2^r+4^r)\mathcal{J}'_{r,n-2}, \quad n\geq 2$$

with the initial conditions $\mathcal{J}'_{r,0}=0$ and $\mathcal{J}'_{r,1}=1$, and the second type of the r-Jacobsthal–Lucas numbers satisfies the recurrence relation

$$\mathcal{K}'_{r,n} = 2^r\mathcal{K}'_{r,n-1} + (2^r+4^r)\mathcal{K}'_{r,n-2}, \quad n\geq 2 \tag{2.1}$$

with the initial conditions $\mathcal{K}'_{r,0}=2^{1-r}$ and $\mathcal{K}'_{r,1}=1$.

Binet formula for the r-Jacobsthal and the r-Jacobsthal–Lucas numbers can be found in the following theorem.

Theorem 2.1. *For any nonnegative integer n, the nth second type of the r-Jacobsthal and the r-Jacobsthal–Lucas numbers are*

$$\mathcal{J}'_{r,n} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \tag{2.2}$$

and

$$\mathcal{K}'_{r,n} = \frac{\lambda_1^n + \lambda_2^n}{\lambda_1 + \lambda_2},$$

respectively.

Proof. Proofs can be done with the similar way to the proof of Theorem 1.4. □

Theorem 2.2. *The generating functions for the sequences $\{\mathcal{J}'_{r,n}\}_{n=0}^{\infty}$ and $\{\mathcal{K}'_{r,n}\}_{n=0}^{\infty}$ are*

$$\sum_{n=0}^{\infty} \mathcal{J}'_{r,n} x^n = \frac{x}{1 - 2^r x - (2^r + 4^r)x^2}$$

and

$$\sum_{n=0}^{\infty} \mathcal{K}'_{r,n} x^n = \frac{2^{1-r} - x}{1 - 2^r x - (2^r + 4^r)x^2},$$

respectively.

Proof. Proofs can be done by using the similar way to the proof of Theorem 1.5. □

The following lemma presents some connections between the r-Jacobsthal and the r-Jacobsthal–Lucas numbers.

Lemma 2.3. *For any positive integers r and n, we have*

$$\mathcal{K}_{r,n} = 2^{1-r}\mathcal{J}_{r,n+1} - \mathcal{J}_{r,n}$$

and

$$\mathcal{K}'_{r,n} = 2^{1-r}\mathcal{J}'_{r,n+1} - \mathcal{J}'_{r,n}.$$

Proof. From the Binet formula for the r-Jacobsthal numbers (1.6), we have

$$2^{1-r}\mathcal{J}_{r,n+1} - \mathcal{J}_{r,n} = \frac{2^{1-r}(\lambda_1^*\lambda_1^{n+1} - \lambda_2^*\lambda_2^{n+1}) - \lambda_1^*\lambda_1^n + \lambda_2^*\lambda_2^n}{\sqrt{4\cdot 2^r + 5\cdot 4^r}}$$
$$= \frac{\lambda_1^*\lambda_1^n(2^{1-r}\lambda_1 - 1) - \lambda_2^*\lambda_2^n(2^{1-r}\lambda_2 - 1)}{\sqrt{4\cdot 2^r + 5\cdot 4^r}}.$$

If we substitute the identities $2^{1-r}\lambda_1 - 1 = 2^{-r}\sqrt{4\cdot 2^r + 5\cdot 4^r}$ and $2^{1-r}\lambda_2 - 1 = -2^{-r}\sqrt{4\cdot 2^r + 5\cdot 4^r}$ into the last expression, we obtain the first identity in the theorem. The second identity can be proved similarly. □

The connections between both types of the r-Jacobsthal numbers and between both types of the r-Jacobsthal–Lucas numbers are given in the following lemma.

Lemma 2.4. *For any positive integers r and n, we have*

$$\mathcal{J}_{r,n} = \mathcal{J}'_{r,n+1} + (2^r + 1)\mathcal{J}'_{r,n}$$

and

$$\mathcal{K}_{r,n} = \mathcal{K}'_{r,n+1} + (2^r + 1)\mathcal{K}'_{r,n}.$$

Proof. The proofs can be done, easily, by using the Binet formulas for the r-Jacobsthal and the r-Jacobsthal–Lucas numbers. □

The next lemma gives the second type of the r-Jacobsthal–Lucas and the r-Jacobsthal–Lucas numbers with negative indices.

Lemma 2.5. *For any positive integers r and n, we have*

$$\mathcal{J}'_{r,-n} = \frac{(-1)^{n+1}}{(2^r + 4^r)^n}\mathcal{J}'_{r,n}$$

and

$$\mathcal{K}'_{r,-n} = \frac{(-1)^n}{(2^r + 4^r)^n}\mathcal{K}'_{r,n}.$$

Proof. From the Binet formula (2.2) for the second type of the r-Jacobsthal numbers, we have

$$\begin{aligned}\mathcal{J}'_{r,-n} &= \frac{\lambda_1^{-n} - \lambda_2^{-n}}{\sqrt{4\cdot 2^r + 5\cdot 4^r}}\\ &= \frac{1}{\sqrt{4\cdot 2^r + 5\cdot 4^r}}\left[\frac{1}{\lambda_1^n} - \frac{1}{\lambda_2^n}\right]\\ &= -\frac{1}{\sqrt{4\cdot 2^r + 5\cdot 4^r}}\left[\frac{\lambda_1^n - \lambda_2^n}{(\lambda_1\cdot\lambda_2)^n}\right].\end{aligned}$$

The equality $\lambda_1 \cdot \lambda_2 = -(2^r + 4^r)$ gives the first identity in lemma. The second identity can be obtained in a similar way. □

Similarly, the first type of the r-Jacobsthal and the r-Jacobsthal–Lucas numbers with negative indices can be obtained as in the following by the help of Lemma 2.4 and Lemma 2.5.

Lemma 2.6. *For any positive integers r and n, we have*

$$\mathcal{J}_{r,-n} = \frac{(-1)^{n+1}}{(2^r + 4^r)^{n-1}}\left[2^{-r}\mathcal{J}'_{r,n} - \mathcal{J}'_{r,n-1}\right]$$

and

$$\mathcal{K}_{r,-n} = \frac{(-1)^{n}}{(2^r + 4^r)^{n-1}}\left[2^{-r}\mathcal{K}'_{r,n} - \mathcal{K}'_{r,n-1}\right].$$

Lemma 2.6 provides us to expand all the results about the r-Jacobsthal--Lucas and the r-Jacobsthal–Lucas numbers to integers. Although Bród restricted r to nonnegative integers, we should emphasize that there is no need such a restriction. Namely, r can be an arbitrary integer.

3. Some properties of r-Jacobsthal and r-Jacobsthal–Lucas numbers

In this section, we give some results for the r-Jacobsthal and the r-Jacobsthal–Lucas numbers starting with Vajda's identities.

THEOREM 3.1. *For any integers, r, n, m and k, we have*

$$\begin{aligned}\mathcal{J}_{r,n+m}\mathcal{J}_{r,n+k} - \mathcal{J}_{r,n}\mathcal{J}_{r,n+m+k} &= (2^r+1)^2\big[\mathcal{J}'_{r,n+m}\mathcal{J}'_{r,n+k} - \mathcal{J}'_{r,n}\mathcal{J}'_{r,n+m+k}\big] \\ &= (2^r+1)^2(-1)^n(2^r+4^r)^n\mathcal{J}'_{r,m}\mathcal{J}'_{r,k}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\mathcal{K}_{r,n+m}\mathcal{K}_{r,n+k} - \mathcal{K}_{r,n}\mathcal{K}_{r,n+m+k} &= (2^r+1)^2\big[\mathcal{K}'_{r,n+m}\mathcal{K}'_{r,n+k} - \mathcal{K}'_{r,n}\mathcal{K}'_{r,n+m+k}\big] \\ &= \frac{(2^r+1)^2(-1)^{n+1}(2^r+4^r)^n(4\cdot 2^r+5\cdot 4^r)}{4^r}\mathcal{J}'_{r,m}\mathcal{J}'_{r,k}.\end{aligned}$$

PROOF. Using the Binet formula (1.6) for the r-Jacobsthal numbers, we have

$$\begin{aligned}\mathcal{J}_{r,n+m}\mathcal{J}_{r,n+k} - \mathcal{J}_{r,n}\mathcal{J}_{r,n+m+k} &= \frac{1}{(4\cdot 2^r+5\cdot 4^r)^2}\big[(\lambda_1^*\lambda_1^{n+m} - \lambda_2^*\lambda_2^{n+m})(\lambda_1^*\lambda_1^{n+k} - \lambda_2^*\lambda_2^{n+k}) \\ &\quad - (\lambda_1^*\lambda_1^{n} - \lambda_2^*\lambda_2^{n})(\lambda_1^*\lambda_1^{n+m+k} - \lambda_2^*\lambda_2^{n+m+k})\big] \\ &= \frac{(2^r+1)^2(\lambda_1\lambda_2)^n}{(4\cdot 2^r+5\cdot 4^r)^2}\big[-\lambda_1^m\lambda_2^k - \lambda_1^k\lambda_2^m + \lambda_1^{m+k} + \lambda_2^{m+k}\big] \\ &= \frac{(2^r+1)^2(-1)^n(2^r+4^r)^n}{(4\cdot 2^r+5\cdot 4^r)^2}\big[(\lambda_1^m - \lambda_2^m)(\lambda_1^k - \lambda_2^k)\big].\end{aligned}$$

The last equality gives the first identity in (3.1). The others can be obtained similarly. □

If we take $k \to -m$, Theorem 3.1 gives Catalan's identities

$$\begin{aligned}\mathcal{J}_{r,n+m}\mathcal{J}_{r,n-m} - \big[\mathcal{J}_{r,n}\big]^2 &= (2^r+1)^2\Big[\mathcal{J}'_{r,n+m}\mathcal{J}'_{r,n-m} - \big[\mathcal{J}'_{r,n}\big]^2\Big] \\ &= (2^r+1)^2(-1)^{n+m+1}(2^r+4^r)^{n-m}\big[\mathcal{J}'_{r,m}\big]^2\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(3.3)\qquad &\mathcal{K}_{r,n+m}\mathcal{K}_{r,n-m}-\left[\mathcal{K}_{r,n}\right]^2\\ &=(2^r+1)^2\left[\mathcal{K}'_{r,n+m}\mathcal{K}'_{r,n-m}-\left[\mathcal{K}'_{r,n}\right]^2\right]\\ &=\frac{(2^r+1)^2(-1)^{n+m}(2^r+4^r)^{n-m}(4\cdot 2^r+5\cdot 4^r)}{4^r}\left[\mathcal{J}'_{r,m}\right]^2.\end{aligned}$$

If we take $m\to 1$, Catalan's identities (3.2) and (3.3) give Cassini's identities

$$\begin{aligned}(3.4)\qquad &\mathcal{J}_{r,n+1}\mathcal{J}_{r,n-1}-\left[\mathcal{J}_{r,n}\right]^2\\ &=(2^r+1)^2\left[\mathcal{J}'_{r,n+1}\mathcal{J}'_{r,n-1}-\left[\mathcal{J}'_{r,n}\right]^2\right]\\ &=(2^r+1)^2(-1)^n(2^r+4^r)^{n-1}\end{aligned}$$

and

$$\begin{aligned}&\mathcal{K}_{r,n+1}\mathcal{K}_{r,n-1}-\left[\mathcal{K}_{r,n}\right]^2\\ &=(2^r+1)^2\left[\mathcal{K}'_{r,n+1}\mathcal{K}'_{r,n-1}-\left[\mathcal{K}'_{r,n}\right]^2\right]\\ &=\frac{(2^r+1)^2(-1)^{n+1}(2^r+4^r)^{n-1}(4\cdot 2^r+5\cdot 4^r)}{4^r}.\end{aligned}$$

In the next theorem d'Ocagne's identities for the r-Jacobsthal and the r-Jacobsthal–Lucas numbers are given.

Theorem 3.2. *For any integers, r, m and n, we have*

$$\begin{aligned}(3.5)\qquad &\mathcal{J}_{r,m}\mathcal{J}_{r,n+1}-\mathcal{J}_{r,m+1}\mathcal{J}_{r,n}\\ &=(2^r+1)^2\left[\mathcal{J}'_{r,m}\mathcal{J}'_{r,n+1}-\mathcal{J}'_{r,m+1}\mathcal{J}'_{r,n}\right]\\ &=(2^r+1)^2(-1)^{m+1}(2^r+4^r)^m\mathcal{J}'_{r,n-m}\end{aligned}$$

and

$$\begin{aligned}&\mathcal{K}_{r,m}\mathcal{K}_{r,n+1}-\mathcal{K}_{r,m+1}\mathcal{K}_{r,n}\\ &=(2^r+1)^2\left[\mathcal{K}'_{r,m}\mathcal{K}'_{r,n+1}-\mathcal{K}'_{r,m+1}\mathcal{K}'_{r,n}\right]\\ &=(2^r+1)^2 2^{-2r}(4\cdot 2^r+5\cdot 4^r)(-1)^m(2^r+4^r)^m\mathcal{J}'_{r,n-m}.\end{aligned}$$

Proof. From the Binet formula (1.6) for the r-Jacobsthal numbers, we have

$$\begin{aligned}\mathcal{J}_{r,m}\mathcal{J}_{r,n+1} - \mathcal{J}_{r,m+1}\mathcal{J}_{r,n} &= \frac{\lambda_1^*\lambda_2^*(-\lambda_1^m\lambda_2^{n+1} - \lambda_1^{n+1}\lambda_2^m + \lambda_1^{m+1}\lambda_2^n + \lambda_1^n\lambda_2^{m+1})}{4\cdot 2^r + 5\cdot 4^r}\\ &= -\frac{(2^r+1)^2\left[\lambda_1^m\lambda_2^n(\lambda_1-\lambda_2) - \lambda_1^n\lambda_2^m(\lambda_1-\lambda_2)\right]}{4\cdot 2^r + 5\cdot 4^r}\\ &= -\frac{(2^r+1)^2(\lambda_1\lambda_2)^m(\lambda_1^{n-m} - \lambda_2^{n-m})}{\sqrt{4\cdot 2^r + 5\cdot 4^r}}.\end{aligned}$$

From the last equality, we obtain the first identity in (3.5). The other identities can be obtained similarly. □

Some connections between the r-Jacobsthal and the r-Jacobsthal–Lucas numbers are presented in the next theorem.

Theorem 3.3. *For any integers, r, m and n, we have*

$$\begin{aligned}&4^r\mathcal{K}_{r,n}^2 + (4\cdot 2^r + 5\cdot 4^r)\mathcal{J}_{r,n}^2 = 2\mathcal{K}_{r,2n+2}, &(3.6)\\ &2^r(\mathcal{K}'_{r,n})^2 + (5\cdot 2^r + 4)(\mathcal{J}'_{r,n})^2 = 2\mathcal{K}'_{r,2n},\\ &4^r\mathcal{K}_{r,n}^2 - (4\cdot 2^r + 5\cdot 4^r)\mathcal{J}_{r,n}^2 = 4(2^r+1)^2(-1)^n(2^r+4^r)^n,\\ &2^r(\mathcal{K}'_{r,n})^2 - (5\cdot 2^r + 4)(\mathcal{J}'_{r,n})^2 = 4(-1)^n(2^r+4^r)^n,\end{aligned}$$

$$\begin{aligned}&\mathcal{J}_{r,m}\mathcal{K}_{r,n} + \mathcal{J}_{r,n}\mathcal{K}_{r,m} = 2^{1-2r}\mathcal{J}_{r,m+n+2}, &(3.7)\\ &\mathcal{J}'_{r,m}\mathcal{K}'_{r,n} + \mathcal{J}'_{r,n}\mathcal{K}'_{r,m} = 2^{1-r}\mathcal{J}'_{r,m+n},\\ &\mathcal{J}_{r,m}\mathcal{K}_{r,n} - \mathcal{J}_{r,n}\mathcal{K}_{r,m} = 2^{1-r}(2^r+1)^2(-1)^n(2^r+4^r)^2\mathcal{J}'_{r,m-n},\\ &\mathcal{J}'_{r,m}\mathcal{K}'_{r,n} - \mathcal{J}'_{r,n}\mathcal{K}'_{r,m} = 2^{1-r}(-1)^n(2^r+4^r)^2\mathcal{J}'_{r,m-n}.\end{aligned}$$

Proof. All the proofs are based on the Binet formula and we prove two of them. We have

$$\begin{aligned}4^r\mathcal{K}_{r,n}^2 + (4\cdot 2^r + 5\cdot 4^r)\mathcal{J}_{r,n}^2 &= (\lambda_1^*\lambda_1^n + \lambda_2^*\lambda_2^n)^2 + (\lambda_1^*\lambda_1^n - \lambda_2^*\lambda_2^n)^2\\ &= 2(\lambda_1^*)^2\lambda_1^{2n} + 2(\lambda_2^*)^2\lambda_2^{2n}\end{aligned}$$

$$= 2\big[(2^r + 1 + \lambda_1)\lambda_1^*\lambda_1^{2n} + (2^r + 1 + \lambda_2)\lambda_2^*\lambda_2^{2n}\big]$$
$$= 2\big[(2^r + 1)(\lambda_1^*\lambda_1^{2n} + \lambda_2^*\lambda_2^{2n}) + (\lambda_1^*\lambda_1^{2n+1} + \lambda_2^*\lambda_2^{2n+1})\big]$$
$$= 2^{r+1}\big[(2^r + 1)\mathcal{K}_{r,2n} + \mathcal{K}_{r,2n+1}\big].$$

The recurrence relation (2.1) gives (3.6).

Now we prove (3.7). From Binet formula for the r-Jacobsthal and the r-Jacobsthal–Lucas numbers, we have

$$\begin{aligned}
&\mathcal{J}_{r,m}\mathcal{K}_{r,n} + \mathcal{J}_{r,n}\mathcal{K}_{r,m}\\
&\quad= \frac{1}{2^r\sqrt{4\cdot 2^r + 5\cdot 4^r}}\big[(\lambda_1^*\lambda_1^m - \lambda_2^*\lambda_2^m)(\lambda_1^*\lambda_1^n + \lambda_2^*\lambda_2^n)\\
&\qquad\qquad + (\lambda_1^*\lambda_1^n - \lambda_2^*\lambda_2^n)(\lambda_1^*\lambda_1^m + \lambda_2^*\lambda_2^m)\big]\\
&\quad= \frac{1}{2^r\sqrt{4\cdot 2^r + 5\cdot 4^r}}\big[2(\lambda_1^*)^2\lambda_1^{m+n} - 2(\lambda_2^*)^2\lambda_2^{m+n}\big]\\
&\quad= \frac{2}{2^r\sqrt{4\cdot 2^r + 5\cdot 4^r}}\big[(2^r + 1 + \lambda_1)\lambda_1^*\lambda_1^{m+n}\\
&\qquad\qquad - (2^r + 1 + \lambda_2)\lambda_2^*\lambda_2^{m+n}\big]\\
&\quad= \frac{1}{2^r\sqrt{4\cdot 2^r + 5\cdot 4^r}}\big[(2^r + 1)(\lambda_1^*\lambda_1^{m+n} + \lambda_2^*\lambda_2^{m+n})\\
&\qquad\qquad + (\lambda_1^*\lambda_1^{m+n+1} + \lambda_2^*\lambda_2^{m+n+1})\big]\\
&\quad= \frac{2}{2^r}\big[\mathcal{J}_{r,m+n+1} + (2^r + 1)\mathcal{J}_{r,m+n}\big].
\end{aligned}$$

From the recurrence relation (1.5), we obtain (3.7). The other identities can be proved similarly. □

By Theorem 3.3, for $r = 0$, we obtain known identities (1.1), (1.2), (1.3), (1.4) for the classical Jacobsthal and Jacobsthal–Lucas numbers.

In the next theorem we give summation formulas for the r-Jacobsthal-Lucas numbers and the second types of the r-Jacobsthal and the r-Jacobsthal-Lucas numbers.

Theorem 3.4. *Let n, r be integers. Then*

$$\sum_{i=0}^{n-1}\mathcal{K}_{r,i} = \frac{\mathcal{K}_{r,n} + (2^r + 4^r)\mathcal{K}_{r,n-1} - 2^r - 2^{1-r} - 4}{2^{r+1} + 4^r - 1}, \tag{3.8}$$

$$\sum_{i=0}^{n-1}\mathcal{J}'_{r,i} = \frac{\mathcal{J}'_{r,n} + (2^r + 4^r)\mathcal{J}'_{r,n-1} - 1}{2^{r+1} + 4^r - 1}, \tag{3.9}$$

$$\sum_{i=0}^{n-1} \mathcal{K}'_{r,i} = \frac{\mathcal{K}'_{r,n} + (2^r + 4^r)\mathcal{K}'_{r,n-1} + 1 - 2^{1-r}}{2^{r+1} + 4^r - 1}. \tag{3.10}$$

PROOF. For (3.8), on account of (1.7) we get

$$\sum_{i=0}^{n-1} \mathcal{K}_{r,i} = \sum_{i=0}^{n-1} (A_1\lambda_1^i + A_2\lambda_2^i) = A_1 \frac{1-\lambda_1^n}{1-\lambda_1} + A_2 \frac{1-\lambda_2^n}{1-\lambda_2}$$

$$= \frac{A_1 + A_2 - (A_1\lambda_2 + A_2\lambda_1) - (A_1\lambda_1^n + A_2\lambda_2^n) + \lambda_1\lambda_2(A_1\lambda_1^{n-1} + A_2\lambda_2^{n-1})}{1 - (\lambda_1 + \lambda_2) + \lambda_1\lambda_2}$$

$$= \frac{A_1 + A_2 - (\lambda_1\lambda_2 - \lambda_2\lambda_1) - \mathcal{K}_{r,n} - (2^r + 4^r)\mathcal{K}_{r,n-1}}{1 - 2^r - (2^r + 4^r)}.$$

By simple calculations we have $A_1 + A_2 = 3 + 2^{1-r}$, $A_1\lambda_2 + A_2\lambda_1 = -(1 + 2^r)$. Hence

$$\sum_{i=0}^{n-1} \mathcal{K}_{r,i} = \frac{4 + 2^{1-r} + 2^r - \mathcal{K}_{r,n} - (2^r + 4^r)\mathcal{K}_{r,n-1}}{-(2^{r+1} + 4^r - 1)}$$

$$= \frac{\mathcal{K}_{r,n} + (2^r + 4^r)\mathcal{K}_{r,n-1} - 2^r - 2^{1-r} - 4}{2^{r+1} + 4^r - 1}.$$

In the same way one can easily prove (3.9) and (3.10). □

4. Matrix generators

Now we give the matrix generators of the numbers $\mathcal{J}_{r,n}$ and $\mathcal{K}_{r,n}$.

THEOREM 4.1. *Let n, r be integers. Then*

$$\begin{bmatrix} \mathcal{J}_{r,n+1} & \mathcal{J}_{r,n} \\ \mathcal{J}_{r,n} & \mathcal{J}_{r,n-1} \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{r,2} & \mathcal{J}_{r,1} \\ \mathcal{J}_{r,1} & \mathcal{J}_{r,0} \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix}^{n-1}. \tag{4.1}$$

PROOF. (by induction on n) It is easily seen that for $n = 1$ the result is obvious. Assuming that the formula (4.1) holds for $n \geq 1$, we will prove it for $n + 1$.

Using induction's hypothesis and the recurrence formula for the r-Jacobsthal numbers, we have

$$\begin{aligned}
&\begin{bmatrix} \mathcal{J}_{r,2} & \mathcal{J}_{r,1} \\ \mathcal{J}_{r,1} & \mathcal{J}_{r,0} \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix} \\
&\qquad = \begin{bmatrix} \mathcal{J}_{r,n+1} & \mathcal{J}_{r,n} \\ \mathcal{J}_{r,n} & \mathcal{J}_{r,n-1} \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix} \\
&\qquad = \begin{bmatrix} 2^r \mathcal{J}_{r,n+1} + (2^r + 4^r)\mathcal{J}_{r,n} & \mathcal{J}_{r,n+1} \\ 2^r \mathcal{J}_{r,n} + (2^r + 4^r)\mathcal{J}_{r,n-1} & \mathcal{J}_{r,n} \end{bmatrix} \\
&\qquad = \begin{bmatrix} \mathcal{J}_{r,n+2} & \mathcal{J}_{r,n+1} \\ \mathcal{J}_{r,n+1} & \mathcal{J}_{r,n} \end{bmatrix},
\end{aligned}$$

which ends the proof. □

As a consequence of Theorem 4.1 we get Cassini's identity (3.4) for the r-Jacobsthal numbers.

COROLLARY 4.2. *Let n, r be integers. Then*

$$\mathcal{J}_{r,n+1}\mathcal{J}_{r,n-1} - \mathcal{J}_{r,n}^2 = (-1)^n (2^r + 4^r)^{n-1} (2^r + 1)^2.$$

PROOF. Calculating determinants in formula (4.1), we obtain

$$\begin{vmatrix} \mathcal{J}_{r,n+1} & \mathcal{J}_{r,n} \\ \mathcal{J}_{r,n} & \mathcal{J}_{r,n-1} \end{vmatrix} = \mathcal{J}_{r,n+1}\mathcal{J}_{r,n-1} - \mathcal{J}_{r,n}^2,$$

$$\begin{vmatrix} \mathcal{J}_{r,2} & \mathcal{J}_{r,1} \\ \mathcal{J}_{r,1} & \mathcal{J}_{r,0} \end{vmatrix} = \begin{vmatrix} 3 \cdot 4^r + 2 \cdot 2^r & 2 \cdot 2^r + 1 \\ 2 \cdot 2^r + 1 & 1 \end{vmatrix} = -(2^r + 1)^2,$$

$$\begin{vmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{vmatrix} = -(2^r + 4^r).$$

Hence we get

$$\begin{aligned}
\mathcal{J}_{r,n+1}\mathcal{J}_{r,n-1} - \mathcal{J}_{r,n}^2 &= -(2^r + 1)^2 (-1)^{n-1} (2^r + 4^r)^{n-1} \\
&= (-1)^n (2^r + 4^r)^{n-1} (2^r + 1)^2,
\end{aligned}$$

which completes the proof. □

Similarly to Theorem 4.1 and Corollary 4.2, we can prove the following results.

Theorem 4.3. *Let n, r be integers. Then*

$$\begin{bmatrix} \mathcal{K}_{r,n+1} & \mathcal{K}_{r,n} \\ \mathcal{K}_{r,n} & \mathcal{K}_{r,n-1} \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{r,2} & \mathcal{K}_{r,1} \\ \mathcal{K}_{r,1} & \mathcal{K}_{r,0} \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix}^{n-1}.$$

Corollary 4.4. *Let n, r be integers. Then*

$$\mathcal{K}_{r,n+1}\mathcal{K}_{r,n-1} - \mathcal{K}_{r,n}^2 = (-1)^{n-1}(2^r + 4^r)^{n-1}(13 + 2^{2-r} + 14 \cdot 2^r + 5 \cdot 4^r).$$

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