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NUMERIC FEM'S SOLUTION FOR SPACE-TIME DIFFUSION PARTIAL DIFFERENTIAL EQUATIONS WITH CAPUTO–FABRIZION AND RIEMANN–LIOUVILLE FRACTIONAL ORDER'S DERIVATIVES

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Abstract. In this paper, we use the finite element method to solve the fractional space-time diffusion equation over finite fields. This equation is obtained from the standard diffusion equation by replacing the first temporal derivative with the new fractional derivative recently introduced by Caputo and Fabrizion and the second spatial derivative with the Riemann–Liouville fractional derivative. The existence and uniqueness of the numerical solution and the result of error estimation are given. Numerical examples are used to support the theoretical results.

1. Introduction

In this paper, we consider the space-time fractional diffusion equation of the form

(1.1)
$$\begin{cases} {}_0^{CF}D_t^{\alpha}u(x,t) = {}_0^{RL}D_x^{1+\beta}u(x,t) + f(x,t), & (x,t) \in [0,T]\times[0,1] = \Omega\times I, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(0,t) = u(1,t) = 0, & t \in I \end{cases}$$

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where ${}_0^{CF}D_t^{\alpha}u(x,t)$ presents the Caputo Fabrizio fractional derivative of order $0 < \alpha < 1$ with respect to the time variable and ${}_0^{RL}D_x^{1+\beta}u(x,t)$ is the Riemann–Liouville fractional derivative where $0 < \beta < 1$ with respect to the space. $f\colon \Omega \times I \to \mathbb{R}$ and $u_0\colon \Omega \to \mathbb{R}$ are given functions.

Fraction theory is of great interest to researchers because of its practical applications in various scientific and engineering fields such as physics, chemistry, economics, electrodynamics, and biology. Although fractional calculus dates back three centuries, its study is being developed and researched, introducing many definitions of fractional integrals and derivatives, and various numerical methods for solving fractional partial differential equations [3]–[5], [7], [8], [11] and [12].

However, the reference to numerical and analytical methods for fractional space-time derivatives of partial differential equations is not as good as that for partial differential equations with only fractional derivatives. In [10], Meerschaert and Tadjeran gave a finite-difference approximation to solve the spatial fractional diffuse advection equation using the Riemann–Liouville fraction derivation. Hejazi *et al.* in [4] proposed a finite volume method to solve the fractional space-time derivatives on both sides of the diffusion advection equation. Li and Xu in [8] proposed a finite-difference spectral approximation method for the time-fractional derivation of the diffusion equation. A numerical approximation is given for nonlinear fractional differential equations by Li *et al.* in [6].

Recently, Caputo and Fabrizio introduced a new definition of nonsingular kernel fractional derivatives, which assumes two different representations for temporal and spatial variables (see [2]).

The purpose of this paper is to propose a finite element method for solving the space-time fractional reaction-diffusion equation. The existence and uniqueness of the solution and the resulting error estimate are given here. This work is organized as follows. In Section 2, prior knowledge about fractional derivatives and fractional derivative spaces is introduced. Semi-discretization is given in Section 3. Error estimates for the finite element scheme are obtained in Section 4. To demonstrate the validity of the theoretical results, numerical examples are given in Section 5.

2. Fractional derivatives and fractional spaces

2.1. Fractional derivatives

In this subsection, we introduce some definitions of fractional derivatives that will be used later in this paper.

First, we recall the usual fractional Caputo derivative of order α, given by:

$$ {}_a^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} u'(x,s) ds, \quad 0 < \alpha < 1, \ a > 0 $$

where $\Gamma(z) = \int_0^{+\infty} x^{z-1} e^{-x} dx$ is Euler's gamma function.

By replacing the kernel function $(t-s)^{-\alpha}$ with the new kernel function $\exp\left[-\alpha \frac{t-s}{1-\alpha}\right]$ and the coefficients $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{1-\alpha}$ where $M(\alpha)$ is a normalization function, so $M(0) = M(1) = 1$, Caputo and Fabrizio gave the following new definition of fractional time derivatives.

Definition 2.1 ([2]). Let $u \in H^1(a,b)$, $b > a$, $\alpha \in (0,1)$. Then the *new fractional Caputo* derivative is defined as follows

$$ D_t^\alpha u(x,t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right] u'(x,s) ds. $$

If the function $u \notin H^1(a,b)$, then the fractional derivative can be defined as

$$ D_t^\alpha u(x,t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right] (u(x,t) - u(x,s)) \, ds. $$

Definition 2.2 ([1]). Losada and Nieto proposed that the *new Caputo fractional derivatives* of order $0 < \alpha < 1$ can be expressed as follows

$$ D_t^\alpha u(x,t) = \frac{1}{1-\alpha} \int_a^t \exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right] u'(x,s) ds. $$

In this paper, for time variables, we will use the new smooth kernel fractional derivative defined by Caputo and Fabrizio in the two definitions above, and for space variables, we will use the usual Riemann–Liouville fractional derivative defined by

Definition 2.3. The *left* and *right fractional Riemann–Liouville derivatives* of order α are given as

$$ {}_a^{Rl} D_x^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\eta)^{n-\alpha-1} u(\eta,t) d\eta, $$

$$ {}_x^{Rl} D_b^\alpha u(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (\eta-x)^{n-\alpha-1} u(\eta,t) d\eta $$

where $n-1 < \alpha < n \in \mathbb{N}$ and $x \in [a,b]$.

2.2. Fractional spaces

In this subsection, we will introduce the fractional spaces with some important properties concerning the space fractional diffusion operator ${}^{Rl}D^{1+\beta}$. We are going to use the notation D^α for convenience.

First we give some notations that will be used: $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, $\|\cdot\|_{L^p}$ denotes the $L^p(\Omega)$ norm and for the special cases $L^2(\Omega)$ and $L^\infty(\Omega)$ norms are denoted by $\|\cdot\|$ and $\|\cdot\|_\infty$ respectively.

For the Sobolev spaces $H^k(\Omega)$ the norm is denoted by $\|\cdot\|_{H^k(\Omega)}$ or $\|\cdot\|_k$, when $u(x,t)$ is defined on the entire time interval $(0,T)$, we define

$$\|u\|_{L^\infty(H^k(\Omega))} = \sup_{0<t<T} \|u(\cdot,t)\|_k\,.$$

Definition 2.4. Let $\alpha > 0$, define the following semi-norms and norms of the left (J_L^α), the right (J_R^α) and the symmetric (J_S^α) fractional derivatives spaces on bounded domain Ω as follows

$$|u|_{J_L^\alpha(\Omega)} = \|D^\alpha u\|_{L^2(\Omega)}\,, \qquad \|u\|^2_{J_L^\alpha(\Omega)} = \|u\|^2_{L^2(\Omega)} + p\,|u|^2_{J_L^\alpha(\Omega)}\,, \tag{2.1}$$

$$|u|_{J_R^\alpha(\Omega)} = \|D^{\alpha*} u\|_{L^2(\Omega)}\,, \qquad \|u\|^2_{J_R^\alpha(\Omega)} = \|u\|^2_{L^2(\Omega)} + p\,|u|^2_{J_R^\alpha(\Omega)}\,, \tag{2.2}$$

$$|u|_{J_S^\alpha(\Omega)} = \|(D^\alpha u, D^{\alpha*})\|_{L^2(\Omega)}\,, \qquad \|u\|^2_{J_S^\alpha(\Omega)} = \|u\|^2_{L^2(\Omega)} + p\,|u|^2_{J_S^\alpha(\Omega)} \tag{2.3}$$

where p will be defined later in the next section.

Definition 2.5. We define the *norm of fractional Sobolev spaces* $H^\alpha(\mathbb{R})$ for $\alpha > 0$ as follows

$$|u|_{H^\alpha(\mathbb{R})} = \||\omega|^\alpha\,\hat{u}\|_{L^2(\mathbb{R})}\,, \quad \|u\|^2_{H^\alpha(\mathbb{R})} = \|u\|^2_{L^2(\mathbb{R})} + p\,|u|^2_{H^\alpha(\mathbb{R})}\,.$$

Denote $H^\alpha(\mathbb{R})$ as the closure of $C^\infty(\mathbb{R})$ with respect to the norm $\|.\|_{H^\alpha(\mathbb{R})}$. Now, for a bounded domain Ω, we define the fractional Sobolev space $H^\alpha(\Omega)$ by

$$H^\alpha(\Omega) = \left\{v \in L^2(\Omega),\ \exists\tilde{v} \in H^\alpha(\mathbb{R}) \text{ such that } \tilde{v}|_\Omega = v\right\}$$

equipped with the norm

$$\|v\|_{\alpha,\Omega} = \inf_{\tilde{v}\in H^\alpha(\mathbb{R}),\tilde{v}|_\Omega = v} \|\tilde{v}\|_{\alpha,\mathbb{R}}\,.$$

Lemma 2.1. *For $\alpha > 0, \alpha \neq n - 1/2$ and $n \in \mathbb{N}$, the spaces $J_L^\alpha(\Omega)$, $J_R^\alpha(\Omega)$, $J_S^\alpha(\Omega)$ and $H^\alpha(\Omega)$ are equal in the sense that their semi-norms and norms are equivalent.*

Lemma 2.2. *For $\alpha > 0, \alpha \neq n - 1/2$ and $n \in \mathbb{N}$, the spaces $J_L^\alpha(\Omega)$, $J_R^\alpha(\Omega)$, $J_S^\alpha(\Omega)$ and $H^\alpha(\Omega)$ denote the closure of $C^\infty(\Omega)$ under their respect norms.*

Lemma 2.3. *For $\alpha > 0, \alpha \neq n - 1/2$ and $n \in \mathbb{N}$, the spaces $J_{L,0}^\alpha(\Omega)$, $J_{R,0}^\alpha(\Omega)$, $J_{S,0}^\alpha(\Omega)$ and $H_0^\alpha(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ under their respect norms.*

In this paper, we use H_0^α to denote the fractional derivative space equipped with the norm $\|\cdot\|_\alpha$ which can be any one of the norms given in Definition 2.4.

Let us denote $X = H_0^{(1+\beta)/2}(\Omega)$ and its dual by $X^* = H_0^{-(1+\beta)/2}(\Omega)$ with the norm $\|.\|_{-(1+\beta)/2}$.

The following properties are useful for the theoretical analysis of finite elements. There exist constants $C_1, C_2 > 0$ such that for $u, v \in X$, $1/2 < \alpha < 1$, we have

$$(2.4)\quad -(D^{2\alpha}u, v) = -(D^\alpha u, D^{\alpha *}v) \leq C_1 \|u\|_\alpha \|v\|_\alpha \ \text{(continuity on } X \times X),$$

$$(2.5)\quad -(D^{2\alpha}u, u) = -(D^\alpha u, D^{\alpha *}u) \geq C_2 \|u\|_\alpha^2 \quad \text{(cœrcivity on } X).$$

Lemma 2.4 (Fractional Poincaré–Friedrichs Inequality). *Let $\Omega \subset \mathbb{R}$ be a bounded domain, then there exists a positive constant C such that we have*

$$\|u\|_{L^2(\Omega)} \leq C \|u\|_{H_0^\alpha(\Omega)}$$

for $u \in H_0^\alpha(\Omega)$ and for $0 < s < \alpha$ we have

$$\|u\|_{H_0^s(\Omega)} \leq C \|u\|_{H_0^\alpha(\Omega)}.$$

3. Time discretization

In this section, we will consider the time discretization of the problem (1.1) and present the variational formulation of the semi-discrete scheme.

First, we are going to use the finite difference approximation to discrete the time-fractional derivative for $0 < \alpha < 1$. Let $\Delta t = T/N$ be the time mesh-size, $t_n = n\Delta t$ for $n = 0, \ldots, N$.

To motivate the construction of the scheme, we use the following formulation: for all $0 \le n \le N-1$, we have

$$
\begin{aligned}
&{}_0^{CF}D_t^{\alpha} u(x, t_{n+1}) = \frac{1}{1-\alpha} \int_0^{t_{n+1}} \exp\left[\frac{-\alpha(t_{n+1}-s)}{1-\alpha}\right] u'(x,s)ds \\
&= \frac{1}{1-\alpha} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \exp\left[\frac{-\alpha(t_{n+1}-s)}{1-\alpha}\right] \frac{u(x,t_{k+1}) - u(x,t_k)}{\Delta t} ds + r_{\Delta t}^{n+1} \\
&= \frac{1}{1-\alpha} \sum_{k=0}^{n} \frac{u(x,t_{n-k+1}) - u(x,t_{n-k})}{\Delta t} \int_{t_k}^{t_{k+1}} \exp\left[\frac{-\alpha s}{1-\alpha}\right] ds + r_{\Delta t}^{n+1} \\
&= \frac{1}{\alpha} \sum_{k=0}^{n} \frac{u(x,t_{n+1-k}) - u(x,t_{n-k})}{\Delta t} \left(\exp\left[\frac{-\alpha t_k}{1-\alpha}\right] - \exp\left[\frac{-\alpha t_{k+1}}{1-\alpha}\right]\right) + r_{\Delta t}^{n+1} \\
&= \frac{1}{\alpha} \sum_{k=0}^{n} \frac{u(x,t_{n+1-k}) - u(x,t_{n-k})}{\Delta t} \left(\exp\left[\frac{-\alpha k \Delta t}{1-\alpha}\right] - \exp\left[\frac{-\alpha(k+1)\Delta t}{1-\alpha}\right]\right) \\
&\quad + r_{\Delta t}^{n+1}.
\end{aligned}
$$

For the sake of simplification, let us introduce the notation

$$
B_k = \exp\left[\frac{-\alpha k \Delta t}{1-\alpha}\right] - \exp\left[\frac{-\alpha(k+1)\Delta t}{1-\alpha}\right]
$$

and define the discrete fractional operator P_t^{α} by

$$
P_t^{\alpha} u(x, t_{n+1}) = \frac{1}{\alpha} \sum_{k=0}^{n} B_k \frac{u(x,t_{n+1-k}) - u(x,t_{n-k})}{\Delta t}.
$$

Then we have

$$
{}_0^{CF}D_t^{\alpha} u(x, t_{n+1}) = P_t^{\alpha} u(x, t_{n+1}) + r_{\Delta t}^{n+1}
$$

where $r_{\Delta t}^{n+1}$ is the truncation error defined by

$$
\begin{aligned}
r_{\Delta t}^{n+1} = \frac{1}{1-\alpha} \sum_{k=0}^{n} \Bigg[&\int_{t_k}^{t_{k+1}} \exp\left[\frac{-\alpha(t_{n+1}-s)}{1-\alpha}\right] u'(x,s)ds \\
&- \frac{u(x,t_{k+1}) - u(x,t_k)}{\Delta t} \int_{t_k}^{t_{k+1}} \exp\left[\frac{-\alpha s}{1-\alpha}\right] ds \Bigg]
\end{aligned}
$$

and $r_{\Delta t}^{n+1}$ is estimated, see [9], by

$$\left\| r_{\Delta t}^{n+1} \right\| \leq C_{u,\alpha} \Delta t^2.$$

The problem (1.1) can be transformed into a semi-discrete variational problem, which is given for $u^{n+1}(x)$ the approximation of $u(x, t_n)$ by:

$$\begin{cases} \text{Find } u^{n+1} \in X \text{ for } n = 0, 1, \cdots, N-1 \text{ such that} \\ (P_t^{\alpha} u^{n+1}, v) + ({}_0^{Rl}D_x^{\beta} u^{n+1}, \nabla v) = (f^{n+1}, v) \quad \forall v \in X. \end{cases}$$

After some adjustment, we have the iterative solution u^{n+1} in the following form : $\forall v \in X$

$$(u^{n+1}, v) + p({}_0^{Rl}D_x^{\beta} u^{n+1}, \nabla v) = \lambda \sum_{k=0}^{n} (B_k - B_{k+1}) (u^{n-k}, v) + \lambda B_n (u^0, v) + p(f^{n+1}, v)$$

where $\lambda = \frac{1}{B_0}$ and $p = a\lambda\Delta t$.

4. Full discretization: Finite element approximation

In this section, we will give the full discrete schemes, examine the existence and uniqueness of the variational solution and finally present the error estimate.

Let S_h denote a uniform partition of Ω which is given by

$$0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = L$$

where m is a positive integer.

Let $h = L/m = x_i - x_{i-1}$ and $\Omega_i = [x_{i-1}, x_i)$ for $i = 1, \cdots, m$. Define the space X_h as the set of piecewise polynomials of order $r, (r \in \mathbb{N})$ on the mesh S_h:

$$X_h = \left\{ v : v|_{\Omega_i} \in P_r(\Omega), v \in C(\Omega) \right\}.$$

Let u_h^{n+1} be the finite element solution at $t = t_{n+1}$ of the problem (1.1), then we derive the full discrete scheme for $0 < \alpha < 1$, $\forall v \in X_h$ by

$$(4.1) \qquad (u_h^{n+1}, v) + p({}_0^{Rl}D_x^{\beta} u_h^{n+1}, \nabla v) = \lambda \sum_{k=0}^{n} (B_k - B_{k+1}) (u_h^{n-k}, v) + \lambda B_n (u^0, v) + p(f^{n+1}, v).$$

LEMMA 4.1 (Existence and uniqueness of the solution). *For a small step size $\Delta t > 0$, there exists a unique solution u_h^{n+1} satisfying* (4.1).

Furthermore, if u_h^{n+1} is a solution, then

$$\left\|u_h^{n+1}\right\|_{(1+\beta)/2} \leq C \left\|\tilde{f}\right\|_{-(1+\beta)/2}.$$

PROOF. Since X_h is a subset of X the existence and uniqueness of the solution is assured by the well-known Lax–Milgram Lemma.

It consists of proving the cœrcivity and continuity of the bilinear form A given by

$$A(u, v) = (u_h^{n+1}, v) + p({}_0^{Rl}D_x^{\beta} u_h^{n+1}, \nabla v)$$

and the continuity of the linear form F given by $F(v) = (\tilde{f}, v)$ where

$$\tilde{f} = \lambda \sum_{k=0}^{n} (B_k - B_{k+1}) (u_h^{n-k}, v) + \lambda B_n (u^0, v) + p(f^{n+1}, v).$$

1. The cœrcivity of A: Using (2.3) and (2.5), we get for any $u_h^{n+1} \in H_0^{(1+\beta)/2}$

$$\begin{aligned} A(u_h^{n+1}, u_h^{n+1}) &= (u_h^{n+1}, u_h^{n+1}) + p({}_0^{Rl}D_x^{\beta} u_h^{n+1}, \nabla u_h^{n+1}) \\ &= \left\|u_h^{n+1}\right\|_{L^2}^2 + p({}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1},\ {}_0^{Rl}D_x^{[(\beta+1)/2]^*} u_h^{n+1}) \\ &= \left\|u_h^{n+1}\right\|_{L^2}^2 + p\left|u_h^{n+1}\right|_{J_S^{(\beta+1)/2}}^2 \\ &= \left\|u_h^{n+1}\right\|_{J_S^{(1+\beta)/2}}^2 . \end{aligned}$$

We get by Lemma 2.1 to Lemma 2.4

$$A(u_h^{n+1}, u_h^{n+1}) \geq C_1 \left\|u_h^{n+1}\right\|_{H_0^{(1+\beta)/2}}^2 \geq C_1 \left\|u_h^{n+1}\right\|_{(1+\beta)/2}^2 .$$

Then A is cœrcive.

2. The continuity of A: Using (2.4), for any $u_h^{n+1}, v \in H_0^{(1+\beta)/2}$

$$\begin{aligned} A(u_h^{n+1}, v) &= (u_h^{n+1}, v) + p({}_0^{Rl}D_x^{\beta} u_h^{n+1}, \nabla v) \\ &= (u_h^{n+1}, v) + p({}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1},\ {}_0^{Rl}D_x^{[(\beta+1)/2]^*} v) \\ &= \int_\Omega |u_h^{n+1}(x) v(x)| dx \\ &\quad + p \int_\Omega |\ {}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1}(x)\ {}_0^{Rl}D_x^{[(\beta+1)/2]^*} v(x)\ | dx. \end{aligned}$$

We get by Cauchy–Schwarz inequality

$$\begin{aligned} A(u_h^{n+1}, v) \leq & \left(\int_\Omega \left| u_h^{n+1}(x) \right|^2 dx \right)^{1/2} \left(\int_\Omega |v(x)|^2 dx \right)^{1/2} \\ & + p \left(\int_\Omega \left| {}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1}(x) \right|^2 dx \right)^{1/2} \left(\int_\Omega \left| {}_0^{Rl}D_x^{[(\beta+1)/2]^*} v(x) \right|^2 dx \right)^{1/2} \end{aligned}$$

$$= \left\| u_h^{n+1} \right\|_{L^2} \|v\|_{L^2} + p \left\| {}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1} \right\|_{L^2} \|{}_0^{Rl}D_x^{[(\beta+1)/2]^*} v\|_{L^2}. \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} 2 \left\| u_h^{n+1} \right\|_{L^2} \|v\|_{L^2} & \left\| {}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1} \right\|_{L^2} \|{}_0^{Rl}D_x^{[(\beta+1)/2]^*} v\|_{L^2} \\ & \leq \left\| u_h^{n+1} \right\|_{L^2}^2 \left\| {}_0^{Rl}D_x^{[(\beta+1)/2]^*} v \right\|_{L^2}^2 + \left\| {}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1} \right\|_{L^2}^2 \|v\|_{L^2}^2 . \end{aligned} \tag{4.3}$$

By Lemmas 2.1–2.4, (2.1), (2.2), (4.2) and (4.3), we obtain

$$\begin{aligned} A^2(u_h^{n+1}, v) \leq & \left[\left\| u_h^{n+1} \right\|_{L^2} \|v\|_{L^2} + p \left\| {}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1} \right\|_{L^2} \left\| {}_0^{Rl}D_x^{[(\beta+1)/2]^*} v \right\|_{L^2} \right]^2 \\ \leq & \left\| u_h^{n+1} \right\|_{L^2}^2 \|v\|_{L^2}^2 + p \left\| u_h^{n+1} \right\|_{L^2}^2 \left\| {}_0^{Rl}D_x^{[(\beta+1)/2]^*} v \right\|_{L^2}^2 \\ & + p \left\| {}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1} \right\|_{L^2}^2 \|v\|_{L^2}^2 \\ & + p^2 \left\| {}_0^{Rl}D_x^{(\beta+1)/2} u_h^{n+1} \right\|_{L^2}^2 \left\| {}_0^{Rl}D_x^{[(\beta+1)/2]^*} v \right\|_{L^2}^2 \end{aligned}$$

$$\le \left(\left\|u_h^{n+1}\right\|_{L^2}^2 + p\left\|{}_0^{Rl}D_x^{(\beta+1)/2}u_h^{n+1}\right\|_{L^2}^2\right)$$

$$\times\left(\|v\|_{L^2}^2 + p\left\|{}_0^{Rl}D_x^{[(\beta+1)/2]^*}v\right\|_{L^2}^2\right)$$

$$\le \left\|u_h^{n+1}\right\|_{J_L^{(1+\beta)/2}}^2 \|v\|_{J_R^{(1+\beta)/2}}^2 \le C_2^2\left\|u_h^{n+1}\right\|_{H_0^{(1+\beta)/2}}^2 \|v\|_{H_0^{(1+\beta)/2}}^2$$

$$\le C_2^2\left\|u_h^{n+1}\right\|_{(1+\beta)/2}^2 \|v\|_{(1+\beta)/2}^2.$$

Then,

$$A(u_h^{n+1}, v) \le C_2\left\|u_h^{n+1}\right\|_{(1+\beta)/2}\|v\|_{(1+\beta)/2}.$$

Hence, A is continuous.

3. The continuity of F: For any $v \in H_0^{(1+\beta)/2}$, we have by using the Cauchy-Schwarz inequality

$$|F(v)| = \left|(\tilde{f}, v)\right| \le \int_\Omega |\tilde{f}(x)v(x)|dx$$

$$\le \left(\int_\Omega |\tilde{f}(x)|^2 dx\right)^{1/2}\left(\int_\Omega |v(x)|^2 dx\right)^{1/2} = \|\tilde{f}\|_{L^2}\|v\|_{L^2}.$$

Using (2.1)–(2.2) and Lemmas 2.1–2.4, we get

$$|F(v)| \le \|\tilde{f}\|_{L^2}\|v\|_{L^2} \le \|\tilde{f}\|_{J_R^{[(\beta+1)/2]^*}}\|v\|_{J_L^{(\beta+1)/2}}$$

$$\le \|\tilde{f}\|_{H_0^{-(\beta+1)/2}}\|v\|_{H_0^{(\beta+1)/2}} \le \|\tilde{f}\|_{-(\beta+1)/2}\|v\|_{(\beta+1)/2}.$$

Moreover, as $\tilde{f} \in X \subset X^*$ is continuous, we can say that $F(\cdot)$ is continuous over X which completes the proof.

4. Estimates of solution u_h^{n+1}: By cœrcivity of A and continuity of F, we have:

$$C_1\left\|u_h^{n+1}\right\|_{(1+\beta)/2}^2 \le A(u_h^{n+1}, u_h^{n+1}) = F(u_h^{n+1})$$

$$\le \|\tilde{f}\|_{-(\beta+1)/2}\|u_h^{n+1}\|_{(\beta+1)/2}.$$

So,

$$\left\|u_h^{n+1}\right\|_{(1+\beta)/2} \le C\|\tilde{f}\|_{-(\beta+1)/2}$$

where $C = C_1^{-1}$. □

4.1. The error estimate

Theorem 4.1. *Assume that the problem* (1.1) *has a unique exact solution* $u(t_{n+1})$ *at* $t = t_{n+1}$ *and* $u_t \in L^2(I, H^{r+1}(\Omega)) \cap L^\infty(I, H^{r+1})$ *is the finite element solution of the problem* (1.1) *with the initial condition* $u^0 \in H^{r+1}(\Omega)$. *Suppose that* $u_{tt} \in L^2(I, L^2(\Omega))$, *then we have*

$$\left\|u(t_{n+1}) - u_h^{n+1}\right\|_{(1+\beta)/2} \leq C_u \left[h^{r+(1-\beta)/2} \left\|u\right\|_{L^\infty(H^{r+1}(\Omega))} + \Delta t^2\right].$$

Proof. Define for $u^{n+1} \in \xi_h$,

$$e^{n+1} = u(t_{n+1}) - u_h^{n+1},$$

$$\overline{E}^{n+1} = u(t_{n+1}) - u^{n+1}$$

and

$$E^{n+1} = u^{n+1} - u_h^{n+1}.$$

So, we get

$$e^{n+1} = \overline{E}^{n+1} + E^{n+1}.$$

The exact solution satisfies at $t = t_{n+1}$ the following formula

$$(u(t_{n+1}), v) + p\left({}_0^{RL}D_x^\beta u(t_{n+1}), \nabla v\right) = r\sum_{k=0}^{n-1} \left(B_k - B_{k+1}\right) u\left(t_{n-k}, v\right) + rB_n\left(u_h^0, v\right) + p\left(f^{n+1}, v\right) - p\left(r_{\Delta t}^{n+1}, v\right). \tag{4.4}$$

Subtracting (4.1) from (4.4), we get

$$\left(e^{n+1}, v\right) + p\left({}_0^{RL}D_x^\beta e^{n+1}, \nabla v\right) = r\sum_{k=0}^{n-1} \left(B_k - B_{k+1}\right)\left(e^{n-k}, v\right) - p\left(r_{\Delta t}^{n+1}, v\right). \tag{4.5}$$

Using $e^{n+1} = \overline{E}^{n+1} + E^{n+1}$ and taking $v = E^{n+1}$ in (4.5) yields to

$$\begin{aligned}(E^{n+1}, E^{n+1}) + p\left({}_0^{RL}D_x^\beta E^{n+1}, \nabla v\right) &= r\sum_{k=0}^{n-1} \left(B_k - B_{k+1}\right)\left(e^{n-k}, E^{n+1}\right) \\ &- \left(\overline{E}^{n+1}, E^{n+1}\right) + p\left({}_0^{RL}D_x^\beta \overline{E}^{n+1}, \nabla E^{n+1}\right) - p\left(r_{\Delta t}^{n+1}, E^{n+1}\right).\end{aligned} \tag{4.6}$$

The left-hand side of (4.6) is equivalent to $\left\|E^{n+1}\right\|_{(1+\beta)/2}^2$, for the first term on the right-hand side, we have

$$r\sum_{k=0}^{n-1}\left(B_k-B_{k+1}\right)\left(e^{n-k},E^{n+1}\right)\leq r\sum_{k=0}^{n-1}\left(B_k-B_{k+1}\right)\left\|e^{n-k}\right\|\left\|E^{n+1}\right\|.$$

Then, we obtain

$$\left(e^{n-k},E^{n+1}\right)\leq\left\|e^{n-k}\right\|\left\|E^{n+1}\right\|\leq\frac{1}{\varepsilon}\left\|e^{n-k}\right\|^2+\varepsilon\left\|E^{n+1}\right\|^2.$$

Next,

$$\left(\overline{E}^{n+1},E^{n+1}\right)\leq\left\|\overline{E}^{n+1}\right\|\left\|E^{n+1}\right\|\leq\frac{1}{\varepsilon}\left\|\overline{E}^{n+1}\right\|^2+\varepsilon\left\|E^{n+1}\right\|^2$$

where

$$\left\|\overline{E}^{n+1}\right\|\leq Ch^{s+1}\left\|u(t_n)\right\|_{r+1}.$$

Next,

$$\begin{aligned}p\left({}_0^{RL}D_x^{\beta}\overline{E}^{n+1},\nabla E^{n+1}\right)&\leq p\left\|\overline{E}^{n+1}\right\|_{(1+\beta)/2}\left\|E^{n+1}\right\|_{(1+\beta)/2}\\&\leq\frac{p^2}{\varepsilon}\left\|\overline{E}^{n+1}\right\|_{(1+\beta)/2}^2+\varepsilon\left\|E^{n+1}\right\|_{(1+\beta)/2}^2\end{aligned}$$

where

$$\left\|\overline{E}^{n+1}\right\|_1\leq Ch^{s}\left\|u(t_n)\right\|_{r+1}.$$

Finally,

$$\begin{aligned}p\left(r_{\Delta t}^{n+1},E^{n+1}\right)&\leq p\left\|r_{\Delta t}^{n+1}\right\|_0\left\|E^{n+1}\right\|\leq\frac{p^2}{\varepsilon}\left\|r_{\Delta t}^{n+1}\right\|^2+\varepsilon\left\|E^{n+1}\right\|^2\\&\leq C\frac{p^2}{\varepsilon}\Delta t^4+\varepsilon\left\|E^{n+1}\right\|^2.\end{aligned}$$

Now combining the above results, we have

$$\begin{aligned}\left\|E^{n+1}\right\|_{(1+\beta)/2}^2\leq{}&\left\|e^{n-k}\right\|^2+h^{2r+2}\left\|u(t_n)\right\|_{r+1}^2+p^2h^{2r}\left\|u(t_n)\right\|_{r+1}^2+p^2\Delta t^4\\&+C\varepsilon\left\|E^{n+1}\right\|^2+\varepsilon\left\|E^{n+1}\right\|_{(1+\beta)/2}^2.\end{aligned}$$

Using

$$\left\|E^{n+1}\right\| \leq C\left\|E^{n+1}\right\|_{(1+\beta)/2},$$

we get

$$\left\|E^{n+1}\right\|_{(1+\beta)/2}^2 \leq \left\|e^{n-k}\right\|^2 + p^2\left[h^{2r}\left\|u(t_n)\right\|_{r+1}^2 + \Delta t^4\right].$$

From the above analysis, we find that

$$\begin{aligned}
\left\|e^{n+1}\right\|_{(1+\beta)/2}^2 &\leq \left\|E^{n+1}\right\|_{(1+\beta)/2}^2 + \left\|\overline{E}^{n+1}\right\|_{(1+\beta)/2}^2 \\
&\leq \left\|e^{n-k}\right\|^2 + p^2\left[h^{2r}\left\|u(t_n)\right\|_{r+1}^2 + \Delta t^4\right] + h^{2r}\left\|u(t_n)\right\|_{r+1}^2 \\
&\leq \left\|e^{n-k}\right\|^2 + C\left[h^{2r}\left\|u(t_n)\right\|_{r+1}^2 + \Delta t^4\right].
\end{aligned}$$

This yields to

$$\left\|e^{n+1}\right\|_{(1+\beta)/2}^2 \leq \left\|e^{n-k}\right\|^2 + C\left[h^{2r}\left\|u\right\|_{L^\infty(H^{r+1}(\Omega))}^2 + \Delta t^4\right].$$

Finally, using mathematical induction, we find the required result

$$\left\|u(t_n) - u_h^n\right\|_1^2 \leq C_u\left[h^{2r}\left\|u\right\|_{L^\infty(H^{r+1}(\Omega))}^2 + \Delta t^4\right]. \qquad \square$$

5. Numerical example

To demonstrate the effectiveness of the theoretical result, we will carry out a numerical example. All the numerical results in the tables are evaluated at $T = 1$ and we compute the errors in L^2 discrete norm.

$$\begin{cases}
{}_0^{CF}D_t^{\alpha}u(x,t) = {}_a^{RL}D_x^{1+\beta}u(x,t) + f(x,t), \quad 0 < \alpha, \beta < 1, \quad (x,t) \in [0,1]^2, \\
u(x,0) = 0, \quad x \in [0,1], \\
u(0,t) = u(1,t) = 0, \quad t \in [0,1]
\end{cases}$$

where $f(x,t) = \dfrac{x^3}{\alpha}\left(1 - \exp\left[\dfrac{-\alpha t}{1-\alpha}\right]\right) - \dfrac{6tx^{2-\beta}}{\Gamma(3-\beta)}$, so the exact solution is given by $u(x,t) = tx^3$.

Set $\Delta x = 1/1000$. Table 1 shows the error between the exact solution and the numerical solution and the convergence order for $\alpha = 0.5$ and different values of $h = \Delta t$ and β. In Table 2, we show the error and the convergence order for $\alpha = 0.6$ and different values of h and β. Table 3 shows the error and the convergence order for $\beta = 0.8$ and different values of h and α. Table 4 shows the error and the convergence order for $\beta = 0.9$ and different values of h and α.

Table 1. The error and convergence order for $\alpha = 0.5$ and different values of β and h

$h = \Delta t$	$\beta = 0.3$	$\beta = 0.3$	$\beta = 0.6$	$\beta = 0.6$	$\beta = 0.9$	$\beta = 0.9$
	error	order	error	order	error	order
1/10	$3.9352E-3$		$1.5978E-3$		$8.0443E-4$	
1/20	$1.8761E-3$	1.07	$6.1251E-4$	1.38	$2.4171E-4$	1.73
1/40	$8.2493E-4$	1.19	$2.1751E-4$	1.49	$6.8348E-5$	1.82
1/80	$3.4868E-4$	1.24	$7.4384E-5$	1.55	$1.8735E-5$	1.87
1/160	$1.4451E-4$	1.27	$2.4969E-5$	1.57	$5.0563E-6$	1.89

Table 2. The error and convergence order for $\alpha = 0.6$ and different values of β and h

$h = \Delta t$	$\beta = 0.3$	$\beta = 0.3$	$\beta = 0.6$	$\beta = 0.6$	$\beta = 0.9$	$\beta = 0.9$
	error	order	error	order	error	order
1/10	$5.0343E-3$		$2.0199E-3$		$9.7356E-4$	
1/20	$2.1149E-3$	1.25	$6.8776E-4$	1.55	$2.6629E-4$	1.87
1/40	$8.7434E-4$	1.27	$2.3021E-4$	1.58	$7.1726E-5$	1.89
1/80	$3.5850E-4$	1.29	$7.6438E-5$	1.59	$1.9179E-5$	1.90
1/160	$1.4635E-4$	1.29	$2.5282E-5$	1.60	$5.1113E-6$	1.91

Table 3. The error and convergence order for $\beta = 0.8$ and different values of α and h

$h = \Delta t$	$\alpha = 0.25$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.75$
	error	order	error	order	error	order
1/10	$3.4389E-4$		$8.0443E-4$		$1.9122E-3$	
1/20	$9.5408E-5$	1.85	$2.4171E-4$	1.73	$6.7487E-4$	1.50
1/40	$2.5710E-5$	1.89	$6.8348E-5$	1.82	$2.0779E-4$	1.70
1/80	$6.8304E-6$	1.91	$1.8735E-5$	1.87	$5.9640E-5$	1.80
1/160	$1.8024E-6$	1.92	$5.0563E-6$	1.89	$1.6520E-5$	1.85

Table 4. The error and convergence order for $\beta = 0.9$ and different values of α and h

$h = \Delta t$	$\alpha = 0.25$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.75$
1/10	$3.7385E-4$		$9.7355E-4$		$3.0920E-3$	
1/20	$9.9622E-5$	1.91	$2.6629E-4$	1.87	$8.5934E-4$	1.85
1/40	$2.6280E-5$	1.92	$7.1726E-5$	1.89	$2.3413E-4$	1.88
1/80	$6.9047E-6$	1.93	$1.9179E-5$	1.90	$6.1368E-5$	1.89
1/160	$1.8116E-6$	1.93	$5.1113E-6$	1.91	$1.6961E-6$	1.90

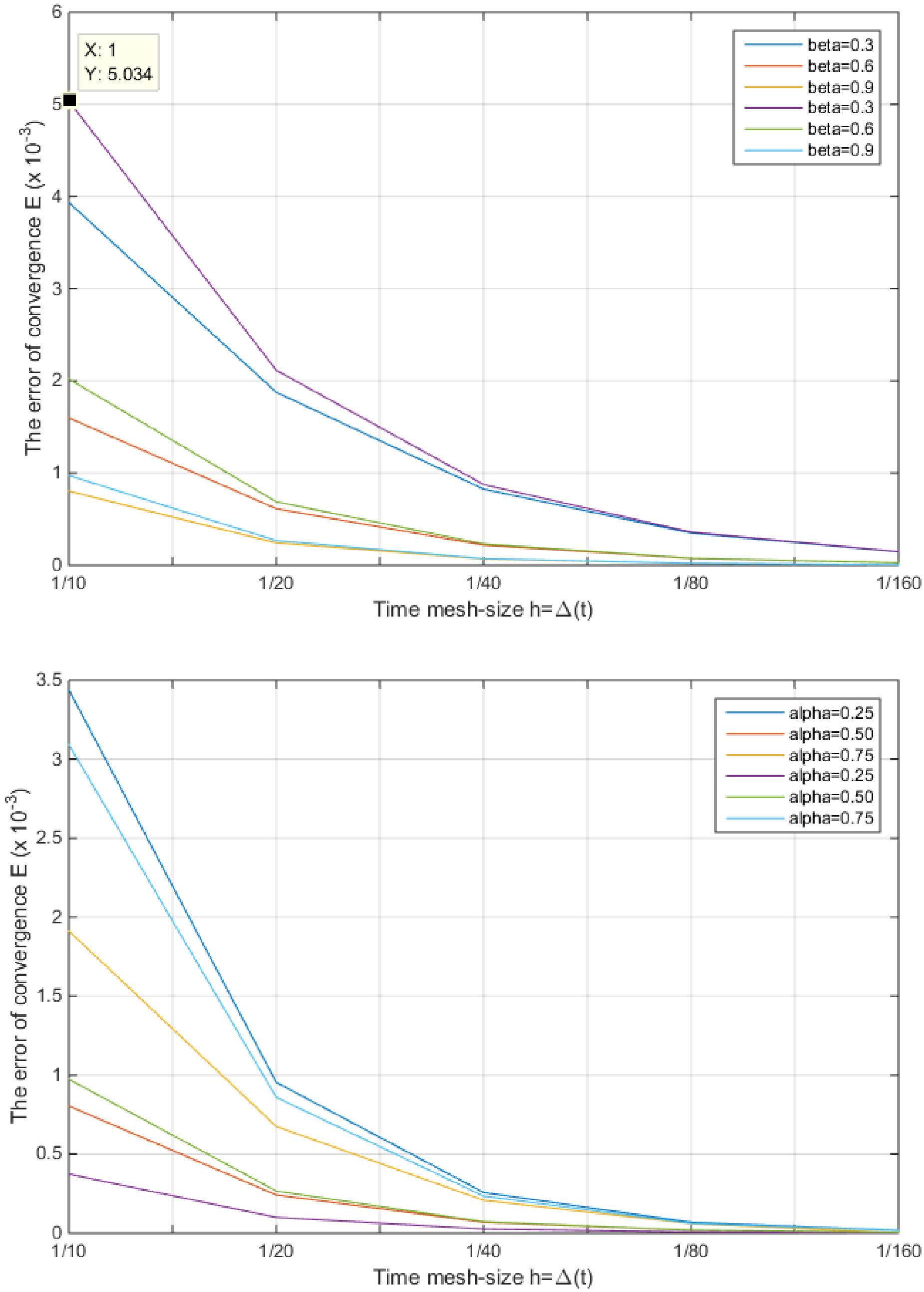

Figure 1. Errors for different values of $\alpha = 0.5$ vs $\beta = 0.8$

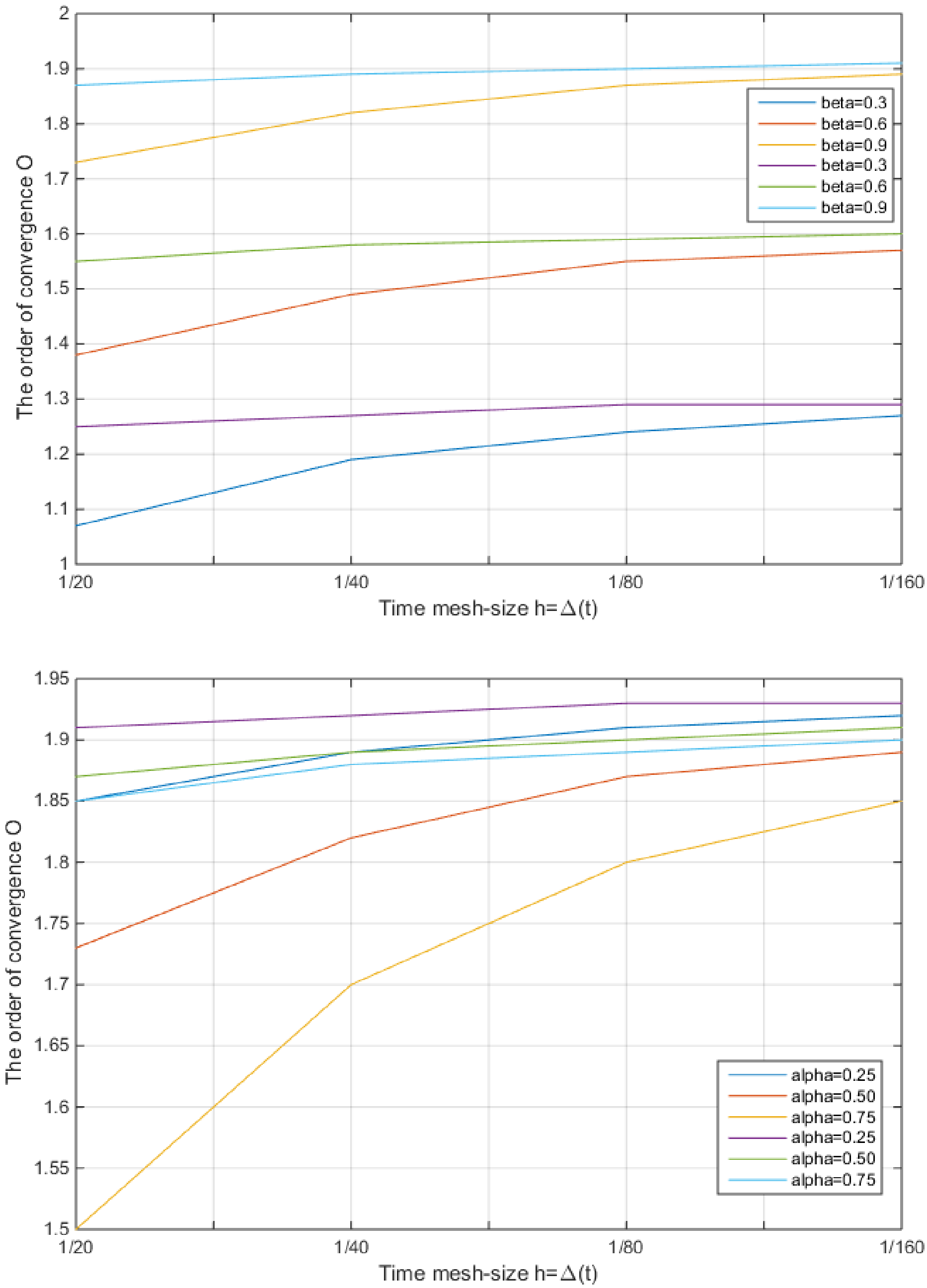

Figure 2. Convergence orders for different values of $\alpha = 0.6$ vs $\beta = 0.9$

Matlab code: appendix

```
function y
clc
clear all
format long
tic
for k=1:5
    alpha=0.5;
    beta=0.9;
    m1=5*2^k;
    n1= 1000;
    E(k)=FEM2(alpha,beta,n1,m1);
end
E
for k=1:4
    O(k)=log2(E(k)/E(k+1)); % calculate the convergence order
end
O
toc
function y=FEM2(alpha,beta,n1,m1)
a=alpha;
B=beta;
n=n1; % discretization in time
tau=1/n;
t=0:tau:1;   %t;
m=m1;% discretization in space
h=1/m; % h
x=0:h:1;     %x;
m0=1-exp(-a*tau/(1-a));
p=a*tau/m0;  % p
r1=p/gamma(3-B)/(h^B);    % r_1
for i=1:m-1
    for j=1:m-1
        if j==i
            g1(i,j)=1;
        elseif j==(i-1)
            g1(i,j)=2^(2-B)-3;
        elseif j>i
            g1(i,j)=0;
        else
            g1(i,j)=C(i-j,B);
         end
    end
end
%g1;
for i=1:m-1
    for j=1:m-1
        if j==i
            g2(i,j)=2^(2-B)-3;
        elseif j==(i+1)
            g2(i,j)=1;
```

```
        elseif j>(i+1)
            g2(i,j)=0;
        else
            g2(i,j)=C(i-j+1,B);
        end
    end
end
%g2;
A=zeros(m-1,m-1);
for i=1:m-1
    A(i,i)=2*h/3;
end
for i=2:m-1
    A(i,i-1)=h/6;
    A(i-1,i)=h/6;
end
%A;
A2=A+r1*(g1-g2);
U0=zeros(1,m-1);
for j=1:n
    for i=1:m-1
        C1= 1/a*(1-exp(-a*t(j+1)/(1-a)));% tn
        C2=6*t(j+1)/gamma(3-B);
        F1=C1*p/h*(x(i+2)^5+x(i)^5-2*x(i+1)^5)/20;%
        F2=p*C2/h/(3-B)/(4-B)*(x(i+2)^(4-B)+x(i)^(4-B)-2*x(i+1)^(4-B));%
        F(i)=F1-F2;
    end
D=U0;
b0=0;
for i=1:j-1
       D=D+w_k(j-i,a,tau)*U(i,:);%
       b0=b0+w_k(i,a,tau)*(j-i)*tau;
end
F(m-1)=F(m-1)+(r1-h/6)*t(j+1)+h*b0/6; % add boundary condition
b=1/m0*A*D'+F';
U1=A2\b;
U(j,:)=U1;    %U;
end
for i=1:m-1
    result1(i)=(x(i+1)^3-U1(i))^2; %
end
result2=sqrt(sum(result1)*h);
y=result2;
end
function y=C(x,L) %%
y=-(x-2)^(2-L)+3*(x-1)^(2-L)-3*x^(2-L)+(x+1)^(2-L);
end
function y=w_k(k,a,tau) %%
y= -2*exp(-a*k*tau/(1-a))+exp(-a*(k-1)*tau/(1-a))...
    +exp(-a*(k+1)*tau/(1-a));
end
end
```

6. Conclusion

This work aims to propose the finite element method to solve the space-time fractional diffusion equations, when we use the Caputo–Fabrizion fractional derivative for the time variables and the Riemann–Liouville for the space variables. To discredit the time fractional derivative, we use the finite difference method, then we use the finite element method to approximate the space fractional derivative to obtain the full discretization schema with convergence order of $O(\Delta t^2 + h^2)$. Through numerical examples, we verified the effectiveness of the proposed method.

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