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FUNCTIONAL EQUATIONS WITH AN ANTI-ENDOMORPHISM FOR FUNCTIONS WITH MULTIDIMENSIONAL CODOMAINS

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Abstract. Let S be a semigroup, $\mathbb{H}$ be the skew field of quaternions, and $\psi\colon S \to S$ be an anti-endomorphism. We determine the general solution of the functional equation

$$g(xy) - g(x\psi(y)) = 2g(x)g(y), \quad x, y \in S,$$

where $g\colon S \to \mathbb{C}$ is the unknown function. And when $S = M$ is a monoid, we solve the functional equation

$$g(xy) + g(x\psi(y)) = 2g(x)g(y), \quad x, y \in M,$$

where $g\colon M \to \mathbb{H}$ is the unknown function.

1. Introduction

Throughout this paper let S denote a semigroup and M a monoid (a semigroup with a neutral element), and let $Y \in \{M, S\}$. The map $\psi\colon Y \to Y$ denotes an anti-endomorphism of S (i.e., $\psi(xy) = \psi(y)\psi(x)$ for all $x, y \in Y$). By ψ^2, we mean $\psi \circ \psi$. Let f be a function on Y. We say that f is ψ-invariant if $f \circ \psi = f$. The function $\mu\colon Y \to \mathbb{C}$ is multiplicative, if $\mu(xy) = \mu(x)\mu(y)$ for all $x, y \in Y$. $\mathbb{H}$ is the skew field of quaternions.

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D'Alembert's classic functional equation

$$g(x+y)+g(x-y)=2g(x)g(y), \quad x,y\in\mathbb{R}, \tag{1.1}$$

for functions $g\colon \mathbb{R}\to\mathbb{C}$ has it roots back in d'Alembert's investigations of vibrating strings [1] from 1750. Kannappan [9] solved the equation (1.1) on abelian groups. His work was extended to general groups, even monoids or semigroups (where inversion is replaced by an involution), by Davison [7], Stetkær [12, 13, 14], Yang [15], and others.

The subject of functional equations with an anti-endomorphism has been introduced since 2020 by Ayoubi and Zeglami in [2] where they characterized the solutions of the functional equation

$$d(xy)+d(x\psi(y))=2d(x)d(y), \quad x,y\in M, \tag{1.2}$$

in which $d\colon M\to\mathbb{C}$ is the unknown function. Later, inspired by Stetkær's paper [14], they [4] solved (1.2) in the setting of semigroups. Furthermore, Ayoubi, Zeglami and Mouzoun proved in [6] that the solutions of the equation

$$g(xy)-g(x\psi(y))=2g(x)g(y), \quad x,y\in M, \tag{1.3}$$

are the functions $g=\frac{1}{2}\mu$, where $\mu\colon M\to\mathbb{C}$ is a multiplicative function satisfying $\mu\circ\psi=0$.

The purposes of the present is to generalize each of the two equations (1.3) and (1.2) at the level of the range set of its unknown functions for the first one and its codomain for the second. Precisely

1) We determine the general solution of the functional equation

$$g(xy)-g(x\psi(y))=2g(x)g(y), \quad x,y\in S, \tag{1.4}$$

where $g\colon S\to\mathbb{C}$ is the unknown function. When ψ is involutive, Stetkær [12, Exercise 9.9] showed that $g=0$ is the only complex-valued solution of the functional equation (1.4). Another contribution in this direction is the paper by Ebanks and Stetkær [8] where they solved the functional equation

$$f(xy)-f(y^{-1}x)=g(x)h(y), \quad x,y\in G,$$

in which $f,g,h\colon G\to\mathbb{C}$ are the unknown functions and G is a group.

2) We solve the functional equation

$$g(xy)+g(x\psi(y))=2g(x)g(y), \tag{1.5}$$

where $g\colon M\to\mathbb{H}$ is the unknown function. Remark 3.3 gives an example showing that non-central solutions of (1.5) exist. This is in contrast to the earlier resul about its complex-valued solutions which are all central. Example 3.4 illustrates the structure of the solutions of d'Alembert equation (1.5) for quaternion-valued functions on the $(ax + b)$-group.

2. Solutions of the equation $g(xy) - g(x\psi(y)) = 2g(x)g(y)$

The following theorem gives the general form of the solutions of the functional equation (1.4).

THEOREM 2.1. *$g\colon S \to \mathbb{C}$ is a solution of* (1.4) *if and only if it has the form*

$$g = \frac{m}{2},$$

where $m\colon S \to \mathbb{C}$ is a multiplicative function satisfying $m \circ \psi = 0$.

PROOF. The result is true for $g = 0$. Let $g\colon S \to \mathbb{C}$ be a non-zero solution of (1.4) and let $x_0 \in S$ be such that $g(x_0) \neq 0$. Let $T(g)$ be the set of non-zero functions $f\colon S \to \mathbb{C}$ that satisfy the functional equation

$$f(xy) - f(x\psi(y)) = 2f(x)g(y), \quad x, y \in S, \tag{2.1}$$

and $f \circ \psi = f$. We examine two cases: $T(g)$ is empty or not.

Case 1: We start with the case where $T(g)$ is empty. Let $x, y \in S$ be arbitrary, we define the function $h\colon S \to \mathbb{C}$ as follows

$$h(a) := g(y)g(xa) - g(x)g(ya), \quad a \in S.$$

Using the fact that g satisfies (1.4) and the definition of h, we will show that h satisfies the equation (2.1) and that $h \circ \psi = h$. For any $a, b \in S$ we have

$$\begin{aligned} h(ab) - h(a\psi(b)) &= g(y)g(xab) - g(x)g(yab) - g(y)g(xa\psi(b)) + g(x)g(ya\psi(b)) \\ &= g(y)\big(g(xab) - g(xa\psi(b))\big) - g(x)\big(g(yab) - g(ya\psi(b))\big) \\ &= 2g(y)g(xa)g(b) - 2g(x)g(ya)g(b) \\ &= 2\big(g(y)g(xa) - g(x)g(ya)\big)g(b) = 2h(a)g(b). \end{aligned}$$

Then h satisfies the equation (2.1). For any $a \in S$ we have

$$\begin{aligned} h(\psi(a)) &= g(y)g(x\psi(a)) - g(x)g(y\psi(a)) \\ &= g(y)\big(g(xa) - 2g(x)g(a)\big) - g(x)\big(g(ya) - 2g(y)g(a)\big) \\ &= g(y)g(xa) - 2g(y)g(x)g(a) - g(x)g(ya) + 2g(x)g(y)g(a) \\ &= g(y)g(xa) - g(x)g(ya) = h(a), \end{aligned}$$

which means that $h \circ \psi = h$. So $h = 0$ because $T(g)$ is empty. From the definition of h, we find that $g(y)g(xa) = g(x)g(ya)$ for all $a, x, y \in S$. Let $m(a) := \frac{g(x_0 a)}{g(x_0)}$, $a \in S$. Then

$$g(xa) = g(x)m(a), \quad a, x \in S, \tag{2.2}$$

from which we get that

$$g(x_0)m(ab) = g(x_0 ab) = g(x_0 a)m(b) = g(x_0)m(a)m(b),$$

for all $a, b \in S$. It follows that m is multiplicative. From (1.4) and (2.2) we get

$$\begin{aligned} g(x_0)\big(m(a) - m(\psi(a))\big) &= g(x_0)m(a) - g(x_0)m(\psi(a)) \\ &= g(x_0 a) - g(x_0\psi(a)) = 2g(x_0)g(a), \quad a \in S, \end{aligned}$$

which implies that

$$g = \frac{m - m \circ \psi}{2}. \tag{2.3}$$

Note that $m \neq m \circ \psi$ because $g \neq 0$. Substituting (2.3) into (1.4) we obtain

$$\begin{aligned} &\frac{m(xy) - m \circ \psi(xy)}{2} - \frac{m(x\psi(y)) - m \circ \psi(x\psi(y))}{2} \\ &\qquad = 2\Big(\frac{m(x) - m \circ \psi(x)}{2}\Big)\Big(\frac{m(y) - m \circ \psi(y)}{2}\Big), \quad x, y \in S. \end{aligned}$$

Then, after some reductions, we find

$$m \circ \psi(x)\big(m(y) + m \circ \psi^2(y) - 2m \circ \psi(y)\big) = 0,$$

for all $x, y \in S$. Since $m \neq m \circ \psi$, it follows from [12, Corollary 3.19] that $m + m \circ \psi^2 \neq 2m \circ \psi$ and hence $m \circ \psi = 0$ and $g = \frac{m}{2}$.

Case 2: $T(g)$ is not empty. Here there is a function l which belongs to $T(g)$. This says that $l\colon S \to \mathbb{C}$ satisfies

$$l(xy) - l(x\psi(y)) = 2l(x)g(y), \quad x, y \in S, \tag{2.4}$$

$l \neq 0$ and $l \circ \psi = l$. Using that $l \circ \psi = l = l \circ \psi^2$ we compute with (2.4) as follows

$$\begin{aligned} 2l(x)g(\psi^2(y)) &= 2l(\psi^2(x))g(\psi^2(y)) = l(\psi^2(x)\psi^2(y)) - l(\psi^2(x)\psi(\psi^2(y))) \\ &= l(\psi^2(xy)) - l(\psi^2(x\psi(y))) = l(xy) - l(x\psi(y)) \\ &= 2l(x)g(y), \quad x, y \in S. \end{aligned}$$

Since $l \neq 0$ we find that $g \circ \psi^2 = g$. Again the equation (2.4) together with $l \circ \psi = l$ tell us that

$$\begin{aligned} l(\psi(x)y) - l(yx) &= l(\psi(x)y) - l(\psi(x)\psi(y)) = 2l(\psi(x))g(y) \\ &= 2l(x)g(y), \quad x, y \in S, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} l(yx) - l(\psi(y)x) &= l(\psi(x)\psi(y)) - l(\psi(x)\psi^2(y)) = 2l(\psi(x))g(\psi(y)) \\ &= 2l(x)g \circ \psi(y), \quad x, y \in S. \end{aligned} \tag{2.6}$$

Summing (2.5) and (2.6) gives us

$$l(\psi(x)y) - l(\psi(y)x) = 2l(x)\big(g(y) + g \circ \psi(y)\big), \quad x, y \in S.$$

From this we get $l(x)(g(y) + g \circ \psi(y)) = -l(y)(g(x) + g \circ \psi(x))$ for all $x, y \in S$. From [12, Exercise 1.1(b)] we read that $g + g \circ \psi = 0$ because $l \neq 0$. Hence $g = -g \circ \psi$. Using this and (1.4) we find that

$$\begin{aligned} g(\psi^2(x)y) &= 2g(\psi^2(x))g(y) + g(\psi^2(x)\psi(y)) \\ &= -2g(\psi^2(x))g(\psi(y)) + g(\psi^2(x)\psi(y)) \\ &= g(\psi^2(x)\psi^2(y)) = g(\psi^2(xy)) = g(xy), \quad x, y \in S. \end{aligned}$$

We follow the same procedure as in [12, Exercise 9.9] to obtain $g = 0$. ☐

Remark 2.2. Theorem 2.1 holds true if we replace $\mathbb{C}$ by a field $\mathbb{K}$ and 2 by a constant $c \in \mathbb{K}^*$.

3. Quaternion-valued solutions of d'Alembert's equation

The following theorem determines the solutions of the functional equation (1.5). In the rest of this section, we denote the neutral element of M by e.

THEOREM 3.1. *The solutions* $g\colon M \to \mathbb{H}$ *of the functional equation* (1.5) *are the following:*

(1) *There exists a multiplicative function* $\mu\colon M \to \mathbb{H}$ *with* $\mu \circ \psi = 0$ *such that*

$$g = \frac{\mu}{2}.$$

(2) *There exists a solution* $d\colon M \to \mathbb{C}$ *of* (1.2) *with* $g(e) = 1$ *such that*

$$g = \mathrm{Re}(d) + \mathrm{Im}(d)\,\mathbf{i}.$$

(3) *There exist a solution* $d\colon M \to \mathbb{C}$ *of* (1.2) *with* $g(e) = 1$ *and* $\mathrm{Im}(d) \neq 0$, $\beta \in \mathbb{R}^*$, *and* $\theta \in \mathbb{R}$ *such that*

$$g = \mathrm{Re}(d) + \frac{\beta - \frac{1}{\beta}}{\beta + \frac{1}{\beta}}\,\mathrm{Im}(d)\,\mathbf{i} - \frac{2\sin(\theta)}{\beta + \frac{1}{\beta}}\,\mathrm{Im}(d)\,\mathbf{j} + \frac{2\cos(\theta)}{\beta + \frac{1}{\beta}}\,\mathrm{Im}(d)\,\mathbf{k}.$$

PROOF. Let $g = q_1 + q_2\,\mathbf{i} + q_3\,\mathbf{j} + q_4\,\mathbf{k}\colon M \to \mathbb{H}$, where q_1, q_2, q_3 and q_4 are real-valued functions on M, be a solution of the functional equation (1.5). We will examine two cases, $g(e) \neq 1$ or $g(e) = 1$.

Case 1: $g(e) \neq 1$. We follow the same procedure as in the proof of [2, Case 1 of Theorem 3.2] to arrive at the solution in case 1 of our statement.

Case 2: $g(e) = 1$. We find, like in the proof of [2, Lemma 3.1(i)], that $g \circ \psi = g$. Using this we obtain, like in the proof of [5, Theorem 5.1], that

$$g(x)g(y) = g(y)g(x) \quad \text{for all } x, y \in S.$$

With this property in mind, we prove in the same way as in the proof of [2, Lemma 3.1(ii) and (iii)] that g is central and that

$$g(x\psi^2(y)z) = g(xyz) \quad \text{for all } x, y, z \in M. \tag{3.1}$$

The matrix representation of quaternions reveals that the matrix function

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

where $a = q_1 + q_2 i$, $b = q_3 + q_4 i$ is a solution of the functional equation (1.5). Then the pair (a, b) satisfies the system

$$a(x\psi(y)) + a(xy) = 2a(x)a(y) - 2b(x)\bar{b}(y), \quad x, y \in M,$$

$$b(x\psi(y)) + b(xy) = 2a(x)b(y) + 2b(x)\overline{a}(y), \quad x, y \in M.$$

Since g is central, ψ-invariant and satisfies (3.1) then so is each of the functions $q_i,\ i \in \{1, 2, 3, 4\}$ and hence we have $a \circ \psi = a$, $b \circ \psi = b$, a and b are central, and the two equalities

$$a(x\psi^2(y)z) = a(xyz),$$

$$b(x\psi^2(y)z) = b(xyz),$$

for all $x, y, z \in M$. We follow the same procedure as in the proof of [10, Theorem 2.3] to arrive at the solution in case 2 or 3. □

Remark 3.2. The central multiplicative functions $\mu\colon S \to \mathbb{H}$ are described in [11, Theorem 4.1].

Remark 3.3. [4, Theorem 3.2] tells us that the solutions of the equation (1.2) are central. This property is not true in general for the solutions of the equation (1.5) as the following illustrates: Let $M = (\mathbb{H}, \cdot)$, $\psi = 0$, and $g_0\colon (\mathbb{H}, \cdot) \to \mathbb{H}$ the function defined by $g_0(q) := \frac{1}{2}q$ for all $q \in \mathbb{H}$. The function g_0 is a solution of the equation (1.5) on $(\mathbb{H}, \cdot)$ and $g_0(\mathbf{ij}) \neq g_0(\mathbf{ji})$.

Example 3.4. Let M be the $(ax + b)$-group from [12, Examples A.17(i)]. Let ψ be the anti-endomorphism defined by $(a, b) \mapsto (a, 0)$ for $(a, b) \in M$. Note that $\mu = 0$ for any multiplicative function $\mu\colon M \to \mathbb{H}$ satisfying $\mu \circ \psi = 0$. Indeed, if $\mu\colon M \to \mathbb{H}$ is multiplicative and $\mu \circ \psi = 0$, then for all $(a, b) \in M$ we have

$$\begin{aligned}\mu\big(a, b\big) &= \mu\big((a, b) \times (1, 0)\big) = \mu\big((a, b) \times \psi(1, 0)\big)\\ &= \mu\big(a, b\big) \times \mu \circ \psi\big(1, 0\big) = \mu\big(a, b\big) \times 0 = 0.\end{aligned}$$

From [12, Example 3.13] we read that the continuous characters on the $(ax+b)$-group are

$$m_\lambda(a, b) = a^\lambda, \quad (a, b) \in M,$$

where $\lambda \in \mathbb{C}$. Note that $m_\lambda \circ \psi = m_\lambda$. Combining [4, Corollary 3.3] with [3, Proposition 4.1(b)] and [12, Corollary 8.18], we find that the non-zero continuous solutions of (1.2) are the following:

$$d_\lambda(a,b) = \frac{m_\lambda(a,b) + m_\lambda \circ \psi(a,b)}{2} = a^\lambda, \quad (a,b) \in M,$$

where $\lambda \in \mathbb{C}$. Let $\lambda = \lambda_1 + \lambda_2\, i$ with $\lambda_1, \lambda_2 \in \mathbb{R}$, then

$$\mathrm{Re}(a^\lambda) = a^{\lambda_1} \cos(\lambda_2 \ln(a)) \text{ and } \mathrm{Im}(a^\lambda) = a^{\lambda_1} \sin(\lambda_2 \ln(a)).$$

From Theorem 3.1 we conclude that the non-zero continuous solutions of (1.5) are the following:

(1) There exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\begin{aligned} g(a,b) &= \mathrm{Re}(d(a,b)) + \mathrm{Im}(d(a,b))\,\mathbf{i} \\ &= a^{\lambda_1} \cos(\lambda_2 \ln(a)) + a^{\lambda_1} \sin(\lambda_2 \ln(a))\,\mathbf{i}, \end{aligned}$$

for all $(a,b) \in M$.

(2) There exist $\lambda_1, \theta \in \mathbb{R}$ and $\lambda_2, \beta \in \mathbb{R}^*$ such that

$$\begin{aligned} g(a,b) = a^{\lambda_1} \cos(\lambda_2 \ln(a)) &+ \frac{\beta - \frac{1}{\beta}}{\beta + \frac{1}{\beta}} a^{\lambda_1} \sin(\lambda_2 \ln(a))\,\mathbf{i} \\ &- \frac{2\sin(\theta)}{\beta + \frac{1}{\beta}} a^{\lambda_1} \sin(\lambda_2 \ln(a))\,\mathbf{j} + \frac{2\cos(\theta)}{\beta + \frac{1}{\beta}} a^{\lambda_1} \sin(\lambda_2 \ln(a))\,\mathbf{k}, \end{aligned}$$

for all $(a,b) \in M$.

Declarations

Conflict of interest. The authors declare that they have no conflict of interest.

Data availability. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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