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ON A GENERALIZED CONJECTURE BY ALZER AND MATKOWSKI

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Abstract. We study a recent conjecture proposed by Horst Alzer and Janusz Matkowski concerning a bilinearity property of the Cauchy exponential difference for real-to-real functions. The original conjecture was affirmatively resolved by Tomasz Małolepszy. We deal with generalizations for real or complex mappings acting on a linear space.

1. Introduction

Alzer and Matkowski [1] recently studied the following functional equation:

$$f(x+y) = f(x)f(y) - \alpha xy, \quad x, y \in \mathbb{R}, \tag{1.1}$$

where $\alpha \in \mathbb{R}$ is a non-zero parameter and $f\colon \mathbb{R} \to \mathbb{R}$ is an unknown function. They proved two theorems on equation (1.1). The first result with a short proof [1, Theorem 1] completely describes solutions of (1.1) in case f has a zero. More precisely, they showed that if f solves (1.1) and it has a zero, then $\alpha > 0$ and either $f(x) = 1 - \sqrt{\alpha}x$, or $f(x) = 1 + \sqrt{\alpha}x$ for $x \in \mathbb{R}$. The second theorem with a longer proof [1, Theorem 2] provides the solutions to equation (1.1) under the assumption that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at least at one point. In this case, there are the same two solutions (clearly, both are differentiable and have a zero). In [1] the authors formulated the following conjecture:

Conjecture (Alzer and Matkowski). *Every solution $f\colon \mathbb{R} \to \mathbb{R}$ of* (1.1) *has a zero.*

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This conjecture has been answered affirmatively by T. Małolepszy, see [4]. In the present note, we will determine the solutions of a more general equation, namely

$$f(x+y) = f(x)f(y) - \phi(x,y), \quad x, y \in X, \tag{1.2}$$

where X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\phi\colon X \times X \to \mathbb{K}$ is a biadditive functional and $f\colon X \to \mathbb{K}$ is a function. The motivation for such a generalization comes from an article by K. Baron and Z. Kominek [2], in which the authors, in connection with a problem proposed by S. Rolewicz [5], studied mappings defined on a real linear space with the additive Cauchy difference bounded from below by a bilinear functional.

2. Main results

In this section, it is assumed that X is a linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\phi\colon X \times X \to \mathbb{K}$ is a biadditive functional and $f\colon X \to \mathbb{K}$. We will consider two situations, depending on the behavior of the biadditive functional ϕ on the diagonal.

Theorem 1. *Assume that ϕ and f solve* (1.2) *and*

$$\exists_{z_0 \in X}\, \phi(z_0, z_0) \neq 0. \tag{2.1}$$

Then there exists a unique constant $a \in \mathbb{K} \setminus \{0\}$ such that

$$f(x) = a\phi(x, z_0) + 1, \quad x \in X, \tag{2.2}$$

and moreover

$$a^2\phi(x, z_0)^2 = \phi(x, x), \quad x \in X. \tag{2.3}$$

Proof. Substituting $y = z_0$ and then $y = -z_0$ in (1.2) we obtain

$$f(x + z_0) = f(x)f(z_0) - \phi(x, z_0), \quad x \in X$$

and

$$f(x - z_0) = f(x)f(-z_0) + \phi(x, z_0), \quad x \in X.$$

Replace x by $x + z_0$ in the latter formula and join it with the former one to arrive at

$$\begin{aligned} f(x) &= f(x + z_0)f(-z_0) + \phi(x + z_0, z_0) \\ &= [f(x)f(z_0) - \phi(x, z_0)]f(-z_0) + \phi(x, z_0) + \phi(z_0, z_0), \quad x \in X. \end{aligned}$$

Denote $c := f(z_0)$, $d := f(-z_0)$ and $\beta = \phi(z_0, z_0) \neq 0$. We get

$$(1 - cd)f(x) = (1 - d)\phi(x, z_0) + \beta, \quad x \in X.$$

As stated in the proof of Theorem 1 in [1], it follows that $f(0) = 1$. The argument works in our case, as well. Indeed, substitution $x = y = 0$ in (1.2) gives us $f(0)^2 = f(0)$, so $f(0) = 0$ or $f(0) = 1$. But $f(0) = 0$ would imply $\beta = 0$, which is a contradiction with the definition of β.

Therefore, from (1.2) applied for $x = z_0$ and $y = -z_0$ we deduce

$$1 = f(z_0 - z_0) = f(z_0)f(-z_0) + \beta,$$

thus $1 - cd = \beta$. Since $\beta \neq 0$, then denoting $a := (1 - d)/\beta$ we get (2.2). The case $a = 0$ is impossible, since it leads to a contradiction with (2.1).

To prove equality (2.3) apply (1.2) with substitution $y = -x$ to obtain

$$f(x)f(-x) = 1 - \phi(x, x), \quad x \in X.$$

Now, use the already proven formula (2.2) to derive (2.3) after some reductions. □

Remark 1. From Theorem 1, we see that under assumption (2.1) and with a fixed functional ϕ there are always either no solutions or exactly two solutions f of (1.2). Indeed, if f is a solution, then it must be of the form (2.2) with some constant $a \in \mathbb{K} \setminus \{0\}$. Substituting this into (1.2) leads us to the equality:

$$a^2\phi(x, z_0)\phi(y, z_0) = \phi(x, y), \quad x, y \in X,$$

which is true for two different values of $a \neq 0$. Therefore, in case there do exist solutions, functional ϕ is of the form

$$\phi(x, y) = a^2 F(x) \cdot F(y), \quad x, y \in X,$$

with an additive, nonzero functional $F\colon X \to \mathbb{K}$, and the two possible functions f are given by (2.2).

We have the following corollary in the real case.

Corollary 1. *Assume that* $\mathbb{K} = \mathbb{R}$ *and*

$$\exists_{z_0 \in X}\ \phi(z_0, z_0) < 0. \tag{2.4}$$

Then equation (1.2) *has no solutions.*

Proof. Inequality (2.4) implies that condition (2.1) holds true. Then, from Theorem 1 we obtain formula (2.3). However, in the real case formula (2.3) implies that $\phi(x, x) \geq 0$ for all $x \in X$, which leads to a contradiction with (2.1). □

In the complex case, every element of the field has a complex root of second order. Therefore, we can state the next corollary.

Corollary 2. *Assume that* $\mathbb{K} = \mathbb{C}$, ϕ *and* f *solve* (1.2), ϕ *satisfies* (2.1) *and* $w\colon X \to \mathbb{C}$ *is a map such that*

$$w^2(x) = \phi(x,x), \quad x \in X.$$

Then

$$f(x) = w(x) + 1, \quad x \in X,$$

or

$$f(x) = -w(x) + 1, \quad x \in X.$$

The next theorem deals with the remaining case for ϕ and is easy to prove.

Theorem 2. *Assume that* ϕ *and* f *solve* (1.2) *and*

$$\forall_{z \in X}\ \phi(z,z) = 0.$$

Then $\phi = 0$ *on* $X \times X$ *and*

$$f(x+y) = f(x)f(y), \quad x, y \in X.$$

Consequently, either $f = 0$ *or there exists an additive functional* $A\colon X \to \mathbb{K}$ *such that* $f = \exp \circ A$.

Proof. It suffices to apply a well-known result, which states that if a multiadditive function vanishes on a diagonal, then it vanishes everywhere, cf. [3, Corollary 15.9.1, p. 448]. The final part follows from the form of solutions of the exponential Cauchy equation, cf. [3, Theorem 13.1.1, p. 343]. $\square$

The following corollary is immediate and offers an alternative proof of the conjecture by Alzer and Matkowski.

Corollary 3 (T. Małolepszy). *Assume that* $\alpha \in \mathbb{R}$ *and* $f\colon \mathbb{R} \to \mathbb{R}$ *solves* (1.1). *Then* $\alpha \geq 0$ *and moreover, in case* $\alpha > 0$ *either* $f(x) = 1 - \sqrt{\alpha}x$, *or* $f(x) = 1 + \sqrt{\alpha}x$ *for* $x \in \mathbb{R}$. *Conversely, both mappings solve* (1.1).

Proof. Firstly, substituting $\alpha = 0$ in (1.1) we obtain the exponential Cauchy's equation, for which the solutions are known. Now assume that $\alpha \neq 0$, $\mathbb{K} = \mathbb{R}$, $X = \mathbb{R}$ and $\phi(x,y) := \alpha xy$. Let $z_0 = 1$. Then $\beta = \phi(1,1) = \alpha \neq 0$. From (2.2) we have

$$f(x) = a\phi(x,1) + 1 = a\alpha x + 1, \quad x \in \mathbb{R}.$$

From (2.3) we obtain

$$a^2(\alpha x)^2 = \phi(x,x) = \alpha x^2, \quad x \in \mathbb{R}.$$

We get $a^2 = 1/\alpha$, so $\alpha > 0$ and $a = \pm 1/\sqrt{\alpha}$. After substitution to the equation for f we arrive at $f(x) = \pm\sqrt{\alpha}x + 1$. Conversely, it is easy to check that both such mappings solve (1.1). $\square$

Our last corollary is a complex counterpart of Corollary 3.

COROLLARY 4. *Assume that $\alpha \in \mathbb{C} \setminus \{0\}$ and $f\colon \mathbb{C} \to \mathbb{C}$ solves*

$$f(x+y) = f(x)f(y) - \alpha xy, \quad x, y \in \mathbb{C}. \tag{2.5}$$

Then either $f(x) = 1 + w_1 x$, or $f(x) = 1 + w_2 x$ for $x \in \mathbb{C}$, where w_1, w_2 are two complex roots of the second order of α. Conversely, both mappings solve (2.5).

PROOF. Assume that $\mathbb{K} = \mathbb{C}$, $X = \mathbb{C}$ and $\phi(x, y) := \alpha xy$. By repeating steps from the previous proof (this time without assuming $\alpha > 0$), we obtain demanded results. $\square$

3. Examples and final remarks

We observe that Theorem 1 generally works only in one direction, that is, the converse implications do not necessarily hold.

EXAMPLE 1. Let X be an inner product space of dimension at least 2 and define $\phi := \langle \cdot, \cdot \rangle$. Then, Theorem 1 implies that the potential solutions $f\colon X \to \mathbb{K}$ of (1.2) are of the form

$$f(x) = \langle x, \xi \rangle + 1, \quad x \in X,$$

with some $\xi \in X$. One can check that such mapping solves (1.2) if and only if

$$\langle x, \xi \rangle \langle y, \xi \rangle = \langle x, y \rangle, \quad x, y \in X,$$

which is impossible if $\dim X \geq 2$.

It may be suspected that in higher dimensions, there are no solutions to (1.2). However, the following example demonstrates that this is not the case.

EXAMPLE 2. Let X be a complex linear space and $A\colon X \to \mathbb{C}$ an additive nonzero functional. Define $\phi(x, y) := -A(x) \cdot A(y)$ for $x, y \in X$. Then, according to Theorem 1 every solution $f\colon X \to \mathbb{C}$ of (1.2) is of the form:

$$f(x) = \gamma A(x) + 1, \quad x \in X$$

with some constant $\gamma \in \mathbb{C}$. A direct calculation shows that f is indeed a solution if and only if $\gamma = \pm i$.

We can choose A in such a way that $A(x) \neq 0$ whenever $x \neq 0$, or such that A has a bigger set of zeros. Therefore, for every complex linear space X there is an abundance of nontrivial solutions (f, ϕ) to (1.2).

This example also illustrates that the assertion of Corollary 1 does not hold in the case of complex spaces, even when the values of ϕ are real (since A may only attain real values, as it does not necessarily have to be linear).

A counterpart of the above example that works in both cases, real and complex, is also possible.

Example 3. Let X be a linear space over the field $\mathbb{K}$ and $A\colon X \to \mathbb{K}$ an additive nonzero functional. Define $\phi(x, y) := A(x) \cdot A(y)$ for $x, y \in X$. Then, similarly every solution $f\colon X \to \mathbb{K}$ of (1.2) is of the form:

$$f(x) = \delta A(x) + 1, \quad x \in X$$

with some constant $\delta \in \mathbb{K}$. Further, f is indeed a solution if and only if $\delta = \pm 1$.

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