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CENTRAL LIMIT THEOREM FOR RANDOM DYNAMICAL SYSTEM WITH JUMPS AND STATE-DEPENDENT JUMP INTENSITY

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Abstract. In this work we focus on a dynamical system with jumps, where the intensity of the jumps depends on the system's state. By verifying the assumptions of the theorem from [4], we show that our model satisfies the central limit theorem.

1. Introduction

Piecewise-deterministic Markov processes (PDMPs) represent a class of stochastic models, that have recently received extensive research attention. They were introduced by Davis [7] as a general and various practical systems in which randomness is limited to jumps. These processes are governed by deterministic semiflows and mentioned jumps which change the state as well the semiflows which will determine evolution of the system. PDMPs have proven to be effective mathematical models for many phenomena, such as gene expression [14] and population dynamics [1]. Results concerning ergodicity and asymptotic stability of these systems are already quite extensively researched [1, 2, 3, 8, 9, 5, 16].

Having the asymptotic stability of such processes, natural questions arise regarding the limit theorems. In [11], we focused on demonstrating a law of the iterated logarithm for some PDMPs in which the intensity of jumps depends on the state of the system. In this paper we prove the central limit theorem (CLT) for this class of processes.

In [4], a criterion for the CLT was introduced, and an example of a model with a constant intensity of jumps was given. The difference between our

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model and the one considered in [4] lies in the assumption that the intensity of jumps follows an exponential distribution with a constant intensity parameter. This assumption seems to be too restrictive concerning biological models, such as gene expression, in which the relation between the intensity of jumps and the state of the system is visible. It turns out that the criterion for the CLT from [4] is also applicable in the case we are considering.

The paper contains three sections. In Section 2, basic definitions and notations are introduced. In Section 3, we start with an intuitive description of the considered model, and then we formulate strict definitions. We also provide conditions that were used in [6] to show exponential ergodicity. The last section contains the proof of the CLT for the model described earlier, and it is strongly influenced by the methods used in [4].

2. Preliminaries

Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}_+ = [0, \infty)$. Let (E, ρ) be a Polish space with the σ-field $\mathcal{B}(E)$ of all Borel subsets of E. By $B(E)$ we denote the space of all bounded, Borel measurable functions $f\colon E \to \mathbb{R}$, equipped with the supremum norm and we list two of its subsets: $C(E)$ and $Lip(E)$ consisting of all continuous functions and Lipschitz-continuous functions, respectively. A continuous function $V\colon E \to \mathbb{R}_+$ is called Lyapunov function if it is bounded on bounded sets and for some $x_0 \in E$

$$\lim_{\rho(x,x_0)\to\infty} V(x) = \infty.$$

Let $\mathcal{M}_s(E)$ be the set of all finite, countably additive functions on $\mathcal{B}(E)$. By $\mathcal{M}_+(E)$ and $\mathcal{M}_1(E)$ we denote the subsets of $\mathcal{M}_s(E)$ consisting of all non-negative measures and all probability measures, respectively. We write $\mathcal{M}_1^V(E)$ for the set of all $\mu \in \mathcal{M}_1(E)$ satisfying $\int_E V(x)\,\mu(dx) < \infty$.

The set $\mathcal{M}_s(E)$ will be considered with the *Fortet-Mourier norm* ([12, 13]), given by

$$||\mu||_{FM} = \sup\{|\langle f, \mu\rangle| : f \in \mathcal{F}_{FM}(E)\} \quad \text{for } \mu \in \mathcal{M}_s(E),$$

where

$$\mathcal{F}_{FM}(E) = \{f \in C(E) : |f(x)| \le 1,\ |f(x) - f(y)| \le \rho(x, y),\ x, y \in E\},$$
$$\langle f, \mu\rangle = \int_E f(x)\mu(dx).$$

As usual, by $B(x, r)$ we denote the open ball in E centered at x and radius $r > 0$. For the fixed set $A \subset E$ we define the indicator function $\mathbb{1}_A\colon E \to \{0, 1\}$ as $\mathbb{1}_A(x) = 1$ for $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise.

A function $\mathcal{K}\colon E \times \mathcal{B}(E) \to [0,1]$ is called a *(sub)stochastic kernel* if for each $A \in \mathcal{B}(E)$, $x \mapsto \mathcal{K}(x,A)$ is a measurable map on E, and for each $x \in E$, $A \mapsto \mathcal{K}(x,A)$ is a (sub)probability Borel measure on $\mathcal{B}(E)$.

An operator $P\colon \mathcal{M}_+(E) \to \mathcal{M}_+(E)$ is called a *Markov operator* if:

- $P(\alpha_1\mu_1 + \alpha_2\mu_2) = \alpha_1 P\mu_1 + \alpha_2 P\mu_2$ for $\alpha_1, \alpha_2 \in \mathbb{R}_+,\ \mu_1, \mu_2 \in \mathcal{M}_+(E)$,
- $P\mu(E) = \mu(E)$ for $\mu \in \mathcal{M}_+(E)$.

A Markov operator P is called *regular* if there is a linear operator $U\colon B(E) \to B(E)$, called the *dual operator* to P, such that

$$\langle f, P\mu\rangle = \langle Uf, \mu\rangle \quad \text{for all } f \in B(E),\ \mu \in \mathcal{M}_+(E). \tag{1}$$

If P is a regular Markov operator then the function $\mathcal{K}\colon E \times \mathcal{B}(E) \to [0,1]$, given by $\mathcal{K}(x,A) = P\delta_x(A)$ for $x \in E$, $A \in \mathcal{B}(E)$, is a stochastic kernel. On the other hand, an arbitrary given stochastic kernel $\mathcal{K}$ defines a regular Markov operator P and its dual operator U by formulas:

$$P\mu(A) = \int_E \mathcal{K}(x,A)\,\mu(dx) \quad \text{for } \mu \in \mathcal{M}_+(E),\ A \in \mathcal{B}(E)$$

and

$$Uf(x) = \int_E f(y)\,\mathcal{K}(x,dy) \quad \text{for } x \in E,\ f \in B(E).$$

Let us note that the operator U can be extended to a linear operator defined on the space of all bounded below Borel functions $\overline{B}(E)$ so that (1) holds for all $f \in \overline{B}(E)$.

A regular Markov operator P is called *Feller* if $Uf \in C(E)$ for every $f \in C(E)$.

We say that $\mu_* \in \mathcal{M}_+(E)$ is *invariant* with respect to P if $P\mu_* = \mu_*$.

If there exists an invariant measure $\mu_* \in \mathcal{M}_1(E)$ and a constant $\beta \in [0,1)$ such that, for every $\mu \in \mathcal{M}_1^V(E)$ and some constant $C(\mu) \in \mathbb{R}$, we have

$$||P^n\mu - \mu_*||_{FM} \le C(\mu)\beta^n \quad \text{for all } n \in \mathbb{N},$$

then μ_* is called *exponentially attracting.* If an operator P has such an exponentially attracting invariant probability measure, then the operator P is said to be *exponentially ergodic.*

It is well known that for every stochastic kernel $\mathcal{K}$ and any fixed measure $\mu \in \mathcal{M}_1(E)$, we can always define on suitable probability space, say $(\Omega, \mathcal{F}, \mathrm{Prob}_\mu)$, a discrete-time homogeneous Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$ for which

$$\mathrm{Prob}_\mu(X_0 \in A) = \mu(A) \quad \text{for } A \in \mathcal{B}(E),$$

$$\mathcal{K}(x,A) = \mathrm{Prob}_\mu(X_{n+1} \in A | X_n = x) \quad \text{for } x \in E,\ A \in \mathcal{B}(E),\ n \in \mathbb{N}_0. \tag{2}$$

Then the Markov operator P corresponding to the kernel (2) describes the evolution of the distributions $\mu_n(\cdot) := \mathrm{Prob}_\mu(X_n \in \cdot)$, that is

$$\mu_{n+1} = P\mu_n \quad \text{for } n \in \mathbb{N}_0.$$

In our further considerations, we will use the symbol $\mathbb{E}_\mu$ for the expectation with respect to Prob_μ. If $\mu = \delta_x$ for some $x \in E$, we will write $\mathbb{E}_x$.

We say that a time-homogeneous Markov chain evolving on the space E^2 (endowed with the product topology) is a *Markovian coupling* of some stochastic kernel $\mathcal{K}\colon E \times \mathcal{B}(E) \to [0,1]$ whenever its stochastic kernel $B\colon E^2 \times \mathcal{B}(E^2) \to [0,1]$ satisfies

$$B(x,y,A\times E) = \mathcal{K}(x,A) \quad \text{and} \quad B(x,y,E\times A) = \mathcal{K}(y,A)$$

for all $x,y \in E$ and $A \in \mathcal{B}(E)$. Let us underline this, that if $Q\colon E^2 \times \mathcal{B}(E^2) \to [0,1]$ is a substochastic kernel satisfying

$$Q(x,y,A\times E) \le \mathcal{K}(x,A) \quad \text{and} \quad Q(x,y,E\times A) \le \mathcal{K}(y,A),$$

for all $x,y \in E$ and $A \in \mathcal{B}(E)$, then we can always construct a Markovian coupling of $\mathcal{K}$ whose transition function B satisfies $Q \le B$. For this purpose, it suffices to define the family $\{R(x,y,\cdot) : \; x,y \in E\}$ of measures on $\mathcal{B}(E^2)$, which on $A \times B \in \mathcal{B}(E^2)$ are given by

$$R(x,y,A\times B) = \frac{(\mathcal{K}(x,A) - Q(x,y,A\times E))(\mathcal{K}(y,B) - Q(x,y,E\times B))}{1 - Q(x,y,E^2)}$$

if $Q(x,y,E^2) < 1$, and $R(x,y,A\times B) = 0$ otherwise. Then $B := Q + R$ is a stochastic kernel satisfying $Q \le B$, and that the Markov chain with transition function B is a coupling of $\mathcal{K}$.

Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a Markov chain evolving in E with transition kernel $\mathcal{K}$ and initial distribution μ. Suppose $\mu_* \in \mathcal{M}_1(E)$ is the unique invariant measure of the Markov operator P corresponding to the kernel given by (2). For any $n \in \mathbb{N}$ and any Borel function $g\colon E \to \mathbb{R}$ define

$$s_n(g) = \frac{g(X_1) + \ldots + g(X_n)}{\sqrt{n}},$$
$$\sigma^2(g) = \lim_{n\to\infty} \mathbb{E}_{\mu_*}(s_n^2(g)).$$

Denote by $\Phi s_n(g)$ the distribution of $s_n(g)$ and put $\overline{g} = g - \langle g, \mu_* \rangle$.

Let $g\colon E \to \mathbb{R}$ be a Borel function such that $\langle g^2, \mu_* \rangle < \infty$. By definition, $\{g(X_n)\}_{n\in\mathbb{N}_0}$ satisfies the CLT condition, if $\sigma^2(\overline{g}) < \infty$ and $\Phi_{s_n}(\overline{g})$ converges weakly to $\mathcal{N}(0, \sigma^2(\overline{g}))$, as $n \to \infty$.

Proving the CLT we will use the following theorem proved in [4].

Theorem 1. *Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a time-homogeneous Markov chain with values in E and let $\mathcal{K}\colon E\times\mathcal{B}(E)\to[0,1]$ be the transition law of $\{X_n\}_{n\in\mathbb{N}_0}$ which satisfies the following conditions:*

(B0) *The Markov operator P corresponding to $\mathcal{K}$ has the Feller property.*

(B1) *There exist a Lyapunov function $V\colon E\to\mathbb{R}_+$ and constants $a\in(0,1)$ and $b\in(0,\infty)$ such that*

$$UV^2(x)\le(aV(x)+b)^2\quad \textit{for every } x\in E.$$

In addition, assume that there is a substochastic kernel $Q\colon E^2\times\mathcal{B}(E^2)\to[0,1]$ satisfying

$$Q(x,y,A\times E)\le\mathcal{K}(x,A)\quad \textit{and}\quad Q(x,y,E\times A)\le\mathcal{K}(y,A)$$

for $x,y\in E$, $A\in\mathcal{B}(E)$.

(B2) *There exist $F\subset E^2$ and $q\in(0,1)$ such that*

$$\operatorname{supp}Q(x,y,\cdot)\subset F\ \ \textit{and}\ \int_{E^2}\rho(u,v)Q(x,y,du\times dv)\le q\rho(x,y)$$

for $(x,y)\in F$.

(B3) *For $G(r)=\{(x,y)\in F:\rho(x,y)\le r\}$, $r>0$, we have*

$$\inf_{(x,y)\in F}Q(x,y,G(q\rho(x,y)))>0.$$

(B4) *There exist constants $\nu\in(0,1]$ and $l>0$ such that*

$$Q(x,y,E^2)\ge1-l\rho(x,y)^\nu\quad \textit{for every } (x,y)\in F.$$

(B5) *There is a coupling $\{(X_n^{(1)},X_n^{(2)})\}_{n\in\mathbb{N}_0}$ of $\mathcal{K}$ with transition law $B\ge Q$ such that for some $R>0$ and*

$$K:=\{(x,y)\in F:V(x)+V(y)<R\}$$

we can find $\zeta\in(0,1)$ and $\overline{C}>0$ satisfying

$$\mathbb{E}_{(x,y)}(\zeta^{-\sigma_K})\le\overline{C}\quad \textit{whenever } V(x)+V(y)<4b(1-a)^{-1},$$
$$\textit{where } \sigma_K=\inf\{n\in\mathbb{N}:X_n\in K\}.$$

Let $\mu\in\mathcal{M}_1^V(E)$ be an initial distribution of $\{X_n\}_{n\in\mathbb{N}_0}$. If $g\in Lip(E)$, then $\{g(X_n)\}_{n\in\mathbb{N}_0}$ satisfies the CLT.

3. Model description and assumptions

Let (Y, ρ) be a Polish space and let $I = \{1, ..., N\}$ for a fixed positive integer N. We define $X := Y \times I$ and consider the space (X, ρ_c), where

$$\rho_c((y_1, i), (y_2, j)) = \rho(y_1, y_2) + c\Psi(i, j), \quad (y_1, i), (y_2, j) \in X, \tag{3}$$

where

$$\Psi(i, j) = \begin{cases} 1 & \text{for } i \neq j, \\ 0 & \text{for } i = j. \end{cases} \tag{4}$$

The constant $c > 0$ will be defined later. Let Θ be a compact interval.

Our considerations are conducted for a discrete-time dynamical system related to the stochastic process $\{(Y(t), \xi(t))\}_{t \in \mathbb{R}_+}$, which evolves through random jumps in the space X. We proceed to provide a description of this process.

Assume that we have a finite collection of maps, called semiflows, $\Pi_i \colon \mathbb{R}_+ \times Y \to Y$, $i \in I$, which are continuous with respect to each variable and satisfy for every $i \in I$ and each $y \in Y$, the following conditions:

$$\Pi_i(0, y) = y \quad \text{and} \quad \Pi_i(s + t, y) = \Pi_i(s, \Pi_i(t, y)) \quad \text{for } s, t \in \mathbb{R}_+.$$

The process $\{Y(t)\}_{t \in \mathbb{R}_+}$ evolves between jumps according to one of the transformation Π_i, whose index i is determined by $\{\xi(t)\}_{t \in \mathbb{R}_+}$.

Let $\pi_{ij} \colon Y \to [0, 1]$, $i, j \in I$ be a matrix of continuous functions such that

$$\sum_{j \in I} \pi_{ij}(y) = 1 \quad \text{for } i \in I,\ y \in Y.$$

Just after every jump, the semiflow is changed from Π_i to Π_j in accordance with π_{ij} and at the moment of a jump the process $\{Y(t)\}_{t \in \mathbb{R}_+}$ shifts to the new state by a function $\omega(\theta, \cdot) \colon Y \to Y$, which is randomly chosen from a given set $\{\omega(\theta, \cdot) : \theta \in \Theta\}$. We assume that $Y \times \Theta \ni (y, \theta) \mapsto \omega(\theta, y) \in Y$ is continuous and that the probability of choosing $\omega(\theta, \cdot)$ is related with density functions $\Theta \ni \theta \mapsto p(\theta, y)$, $y \in Y$, such that $(\theta, y) \mapsto p(\theta, y)$ is continuous. Moreover, we assume that the intensity of jumps is given by a Lipschitz-continuous function $\lambda \colon Y \to (0, \infty)$, which satisfies the following conditions:

$$\underline{\lambda} = \inf_{y \in Y} \lambda(y) > 0 \quad \text{and} \quad \overline{\lambda} = \sup_{y \in Y} \lambda(y) < \infty. \tag{5}$$

The evolution of $\{(Y(t), \xi(t))\}_{t \in \mathbb{R}_+}$ can be described as follows. Assume that the process starts at some point $(y_0, i_0) \in Y \times I$. Then

$$Y(t) = \Pi_{i_0}(t, y_0) \quad \text{and} \quad \xi(t) = i_0, \quad 0 \leq t < t_1,$$

where t_1 depends on y_0 and i_0. At time t_1 the process $\{Y(t)\}_{t\in\mathbb{R}_+}$ jumps to the new position

$$y_1 := \omega(\theta_1, Y(t_1-)) = \omega(\theta_1, \Pi_{i_0}(t_1, y_0)),$$

where $\theta_1 \in \Theta$ is randomly chosen in accordance with the distribution given by density $\theta \mapsto p(\theta, \Pi_{i_0}(t_1, y_0))$. In the next step, we choose randomly $i_1 \in I$ such that the probability of choosing i_1 is given by $\pi_{i_0 i_1}(y_1)$. Then

$$Y(t) = \Pi_{i_1}(t - t_1, y_1) \quad \text{and} \quad \xi(t) = i_1, \quad t_1 \le t < t_2.$$

We repeat the above steps with the point (y_1, i_1) instead of (y_0, i_0). The next steps are defined by induction.

Assuming that $t_0 = 0$, we get

$$Y(t) = \Pi_{i_n}(t - t_n, y_n) \quad \text{and} \quad \xi(t) = i_n, \quad t_n \le t < t_{n+1},\ n \in \mathbb{N}_0.$$

Let us emphasize that in this work we study only the sequence of random variables given by the post-jump locations of the process $\{(Y(t), \xi(t))\}_{t\in\mathbb{R}_+}$, namely the process $\{(Y_n, \xi_n)\}_{n\in\mathbb{N}_0}$, where $Y_n = Y(\tau_n)$, $\xi_n = \xi(\tau_n)$ for $n \in \mathbb{N}_0$ and τ_n is a random variable describing the jump time t_n.

We can now give the formal description of the model. Let us set a probability space $(\Omega, \mathcal{F}, \mathrm{Prob}_\mu)$ and define $\{(Y_n, \xi_n)\}_{n\in\mathbb{N}_0}$ as follows. Let $(Y_0, \xi_0)\colon \Omega \to X$ be a random variable with arbitrary and fixed distribution $\mu \in \mathcal{M}_1(X)$. Further, let us define by induction the sequences of random variables $\{\tau_n\}_{n\in\mathbb{N}_0}$, $\{\xi_n\}_{n\in\mathbb{N}}$, $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$, which describe the sequences $\{t_n\}_{n\in\mathbb{N}}$, $\{i_n\}_{n\in\mathbb{N}}$, $\{\theta_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ respectively, such that the following conditions are fulfilled:

- The sequence $\tau_n\colon \Omega \to [0, \infty)$, $n \in \mathbb{N}_0$, where $\tau_0 = 0$, is strictly increasing such that $\tau_n \to \infty$ a.e., and the increments $\Delta\tau_n = \tau_n - \tau_{n-1}$ are mutually independent and their conditional distributions are given by

  $$\mathrm{Prob}_\mu(\Delta\tau_{n+1} \le t \,|\, Y_n = y \text{ and } \xi_n = i) = 1 - e^{-L(t,y,i)} \quad \text{for } t \in \mathbb{R}_+,$$

  where $y \in Y$, $i \in I$ and L is given by

  $$L(t, y, i) = \int_0^t \lambda(\Pi_i(s, y))ds. \tag{6}$$

- The sequence $\xi_n\colon \Omega \to I$, $n \in \mathbb{N}$ satisfies the following condition

  $$\mathrm{Prob}_\mu(\xi_n = j \mid Y_n = y,\ \xi_{n-1} = i) = \pi_{ij}(y) \quad \text{for } i, j \in I,\ y \in Y.$$

- $\eta_n\colon \Omega \to \Theta$, $n \in \mathbb{N}$, is defined as follows

  $$\mathrm{Prob}_\mu(\eta_{n+1} \in D \mid \Pi_{\xi_n}(\Delta\tau_{n+1}, Y_n) = y) = \int_D p(\theta, y)\, d\theta$$

  for all $D \in \mathcal{B}(\Theta)$ and $y \in Y$.

- $Y_n\colon \Omega \to Y$, $n \in \mathbb{N}$, are given in following way

$$Y_{n+1} = \omega(\eta_{n+1}, \Pi_{\xi_n}(\Delta\tau_{n+1}, Y_n)) \quad \text{for } n \in \mathbb{N}_0.$$

Furthermore, for

$$U_0 = (Y_0, \xi_0), \quad U_k = (Y_0, \tau_1, \dots, \tau_k, \eta_1, \dots, \eta_k, \xi_0, \dots, \xi_k) \quad \text{for } k \in \mathbb{N},$$

we assume that for every $k \in \mathbb{N}_0$, the random variables ξ_{k+1} and η_{k+1} are conditionally independent of U_k given $\{Y_{k+1} = y,\ \xi_k = i\}$ and $\{\Pi_{\xi_k}(\Delta\tau_{k+1}, Y_k) = y\}$, respectively. We require also that ξ_{k+1}, η_{k+1} and $\Delta\tau_{k+1}$ are mutually conditionally independent given U_k, and that $\Delta\tau_{k+1}$ is independent of U_k.

We can easily check that the process $\{(Y_n, \xi_n)\}_{n\in\mathbb{N}_0}$ is a time-homogeneous Markov chain with phase space X such that the evolution of the distributions $\mu_n(\cdot) := \mathrm{Prob}_\mu((Y_n, \xi_n) \in \cdot)$ can be described by the Markov operator $P\colon \mathcal{M}_+(E) \to \mathcal{M}_+(E)$ given by

$$(7) \quad P\mu(A) = \sum_{j\in I} \int_\Theta \int_X \int_0^\infty \mathbb{1}_A(\omega(\theta, \Pi_i(t,y)), j)\lambda(\Pi_i(t,y))e^{-L(t,y,i)} \times \pi_{ij}(\omega(\theta, \Pi_i(t,y))p(\theta, \Pi_i(t,y)))dt d\mu(dy, di) d\theta$$

for $\mu \in \mathcal{M}_+(E)$, $A \in \mathcal{B}(E)$. Its dual operator $U\colon B(E) \to B(E)$ is given by

$$Uf(y,i) = \sum_{j\in I} \int_\Theta \int_0^\infty f(\omega(\theta, \Pi_i(t,y)), j))\lambda(\Pi_i(t,y))e^{-L(t,y,i)} \times \pi_{ij}(\omega(\theta, \Pi_i(t,y)))p(\theta, \Pi_i(t,y)) dt d\theta$$

for $x \in E$, $f \in B(E)$, where L is defined in (6).

We apply the following assumptions:

(A1) There is $y_* \in Y$ such that

$$\sup_{y\in Y} \int_0^\infty e^{-\underline{\lambda} t} \int_\Theta \rho(\omega(\theta, \Pi_i(t, y_*)), y_*) p(\theta, \Pi_i(t,y)) d\theta\, dt < \infty \quad \text{for } i \in I.$$

(A2) There exist constants $\gamma \in \mathbb{R}$, $L > 0$, and a bounded on bounded subsets of Y function $\mathcal{L}\colon Y \to \mathbb{R}_+$ such that the following inequality is satisfied

$$\rho(\Pi_i(t, y_1), \Pi_j(t, y_2)) \le L e^{\gamma t} \rho(y_1, y_2) + t\, \mathcal{L}(y_2) \Psi(i,j)$$

for $t \in \mathbb{R}_+$, $y_1, y_2 \in Y$, $i, j \in I$, where $\Psi(i,j)$ is given by (4).

(A3) There exists a constant $M > 0$ such that

$$\int_\Theta \rho(\omega(\theta, y_1), \omega(\theta, y_2))\, p(\theta, y_1) d\theta \le M \rho(y_1, y_2) \quad \text{for } y_1, y_2 \in Y.$$

(A4) There exists $S > 0$ such that

$$|\lambda(y_1) - \lambda(y_2)| \le S\rho(y_1, y_2) \quad \text{for } y_1, y_2 \in Y.$$

(A5) There exists $T > 0$ and $W > 0$ such that

$$\sum_{j \in I} |\pi_{ij}(y_1) - \pi_{ij}(y_2)| \le T\rho(y_1, y_2) \quad \text{for } y_1, y_2 \in Y,\ i \in I,$$

$$\int_{\Theta} |p(\theta, y_1) - p(\theta, y_2)|\, d\theta \le W\rho(y_1, y_2) \quad \text{for } y_1, y_2 \in Y.$$

(A6) There exists $\epsilon_\pi > 0$ and $\epsilon_p > 0$ such that

$$\sum_{j \in I} \min\{\pi_{i_1 j}(y_1), \pi_{i_2 j}(y_2)\} \ge \epsilon_\pi \quad \text{for } y_1, y_2 \in Y,\ i_1, i_2 \in I,$$

$$\int_{\Theta(y_1, y_2)} \min\{p(\theta, y_1), p(\theta, y_2)\} d\theta \ge \epsilon_p \quad \text{for } y_1, y_2 \in Y,$$

where $\Theta(y_1, y_2) = \{\theta \in \Theta :\ \rho(\omega(\theta, y_1), \omega(\theta, y_2)) \le \rho(y_1, y_2)\}$.

As it was written at the beginning of this section, the constant c appearing in (3) is needed to be sufficiently large and depends on constants appearing in conditions (A1)–(A4). For more details, we refer to [6].

In [6] it is shown, that if the conditions (A1)–(A6) hold and the constants occurring in them satisfy the inequality

$$LM\overline{\lambda} + \gamma < \underline{\lambda}, \tag{8}$$

then the operator P given by (7) is exponentially ergodic.

We would like to justify the conditions (A1)–(A6) stated above. The condition (A2) is met by a quite large class of semiflows defined on reflexive Banach spaces which can be generated by some differential equations engaging dissipative operators [10]. It happens often [5] then that the condition (A1) is a consequence of the conditions (A2) and (A3). The conditions (A3)–(A6) concerning assumptions such as contractivity are quite natural and in our setting commonly used regarding ergodic properties. One may consult [13, 15] for details.

4. The main result

In this section we prove the CLT for model $\{(Y_n, \xi_n)\}_{n \in \mathbb{N}_0}$ described in the previous section. Our proof is strongly based on the proof of Theorem 4.1 in [4]. The proof will require extensions of the conditions (A1) and (A3) of the model, namely:

(A1)′ There exists $\tilde{y} > 0$ such that

$$\sup_{y\in Y}\int_0^\infty e^{-\underline{\lambda} t}\int_\Theta [\rho(\omega(\theta,\Pi_i(t,\tilde{y})),\tilde{y})]^2 p(\theta,\Pi_i(t,y))d\theta dt < \infty \quad \text{for } i \in I.$$

(A3)′ There exists $M' > 0$ such that

$$\int_\Theta [\rho(\omega(\theta,y_1),\omega(\theta,y_2))]^2 p(\theta,y_1)d\theta \le M'[\rho(y_1,y_2)]^2 \quad \text{for } y_1,y_2 \in Y.$$

It is important to notice that (A1)′, (A3)′ are truly more restrictive than the base conditions (A1) and (A3), respectively.

Let us also define a function $V\colon X \to \mathbb{R}_+$ by

$$V(y,i) = \rho(y,\tilde{y}) \quad \text{for } (y,i) \in X, \tag{9}$$

where $\tilde{y}$ is the constant found by the condition (A1)′.

THEOREM 2. *Let* $\{(Y_n,\xi_n)\}_{n\in\mathbb{N}_0}$ *be the Markov chain with transition law corresponding to the Markov operator given by* (7) *and let* (A1)′, (A2), (A3)′, (A4)–(A6) *hold with constants satisfying*

$$\Big(\frac{\overline{\lambda}}{\underline{\lambda}}L\Big)^2 M' + \frac{2\gamma}{\underline{\lambda}} < 1. \tag{10}$$

Let $g \in Lip(X)$. *If the initial distribution* μ *of the chain* $\{g(Y_n,\xi_n)\}_{n\in\mathbb{N}_0}$ *belongs to* $\mathcal{M}_1^V(X)$, *where* V *is defined by* (9), *then* $\{g(Y_n,\xi_n)\}_{n\in\mathbb{N}_0}$ *passes the CLT.*

PROOF. Before we proceed to justifying that the conditions (B0)–(B5) of Theorem 1 are met, we check that the inequality (10) implies (8) with $M := \sqrt{M'}$.

From (10) we see that $\frac{\gamma}{\underline{\lambda}} < \frac{1}{2}$. Let us set $M := \sqrt{M'}$. Proving by contradiction let us assume that

$$LM\overline{\lambda} + \gamma \ge \underline{\lambda}.$$

We have

$$\frac{LM\overline{\lambda}}{\underline{\lambda}} \ge 1 - \frac{\gamma}{\underline{\lambda}}.$$

Using the Bernoulli inequality we obtain

$$\Big(\frac{LM\overline{\lambda}}{\underline{\lambda}}\Big)^2 \ge \Big(1 - \frac{\gamma}{\underline{\lambda}}\Big)^2 \ge 1 - \frac{2\gamma}{\underline{\lambda}},$$

which contradicts condition (10).

Given that conditions (A1)′ and (A3)′ imply (A1) and (A3), respectively, studying the proof of [6, Theorem 3.1], we conclude that the assumptions (A1)′, (A2), (A3)′, (A4)–(A6) guarantee that all conditions (B0)–(B5) beyond (B1) are fulfilled. It remains to show that (B1) is also met.

We start from describing the left-hand side of this condition.

$$UV^2(y,i) = \int_\Theta \int_0^\infty [\rho(\omega(\theta, \Pi_i(t,y), \tilde{y}))]^2 \lambda(\Pi_i(t,y)) e^{-L(t,y,i)} p(\theta, \Pi_i(t,y)) dt d\theta.$$

Let us fix $(y,i) \in X$ and define $\nu \in \mathcal{M}_1(\mathbb{R}_+ \times \Theta)$ as

$$\nu(A) = \int_0^\infty \lambda(\Pi_i(t,y)) e^{-L(t,y,i)} \int_\Theta \mathbb{1}_A(t,\theta) p(\theta, \Pi_i(t,y)) d\theta dt$$

and a function $\varphi_0 \colon \mathbb{R}_+ \times \Theta \to \mathbb{R}$ by

$$\varphi_0(t,\theta) = \rho(\omega(\theta, \Pi_i(t,y)), \tilde{y}) \quad \text{for } (t,\theta) \in \mathbb{R}_+ \times \Theta.$$

Now, using the triangle inequality together with the Minkowski inequality we get

$$\begin{aligned}(UV^2(y,i))^{\frac{1}{2}} &= \Big[\int_{\mathbb{R}_+ \times \Theta} \varphi_0^2(t,\theta) \nu(dt \times d\theta)\Big]^{\frac{1}{2}} \\ &\leq \Big[\int_{\mathbb{R}_+ \times \Theta} [\rho(\omega(\theta \Pi_i(t,y))), \omega(\theta, \Pi_i(t,\tilde{y}))]^2 \nu(dt \times d\theta)\Big]^{\frac{1}{2}} \\ &\quad + \Big[\int_{\mathbb{R}_+ \times \Theta} [\rho(\omega(\theta, \Pi_i(t,\tilde{y}))), \tilde{y}]^2 \nu(dt \times d\theta)\Big]^{\frac{1}{2}}.\end{aligned}$$

To shorten the notation, we denote

$$\begin{aligned}I_1 &= \int_{\mathbb{R}_+ \times \Theta} [\rho(\omega(\theta, \Pi_i(t,y)), \omega(\theta, \Pi_i(t,\tilde{y})))]^2 \nu(dt \times d\theta), \\ I_2 &= \int_{\mathbb{R}_+ \times \Theta} [\rho(\omega(\theta, \Pi_i(t,\tilde{y}))), \tilde{y}]^2 \nu(dt \times d\theta).\end{aligned}$$

Using conditions (A2) and (A3)′ we have

$$\begin{aligned}I_1 &\leq \int_0^\infty \lambda(\Pi_i(t,y)) e^{-L(t,y,i)} M' [\rho(\Pi_i(t,y), \Pi_i(t,\tilde{y}))]^2 dt \\ &\leq \int_0^\infty \lambda(\Pi_i(t,y)) e^{-L(t,y,i)} M' [L e^{\lambda t} \rho(y,\tilde{y})]^2 dt \\ &\leq \overline{\lambda} L^2 M' [\rho(y,\tilde{y})]^2 \int_0^\infty e^{-(\underline{\lambda} - 2\gamma)t} dt,\end{aligned}$$

where in the last inequality we used (5).

We obtain that

$$I_1 \leq \frac{\overline{\lambda} L^2 M'}{\underline{\lambda} - 2\gamma} V^2(y,i).$$

Using (10) we check that

$$0 < \frac{\overline{\lambda} L^2 M'}{\underline{\lambda} - 2\gamma} \leq \frac{\overline{\lambda}^2 L^2 M'}{\underline{\lambda}(\underline{\lambda} - 2\gamma)} < 1.$$

Obviously I_2 is finite because of the condition (A1)$'$. Finally, setting the constant to be

$$a = \sqrt{\frac{\overline{\lambda} L^2 M'}{\underline{\lambda} - 2\gamma}}$$

and

$$b = \sup_{y \in Y} \Big(\int_0^\infty e^{-\underline{\lambda} t} \int_\Theta [\rho(\omega(\theta, \Pi_i(t,\tilde{y})), \tilde{y})]^2 p(\theta, \Pi_i(t,y)) d\theta dt \Big)^{\frac{1}{2}}$$

we showed that

$$UV^2(y,i) \leq (aV(y,i) + b)^2.$$

This establishes (B1) and completes the proof of the theorem. □

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