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CENTRAL LIMIT THEOREM FOR RANDOM "CONTRACTIVE" FUNCTIONS ON INTERVAL

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Abstract. We use the approach from Czudek and Szarek (see [1]) to prove the central limit theorem for a stationary Markov chain generated by an iterative function system for a family of increasing, injective functions on $[0, 1]$ with "contractive" properties. We introduce a new approach to prove existence of an unique invariant measure using e-property (see [2]).

1. Introduction

We are given a family of continuous functions $\{f_i : [0, 1] \to [0, 1] : i \in \{1, \dots, N\}\}$ that are increasing and injective, possessing the following "contractive" properties:

(1) $\forall_{x\in(0,1)} \exists_{i,j\in\{1,\dots,N\}}\ f_i(x) < x < f_j(x)$,
(2) $\exists_{i\in\{1,\dots,N\}}\ f_i(0) > 0$,
(3) $\exists_{i\in\{1,\dots,N\}}\ f_i(1) < 1$.

We are also given a probability vector $(p_1, \dots, p_N)$. The family $(f_1, \dots, f_N; p_1, \dots, p_N)$ is called an iterated function system with probabilities.

This system generates a Markov chain, the distribution of which we will investigate. First, we are interested in whether this chain possesses a unique, non-atomic invariant measure. We will show that every measure will converge weakly to the unique invariant measure. Finally, we will prove the central limit theorem for this chain.

We use the approach from [1] as arguments for proving the properties turn out to be similar and easier to grasp. However, we took a different approach

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to proving the existence of a unique invariant measure as we use e-property introduced in [2] which made the proof quicker.

2. Theoretical base

Let (S,d) be a metric space. By $(C(S), \|\cdot\|)$, we denote the family of all continuous and bounded functions $f\colon S \to \mathbb{R}$, equipped with the supremum norm $\|\cdot\|$.

Let $\mathcal{M}(S)$ denote the set of all finite measures on the σ-algebra $\mathcal{B}(S)$ of Borel subsets of the set S. Let $\mathcal{M}_1(S) \subset \mathcal{M}(S)$ denote the subset of all probability measures on S.

An operator $P\colon \mathcal{M}(S) \to \mathcal{M}(S)$ is called a Markov operator if it satisfies the following conditions:

(1) $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$ for $\lambda_1, \lambda_2 \geq 0$, $\mu_1, \mu_2 \in \mathcal{M}(S)$,
(2) $P\mu(S) = \mu(S)$ for $\mu \in \mathcal{M}(S)$.

A Markov operator P is called a Feller operator if there exists a linear operator $U\colon C(S) \to C(S)$ such that $\int_S Uf d\mu = \int_S f dP\mu$ for every $f \in C(S)$ and $\mu \in \mathcal{M}(S)$.

A measure μ_* is called P-invariant for the Markov operator P if $P\mu_* = \mu_*$.

A Markov operator P is called asymptotically stable if it has a unique invariant measure $\mu_* \in \mathcal{M}_1(S)$ and, moreover, for every measure $\mu \in \mathcal{M}_1(S)$, the sequence $(P^n\mu)_{n\in\mathbb{N}}$ converges weakly to μ_*, i.e.,

$$\forall_{f\in C(S)} \lim_{n\to\infty} \int_S f(x) P^n\mu(\mathrm{d}x) = \int_S f(x)\mu_*(\mathrm{d}x).$$

THEOREM 2.1 (Krylov-Bogolyubov [3]). *Let (S,d) be a compact metric space, and let $P\colon \mathcal{M}(S) \to \mathcal{M}(S)$ be a Feller operator defined on finite Borel measures on this space.*

Then P has at least one invariant probability measure, i.e., there exists a measure $\mu \in \mathcal{M}_1(S)$ such that for any $A \in \mathcal{B}(S)$,

$$P\mu(A) = \mu(A).$$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A set $A \in \mathcal{F}$ is called an atom if, for any $B \in \mathcal{F}$, $B \subset A$, and $\mu(B) < \mu(A)$, we have $\mu(B) = 0$. A measure is called non-atomic if the measure space has no atoms, i.e., if $A, B \in \mathcal{F}$ and $B \subset A$, then $\mu(A) > \mu(B) > 0$.

We will consider the following Feller operator. Assume that $f_i\colon [0,1] \to [0,1]$ for $i \in \{1,\dots,N\}$ are continuous functions and let $(p_1,\dots,p_N)$ be a probability vector. The family $(f_1,\dots,f_N; p_1,\dots,p_N)$ generates a Markov operator $P\colon \mathcal{M}(S) \to \mathcal{M}(S)$ of the form

$$P\mu(A) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(A)) \quad \text{for } A \in \mathcal{B}([0,1]).$$

This operator is a Feller operator, whose predual operator is the operator $U\colon C([0,1]) \to C([0,1])$ given by the formula

$$U\phi(x) = \sum_{i=1}^{N} p_i \phi(f_i(x)) \quad \text{for } \phi \in C([0,1]), x \in [0,1].$$

Let H be the space of continuous functions $f\colon [0,1] \to [0,1]$ that are increasing and injective. Let $\{f_1, \dots, f_N\} \subset H$ be a finite set of functions satisfying the following properties:

(1) $\forall_{x \in (0,1)} \exists_{i,j \in \{1,\dots,N\}}\ f_i(x) < x < f_j(x)$,
(2) $\exists_{i \in \{1,\dots,N\}}\ f_i(0) > 0$,
(3) $\exists_{i \in \{1,\dots,N\}}\ f_i(1) < 1$

and $(p_1, \dots, p_N)$ is a probability vector. f_d denotes a function satisfying condition (2), and f_g denotes a function satisfying condition (3). The corresponding probabilities are denoted by p_d and p_g.

The system $(f_1, \dots, f_N; p_1, \dots, p_N)$ has a corresponding Markov operator P defined as before. For each measure $\nu \in \mathcal{M}_1(S)$, we describe the Markov chain (X_n) with the transition probability $\pi(x, A) = P\delta_x(A)$ for $x \in [0,1]$, $A \in \mathcal{B}([0,1])$, and initial distribution ν using the probabilistic measure $\mathbb{P}_\nu$ on the space $([0,1]^{\mathbb{N}}, \mathcal{B}([0,1])^{\otimes \mathbb{N}})$ such that:

$$\mathbb{P}_\nu[X_{n+1} \in A | X_n = x] = \pi(x, A) \quad \text{and} \quad \mathbb{P}_\nu[X_0 \in A] = \nu(A),$$

where $x \in [0,1]$, $A \in \mathcal{B}([0,1])$.

Since the interval $[0,1]$ is a compact set, the Markov operator P has at least one invariant measure μ_*, which is a consequence of Krylov-Bogoliubov's theorem. We now need to show that there is only one invariant measure. We also need to show that it is nonatomic.

3. The invariant measure

We introduce a few theorems and lemmas needed to prove atomlessness and existence of a unique measure.

THEOREM 3.1 ([4], Corollary 2.13). *Let $\omega \to (g_\omega^n)_{n \geq 0}$ be a non-degenerate random walk generated by a subgroup $G \subset Homeo([0,1])$, such that:*

(1) *There is no non-trivial interval $I \subset [0,1]$ invariant under G.*

(2) *There exists at least one probability measure μ on $(0,1)$ that is stationary for the random walk.*

Then there exists a $q<1$ such that for every $x \in (0,1)$, there exists a neighborhood I_x of x, such that for $\mathbb{P}$ almost all $i \in \Sigma$, the following holds:

$$|g_i^n(I_x)| \leq q^n \quad \textit{for every } n \in \mathbb{N}.$$

Moreover, if the first condition is violated, the theorem holds for

$$x \in (\inf(\operatorname{supp}\mu), \sup(\operatorname{supp}\mu)).$$

Remark 3.1. Note that the theorem is also true on the interval $[a,b]$.

Theorem 3.2. *There exists a $q<1$ such that for every $x \in (0,1)$, there exists a neighborhood I_x of x, such that for $\mathbb{P}$ almost all $i \in \Sigma$, the following holds:*

$$|f_i^n(I_x)| \leq q^n \quad \textit{for every } n \in \mathbb{N}.$$

Proof. Let $g_i \in Homeo([-\varepsilon, 1+\varepsilon])$, such that for $x \in [0,1]$ we have $g_i(x) = f_i(x)$. This family does not satisfy the first assumption of the previous theorem, because $[0,1]$ is invariant under G. We will prove that there exists a stationary measure for the random walk generated by G.

By Krylov–Bogoliubov's theorem, we know that the Markov operator generated by the family $(f_1, \ldots, f_n, p_1, \ldots, p_n)$ has an invariant measure μ. Let $\mu^* \in M([-\varepsilon, 1+\varepsilon])$ be defined by $\mu^*(A) = \mu(A \cap [0,1])$. It is easy to verify that μ^* is an invariant measure for the operator P_G generated by G. We will show that μ^* is a stationary measure for the random walk generated by G. We need to show that for every measurable set A, the following holds:

$$\int_{[-\varepsilon,1+\varepsilon]} \pi(x,A)\, d\mu^*(x) = \mu^*(A).$$

However,

$$\begin{aligned}\int_{[-\varepsilon,1+\varepsilon]} \pi(x,A)\, d\mu^*(x) &= \int_{[-\varepsilon,1+\varepsilon]} P_G\delta_x(A)\, d\mu^*(x)\\ &= \sum_{i=1}^n p_i \int_{[-\varepsilon,1+\varepsilon]} \delta_x(g_i^{-1}A)\, d\mu^*(x).\end{aligned}$$

But the last integral is $\mu^*(g_i^{-1}A)$, so μ^* is a stationary measure. From the definition of μ^*, it follows that $\operatorname{supp}(\mu^*) = \operatorname{supp}(\mu)$, and later in the work, we will prove that $\{0,1\} \in \operatorname{supp}(\mu)$, so by the previous theorem, the statement holds. $\square$

Before we proceed to prove the uniqueness of the invariant measure, we will prove that invariant measures must be atomless, and their supports contain the endpoints of the interval.

Lemma 3.1. *Every invariant measure of the operator P is atomless.*

Proof. Suppose there exists an invariant measure μ^* for the operator P that has an atom. Let $a \in (0,1)$ be a point such that

$$\mu^*(\{a\}) = \sup_{x \in (0,1)} \mu^*(\{x\}).$$

Then a is an atom of the measure μ^*, which means that $\mu^*(\{a\}) > 0$. Since μ^* is invariant under the operator P, we know that

$$\mu^*(\{a\}) = \sum_{i=1}^{n} p_i \mu^*(\{f_i^{-1}(a)\}).$$

For each function f_i, there is a point $f_i^{-1}(a)$ such that the measure μ^* assigns the same value $\mu(\{a\})$, because $f_i^{-1}(\{a\})$ for $i \in \Sigma^*$ forms an infinite set of points mapped to a. This means that for each i, we have $\mu^*(\{f_i^{-1}(a)\}) = \mu^*(\{a\})$, which suggests that $\mu(\{a\})$ should be split between infinitely many different points $f_i^{-1}(a)$. However, this leads to a contradiction with the property that μ^* is a finite measure, as it would imply that $\mu^*((0,1)) = \infty$, which is impossible. Thus, we arrive at a contradiction, and therefore the measure μ^* cannot have atoms. Hence, every invariant measure of the operator P is atomless. □

Lemma 3.2. *Every invariant measure of the operator P contains the points 0 and 1 in its support.*

Proof. Suppose that $0 \notin \operatorname{supp}(\mu)$. Then, by the atomlessness of μ, there exists the largest $a > 0$ such that $\mu([0,a]) = 0$. However, from the invariance of μ, we obtain:

$$0 = \mu([0,a]) = \sum_{i=1}^{n} p_i \mu(f_i^{-1}([0,a])).$$

This implies that for every i, we have $\mu(f_i^{-1}([0,a])) = 0$. However, there exists some i for which $f_i(a) < a$, which implies that $[0,a] \subset f_i^{-1}([0,a])$, contradicting the maximality of a. A similar proof can be conducted for 1. □

Definition 3.1 ([2]). Let S be a compact metric space. We say that a Feller operator P has the *e-property at the point $x \in S$* if for every Lipschitz function $\phi\colon S \to \mathbb{R}$, the following holds:

$$\lim_{y\to x} \sup_{n\in\mathbb{N}} |U^n\phi(x) - U^n\phi(y)| = 0.$$

If P has the e-property at every point, we say that P has the *e-property.*

THEOREM 3.3 ([3, Lemma 2.8]). *Let P be a Feller operator that possesses the e-property. Then for any two distinct ergodic measures $\mu, \nu \in M_1(S)$, the following holds:*

$$\operatorname{supp}\mu \cap \operatorname{supp}\nu = \emptyset.$$

In particular, from the proof of the lemma, it follows that if P has the e-property at x, then $x \notin \operatorname{supp}\mu \cap \operatorname{supp}\nu$.

We now proceed with the proof of the uniqueness of the invariant measure.

THEOREM 3.4. *There exists exactly one invariant measure for the Feller operator P.*

PROOF. We know that if μ is an invariant measure, then $0 \in \operatorname{supp}\mu$. Therefore, we only need to check that the operator P has the e-property at 0. Let $\varepsilon > 0$ be given. We need to show that there exists a $\delta > 0$ such that if $h < \delta$, then for every $n \geq N_0$, the following holds:

$$\Big|\sum_{i\in\Sigma_n} p_i f_i(h) - \sum_{i\in\Sigma_n} p_i f_i(0)\Big| = \sum_{i\in\Sigma_n} p_i|f_i([0,h])| < \varepsilon.$$

Let N_1 be large enough such that $(1-p_d)^{N_1} < \varepsilon/3$. Let $\Sigma_n^1 = \{i \in \Sigma_n : f_{i_k} \neq f_d \text{ for } 1 \leq k \leq N_1\}$. Note that $\mathbb{P}_n(\Sigma_n^1) = (1-p_d)^{N_1}$. Let $\Sigma_n^2 = \{i \in \Sigma_n : i_{|N_1} \notin \Sigma_{N_1}^1\}$. For $i \in \Sigma_n^2$, let $x_i = f_i(0)$. Since $0 < x_i < 1$, there exists a neighborhood J_i such that for almost all $j \in \Sigma$, we have $|f_j^k(J_i)| < q^k$. Let $\Sigma^3 = \{j \in \Sigma : \text{ for every } i \in \Sigma_n^2, |f_j^k(J_i)| < q^k\}$. Clearly, $\mathbb{P}(\Sigma^3) = 1$. Choose δ small enough such that for every $i \in \Sigma_n^2$, we have $f_i([0,\delta]) \subset J_i$. Let $N_2 > N_1$ be sufficiently large such that $q^{N_2-N_1} < \varepsilon/3$. Let $\Sigma_n^4 = \{j^* = (ij)_{|n} : i \in \Sigma_n^2 \text{ and } j \in \Sigma^3\}$ and $\Sigma_n^5 = \{j^* = (ij)_{|n} : i \in \Sigma_n^2 \text{ and } j \notin \Sigma^3\}$. Let $N_3 > N_2$ be sufficiently large such that $\mathbb{P}_{N_3}(\Sigma_n^5) < \varepsilon/3$. For $n > N_3$, the following holds:

$$\begin{aligned}\sum_{i\in\Sigma_n} p_i|f_i([0,h])| &= \sum_{i\in\Sigma_n^1} p_i|f_i([0,h])| + \sum_{i\in\Sigma_n^4} p_i|f_i([0,h])| + \sum_{i\in\Sigma_n^5} p_i|f_i([0,h])|\\ &\leq \sum_{i\in\Sigma_n^1} p_i + \sum_{i\in\Sigma_n^4} p_i q^{n-N_1} + \sum_{i\in\Sigma_n^5} p_i\\ &\leq (1-p_d)^{N_1} + q^{N_2-N_1} + \sum_{i\in\Sigma_n^5} p_i \leq \varepsilon.\end{aligned}$$

Consequently, P has the e-property at 0, so if μ and ν are two distinct ergodic measures, then $0 \notin \operatorname{supp} \mu \cap \operatorname{supp} \nu$ and on the other hand, $0 \in \operatorname{supp} \mu \cap \operatorname{supp} \nu$ – a contradiction, which completes the proof. □

We will now proceed with the proof of the asymptotic stability of the operator P.

THEOREM 3.5 ([1]). *Let μ_* be the unique invariant measure of the operator P. Then every probabilistic measure μ converges weakly to μ_*. For continuous functions ϕ, we have:*

$$\lim_{n\to\infty} \langle P^n \mu, \phi\rangle = \langle \mu_*, \phi\rangle.$$

PROOF. For $\psi \in C([0,1])$, define the sequence of random variables on $(\Sigma, \mathbb{R})$ by

$$\xi_n^{\psi}(i) = \langle \mu_*, \psi \circ f_{(i_n, i_{n-1}, \ldots, i_1)}\rangle.$$

We will show that (ξ_n^{ψ}) is a bounded martingale. The boundedness is obvious, so it is sufficient to show that

$$\mathbb{E}(\xi_{n+1}^{\psi} | \xi_1^{\psi}, \ldots, \xi_n^{\psi}) = \xi_n^{\psi}.$$

Notice that $U \in \sigma(\xi_1^{\psi}, \ldots, \xi_n^{\psi})$ has the form $U_n \times \Sigma_1 \times \Sigma$ for some $U_n \in \Sigma_n$. We have:

$$\begin{aligned}
\int_U \mathbb{E}(\xi_{n+1}^{\psi} | \xi_1^{\psi}, \ldots, \xi_n^{\psi})\, d\mathbb{P} &= \int_U \xi_{n+1}^{\psi}\, d\mathbb{P}_{n+1} = \sum_{i \in U_{n+1}} p_i \langle \mu_*, \psi \circ f_i\rangle \\
&= \sum_{i\in\Sigma_1} p_i \langle \mu_*, \sum_{j \in U_n} p_j \psi \circ f_j(f_i(x))\rangle \\
&= \Big\langle \mu_*, U\Big(\sum_{j\in U_n} p_j \psi \circ f_j\Big)\Big\rangle = \langle \mu_*, \sum_{j\in U_n} p_j \psi \circ f_j\rangle \\
&= \int_{U_n} \xi_n^{\psi}\, d\mathbb{P}_n = \int_U \xi_n^{\psi}\, d\mathbb{P}.
\end{aligned}$$

This completes this part of the proof. By the Martingale Convergence Theorem, we know that ξ_n^{ψ} converges for almost every $i \in \Sigma$. The space of continuous functions on the interval is central, so there exists a set $\Sigma_0 \subset \Sigma$ of full measure such that for every $i \in \Sigma_0$, the sequence ξ_n^{ψ} converges. This follows from the fact that we can specify such a set for the center. From the Riesz Representation Theorem, we know that for $i \in \Sigma_0$, there exists a probabilistic measure μ_i such that

$$\lim_{n\to\infty} \xi_n^{\psi}(i) = \langle \mu_i, \psi\rangle. \tag{3.1}$$

We will show that for almost every i, the measure $\mu_i = \delta_{v(i)}$ for some $v(i)$. It is sufficient to show that for $\varepsilon > 0$, there exists a set $\Sigma_\varepsilon \subset \Sigma$ of full measure such that for each $i \in \Sigma_\varepsilon$, there exists an interval I of length less than ε such that

$$\mu_i(I) \geq 1 - \varepsilon.$$

Then for $i \in \Sigma^1 = \bigcap_{n=1}^{\infty} \Sigma_{1/n}$, we have $\mu_i = \delta_{v(i)}$. The set Σ^1 is a set of full measure. For a fixed $\varepsilon > 0$, choose a natural number l such that $1/l < \varepsilon$. Let the interval $[a, b]$ satisfy

$$\mu_*[a, b] > 1 - \varepsilon$$

and contain $f_d(f_g(0))$ and $f_d(f_g(1))$. We can specify $j \in \Sigma$ such that $f_{j_n}(b) \to 0$. Notice that there exists an n_1 such that $f_{j_{n_1}}(b) < a$. Let $i_1 = j_{n_1}$. In general, for $k \leq l$, there exists n_k such that

$$f_{j_{n_k}}(b) < f_{j_{n_{k-1}}}(a).$$

Let $i_k = j_{n_k}$. Notice that the intervals $J_n = f_{i_n}[a, b]$ are disjoint. It follows that for some n^*, we have

$$|J_{n^*}| < 1/l < \varepsilon/2.$$

Notice that

$$\Sigma^1 = \{i \in \Sigma : \text{ for infinitely many } n,\ |f_{i_n}[a, b]| < \varepsilon/2\}.$$

is a set of full measure. This is true because if $i \notin \Sigma^1$, there exists N such that for $n > N$ $|f_{i_n}[a, b]| \geq \varepsilon/2$. However, with probability 1, the sequence $\sigma^N i$ contains the fragment $(i_d, i_g)i_{n^*}$. So with probability 1, there exists $n' > N$ such that $|f_{i_{n'}}[a, b]| < \varepsilon/2$. By the compactness of the interval, it follows that for infinitely many n, the image $f_{i_{n'}}[a, b] \in I$, where I is an interval of length ε. Notice that $\Sigma^1 \subset \Sigma_\varepsilon$, which completes this part of the proof.

To show the asymptotic stability, it is sufficient to show that for a Lipschitz function ψ and any points $x, y \in (0, 1)$, the following holds:

$$\lim_{n\to\infty} |\langle P^n \delta_x, \psi\rangle - \langle P^n \delta_y, \psi\rangle| = 0.$$

This is true because for any measure $\mu \in M_1(0, 1)$, we have

$$|\langle P^n \mu, \psi\rangle - \langle \mu_*, \psi\rangle| \leq \int_{(0,1)} \int_{(0,1)} |\langle P^n \delta_x, \psi\rangle - \langle P^n \delta_y, \psi\rangle| \mu(dx)\mu_*(dy).$$

Let us fix points x and y belonging to the interval $(0, 1)$, with $x < y$. Choose any $\varepsilon > 0$. Since the measure μ^* is invariant, according to the proof of uniqueness from Theorem 4.4, its support contains the points 0 and 1, which means that $\mu^*((0,x)) > 0$ and $\mu^*((y,1)) > 0$. We know that for almost every sequence $i = (i_1, i_2, \dots) \in \Sigma$ with respect to the measure $\mathbb{P}$, the sequence of measures

$$\mu^* \circ f^{-1}_{(i_1,i_2,\dots,i_n)}((v(i) - \frac{\varepsilon}{2}, v(i) + \frac{\varepsilon}{2}) \cap (0,1)) \to 1$$

as $n \to \infty$. Since $\mu^*((0,x)) > 0$ and $\mu^*((y,1)) > 0$, we can find points $u_n \in (0,x)$ and $v_n \in (y,1)$ such that for sufficiently large n, we have $u_n, v_n \in f^{-1}_{(i_1,i_2,\dots,i_n)}((v(i) - \frac{\varepsilon}{2}, v(i) + \frac{\varepsilon}{2}) \cap (0,1))$. Then it follows that $f_{(i_1,\dots,i_n)}(u_n)$ and $f_{(i_1,\dots,i_n)}(v_n)$ lie in the interval $(v(i) - \frac{\varepsilon}{2}, v(i) + \frac{\varepsilon}{2})$, and in particular, for sufficiently large n, $f_{(i_1,\dots,i_n)}(x)$ and $f_{(i_1,\dots,i_n)}(y)$ lie in this interval. Since $\varepsilon > 0$ was arbitrary, we conclude that for almost every $i = (i_1, i_2, \dots) \in \Sigma$ with respect to $\mathbb{P}$, the following convergence holds:

$$\lim_{n\to\infty} |f_{(i_1,\dots,i_n)}(x) - f_{(i_1,\dots,i_n)}(y)| = 0.$$

By equation (3.1) and the fact that $\langle P^n\delta_z, \phi\rangle = U^n\phi(z)$ for $z \in [0,1]$, we have:

$$\begin{aligned} (3.2)\quad |\langle P^n\delta_x, \phi\rangle - \langle P^n\delta_y, \phi\rangle| &= |U^n\phi_x - U^n\phi_y| \\ &\le L \sum_{(i_1,\dots,i_n)\in\Sigma_n} |f_{(i_1,\dots,i_n)}(x) - f_{(i_1,\dots,i_n)}(y)| p_{i_1}\cdots p_{i_n}, \end{aligned}$$

where L is the Lipschitz constant for the function ϕ. We will analyze why the right-hand side of the above inequality tends to zero as $n \to \infty$. For $i = (i_1, \dots, i_n, \dots)$, define $g_n(i) := |f_{i_n}(x) - f_{i_n}(y)|$, where $i_n = (i_1, \dots, i_n)$. Then, $g_n(i) \to 0$ as $n \to \infty$ for almost every $i \in \Sigma$ with respect to $\mathbb{P}$. From the construction of the probability measures $\mathbb{P}$ and $\mathbb{P}^n$ for $n \in \mathbb{N}$, it follows that the following relationship holds

$$\mathbb{P}(B_n \times \Sigma_1 \times \Sigma_1 \times \dots) = \mathbb{P}_n(B_n) \quad \text{for } B_n \in \Sigma_n.$$

Since $g_n(i)$ depends solely on the first n coordinates, it follows that

$$\begin{aligned} \int_\Sigma g_n(i) d\mathbb{P}(i) &= \int_{\Sigma_n} g_n(i) d\mathbb{P}_n(i) \\ &= \sum_{(i_1,\dots,i_n)\in\Sigma_n} |f_{(i_1,\dots,i_n)}(x) - f_{(i_1,\dots,i_n)}(y)| p_{i_1} \cdot \dots \cdot p_{i_n}. \end{aligned}$$

As a result, according to Lebesgue's theorem, we have

$$\lim_{n\to\infty} \sum_{(i_1,\dots,i_n)\in\Sigma^n} |f_{(i_1,\dots,i_n)}(x) - f_{(i_1,\dots,i_n)}(y)| p_{i_1} \cdot \dots \cdot p_{i_n} = 0,$$

which proves inequality (3.2) and thus completes the proof. $\square$

4. Central Limit Theorem

LEMMA 4.1 ([1]). *Let the family* $(f_1, \dots, f_N; p_1, \dots, p_N)$ *be a contracting iterated function system, and let* $a \in (0, \frac{1}{2})$. *Then there exists* $r \in \mathbb{N}$ *and* $\Omega \subseteq \Sigma$ *with probability* $\mathbb{P}(\Omega) > 0$ *such that* $J = [a, 1-a]$ *and we get*

$$\sum_{n=1}^{\infty} |f_i^n(J)| \le r + \frac{q}{1-q}$$

for every $i \in \Omega$, *where* q *is a constant given by Theorem* 3.2.

LEMMA 4.2. *Let* $J = [a, 1-a]$ *be such that* $f_d(f_g(0)), f_d(f_g(1)) \in J$. *Let* $E_n = \{i \in \Sigma : f_i^{\lfloor n^{1/4} \rfloor}[0,1] \in J\}$. *Then, for* $n \ge 16$, *there exists* $\beta = p_d p_g > 0$ *such that* $\mathbb{P}(E_n) \ge \beta$.

LEMMA 4.3. *Let* $\mathcal{A} \subset \Sigma$ *be a set such that* $\mathbb{P}(\mathcal{A}) \ge \beta$ *for some* $\beta > 0$ *and let* $k, n \in \mathbb{N}$ *with* $k < n$. *Then, there exists a set* $A \subset \Sigma_n$ *such that* $\mathbb{P}_n(\Sigma_n \backslash A) \le (1-\beta)^k$ *and for any* $i \in A$ *there exist* $i_1, i_2, \dots, i_k \in \Sigma_*$ *such that* $i = i_1 i_2 \dots i_k$ *and for* $j = 1, \dots, k$ *at least one of the sequences* $i_j, \sigma i_j, \dots, \sigma^{k-1} i_j$ *is dominated by* A.

THEOREM 4.1 (Central Limit Theorem). *Let* X_n *be a stationary Markov chain generated by a random walk with the initial distribution given by* μ_*. *If* $\phi \colon [0,1] \to \mathbb{R}$ *is a Lipschitz function such that* $\int_{[0,1]} \phi d\mu_* = 0$, *then the process* $\phi(X_n)$ *satisfies the central limit theorem. That is,*

$$\sigma^2 = \lim_{n \to \infty} \mathbb{E} \left(\frac{\phi(X_0) + \cdots + \phi(X_n)}{\sqrt{n}} \right)^2$$

exists, and

$$\frac{\phi(X_0) + \cdots + \phi(X_n)}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \sigma) \quad \textit{as} \quad n \to \infty.$$

PROOF. From the uniqueness of the ergodic measure μ_*, we know that the chain is ergodic. Therefore, by Theorem 1 from [5], it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-3/2} \| \sum_{j=1}^{n} U^j \phi \|_{L^2} < \infty.$$

Let E_n be as in Lemma 4.2. Clearly, $\mathbb{P}(E_n) \ge \beta > 0$. Let Ω be as in Lemma 4.1. Let

$$\mathcal{A} = \{ i j_{|n} : i \in E_n^{\lfloor n^{1/4} \rfloor}, j \in \Omega \}.$$

Clearly, $\mathbb{P}(\mathcal{A}) \geq \alpha > 0$ for some α (independent of n). Then, by Lemma 4.3, for $k = \lfloor n^{1/8} \rfloor$, we get a sequence $A_n \in \Sigma_n$ satisfying $\mathbb{P}(B_n = \Sigma_n / A_n) \leq (1-\alpha)^{\lfloor n^{1/8} \rfloor}$. Moreover, if $i \in A_n$, then $i = i_1 \dots i_k$, where at least one of $i_m, \sigma i_m, \dots, \sigma^{k-1} i_m$ is dominated by $\mathcal{A}$. Therefore, for any x, y, we have:

$$\sum_{j=1}^{|i_m|} |f_i^j(x) - f_i^j(y)| \leq \lfloor n^{1/8} \rfloor + \lfloor n^{1/4} \rfloor + r + \frac{q}{1-q} \leq 2(\lfloor n^{1/4} \rfloor + c).$$

Notice that

$$\sum_{j=1}^{n} |f_i^j(x) - f_i^j(y)| \leq 2k(\lfloor n^{1/4} \rfloor + c) \leq C \lfloor n^{3/4} \rfloor.$$

Let L be the Lipschitz constant of the function ϕ. Since

$$\int_{[0,1]} \phi d\mu_* = \int_{[0,1]} U^j \phi d\mu_* = 0,$$

we have:

$$\int_{[0,1]} \Big| \sum_{j=1}^{n} U^j \phi(x) \Big|^2 d\mu_*(x) \leq \int_{[0,1]} \int_{[0,1]} \Big(\sum_{j=1}^{n} |U^j \phi(x) - U^j \phi(y)| \Big)^2 d\mu_*(x) \mu_*(y).$$

We know that

$$\begin{aligned} \sum_{j=1}^{n} |U^j \phi(x) - U^j \phi(y)| &\leq \int_{A_n} \sum_{j=1}^{n} |f_i^j \phi(x) - f_i^j \phi(y)| d\mathbb{P}(i) \\ &\quad + \int_{B_n} \sum_{j=1}^{n} |f_i^j \phi(x) - f_i^j \phi(y)| d\mathbb{P}(i) \\ &\leq LCn^{3/8} + 2nL\mathbb{P}(B_n) \leq C_* n^{3/8}. \end{aligned}$$

This implies that

$$\int_{[0,1]} \int_{[0,1]} \Big(\sum_{j=1}^{n} |U^j \phi(x) - U^j \phi(y)| \Big)^2 d\mu_*(x) \mu_*(y) \leq C_*^2 n^{3/4}.$$

Therefore, $\| \sum_{j=1}^{n} U^j \phi \|_{L^2} < C^* n^{3/8}$. By Theorem 1 from [5], we obtain the result. □

References

[1] K. Czudek and T. Szarek, *Ergodicity and central limit theorem for random interval homeomorphisms*, Israel J. Math. **239** (2020), no. 1, 75–98. DOI: 10.1007/s11856-020-2046-4.

[2] T. Komorowski, Sz. Peszat, and T. Szarek, *On ergodicity of some Markov processes*, Ann. Probab. **38** (2010), no. 4, 1401–1443. DOI: 10.1214/09-AOP513.

[3] G. Łuczyńska, *Ergodyczność oraz twierdzenia graniczne dla stochastycznych układów dynamicznych na okręgu*, PhD thesis, Uniwersytet Gdański, Gdańsk, 2022.

[4] D. Malicet, *Random walks on* $Homeo(S^1)$, Comm. Math. Phys. **356** (2017), no. 3, 1083–1116.

[5] M. Maxwell and M. Woodroofe, *Central limit theorems for additive functionals of Markov chains*, Ann. Probab. **28** (2000), no. 2, 713–724.

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