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# DERIVATION PAIRS ON RINGS AND RNGS

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**Abstract.** We generalize a classical result about derivation pairs on function algebras. Specifically, we describe the forms of derivation pairs on rings and rngs (non-unital rings) which are not assumed to be commutative. The proofs are based on knowledge of the solutions of the sine addition formula on a semigroup. Examples are given to illustrate the results.

## 1. Introduction

Motivation for this article comes from a classical result in the theory of function algebras. Let $\mathcal{A}$ be an associative unital algebra over the field $\mathbb{C}$ of complex numbers. A derivation pair on $\mathcal{A}$ is defined to be a pair of linear functionals $f, g\colon \mathcal{A} \to \mathbb{C}$ satisfying

$$f(xy) = f(x)g(y) + g(x)f(y), \quad x, y \in \mathcal{A}. \tag{1.1}$$

Let $\widehat{\mathcal{A}}$ denote the set of multiplicative linear functionals $m\colon \mathcal{A} \to \mathbb{C}$ such that $m \neq 0$. If $g \in \widehat{\mathcal{A}}$ and $(f, g)$ is a derivation pair, then $f$ is called a point derivation on $\mathcal{A}$ at $g$. The following result seems to have been found first by Glaeser [2] for commutative $\mathcal{A}$. It was later rediscovered by Zalcman [4], and Stetkær [3, Theorem 4.10] noted that the commutativity assumption on $\mathcal{A}$ can be deleted.

Proposition 1.1. *Let $\mathcal{A}$ be a complex associative unital algebra. Any pair $f, g$ of linear functionals satisfying* (1.2) *on $\mathcal{A}$ with $f \neq 0$ has one of the forms*
(a) $f = \gamma m$ *and* $g = m/2$,
(b) $f = \gamma(m_2 - m_1)$ *and* $g = (m_1 + m_2)/2$, *or*

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(c) *$f$ is a point derivation at $m$ and $g = m$,*
*where $\gamma \in \mathbb{C} \setminus \{0\}$ and $m, m_1, m_2 \in \widehat{\mathcal{A}}$ with $m_1 \neq m_2$.*

Our goal is to prove results similar to Proposition 1.1 on more general algebraic structures which need not possess a vector space structure.

$R$ is *rng* (or *non-unital ring*) if $(R, +)$ is an Abelian group, $(R, \cdot)$ is a semigroup, and multiplication distributes (on both sides) over addition. If $R$ has a multiplicative identity element then $R$ is a *ring*. We denote the multiplicative identity by 1 (or $1_R$ if there is more than one ring in the picture). We do not assume that $(R, \cdot)$ is commutative.

If $R$ and $R'$ are rngs, we say that a function $f\colon R \to R'$ is *additive* if $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. A function $f\colon R \to R'$ is *multiplicative* if $f(xy) = f(x)f(y)$ for all $x, y \in R$. If $f\colon R \to R'$ is both additive and multiplicative, then $f$ is a *rng homomorphism* (so we include the trivial rng homomorphism $f = 0$). Let $Hom(R, R')$ denote the set of all rng homomorphisms of $R$ into $R'$, and let $\widehat{Hom}(R, R')$ denote the set of non-trivial homomorphisms.

If $R$ and $R'$ are rings, then $f\colon R \to R'$ is a *ring homomorphism* provided $f$ is a rng homomorphism and $f(1_R) = 1_{R'}$. For consistency of notation we let $\widehat{Hom}(R, R')$ denote the set of all ring homomorphisms of $R$ into $R'$, since the zero mapping is excluded.

Let $f, g\colon R \to R'$ where $R, R'$ are rngs. Generalizing the definition used in the theory of function algebras, we say that $(f, g)$ is a *derivation pair* (from $R$ into $R'$) if $f$ is additive and

$$f(xy) = f(x)g(y) + g(x)f(y), \quad x, y \in R. \tag{1.2}$$

Obviously this equation is identical to (1.1) except for the domain and codomain of the functions. If $\phi \in Hom(R, R')$ and $(f, \phi)$ is a derivation pair, then we say that $f\colon R \to R'$ is a *point derivation* at $\phi$.

We observe that if $R'$ is a ring, $R$ is a sub-ring of $R'$, and $f$ is a point derivation (from $R$ into $R'$) at the identity function, then $f$ is simply a derivation from $R$ into $R'$.

Equation (1.2) is known in the functional equations literature as the sine addition formula on the semigroup $(R, \cdot)$. Here we use different terminology since $R$ also has an additive structure and $f$ is also assumed to be additive.

We do not assume that a rng is equipped with a vector space structure, nor do we assume anything about the function $g$ other than (1.2) (i.e. we do not assume that $g$ is additive). Our main results are Theorem 3.2 and Corollary 3.3, which generalize Proposition 1.1 to the setting of rngs and rings, respectively. We illustrate the application of these results with some examples on rngs and rings in the final section of the paper.

## 2. Preliminaries

If $X$ is a topological space and $R$ is a topological rng, let $C(X,R)$ denote the algebra of continuous functions mapping $X$ into $R$. Let $C(X) = C(X,\mathbb{C})$.

For any rng $R$ let $R^* := R \setminus \{0\}$.

For any semigroup $S$ define $S^2 := \{xy \mid x,y \in S\}$.

A *domain* is a rng in which $ab = 0$ implies that $a = 0$ or $b = 0$.

We use a basic result about the sine addition formula on a semigroup $S$. A function $m\colon S \to \mathbb{C}$ is multiplicative if $m(xy) = m(x)m(y)$ for all $x,y \in S$. The following result is a corollary of [3, Theorem 4.1].

Proposition 2.1. *Let $S$ be a semigroup, and suppose $f,g\colon S \to \mathbb{C}$ satisfy the sine addition formula*

$$f(xy) = f(x)g(y) + g(x)f(y), \quad x,y \in S,$$

*with $f \neq 0$. Then one of the following three cases holds, where $m, m_1, m_2\colon S \to \mathbb{C}$ are multiplicative functions with $m_1 \neq m_2$ and $m \neq 0$.*

(a) *There exists $\alpha \in \mathbb{C}^*$ such that $f = \alpha(m_1 - m_2)$ and $g = (m_1 + m_2)/2$.*

(b) *$g = m$ and $f$ is a (nonzero) solution of $f(xy) = f(x)m(y) + m(x)f(y)$ for all $x,y \in S$.*

(c) *$S \neq S^2$, $g = 0$, $f(xy) = 0$ for all $x,y \in S$, and there exists $x_0 \in S \setminus S^2$ such that $f(x_0) \neq 0$.*

*Cases* (a), (b), *and* (c) *are mutually exclusive.*

*Furthermore, if $S$ is a topological semigroup and $f \in C(S)$, then $g$, $m_1$, $m_2$, $m \in C(S)$.*

Note that case (c) cannot occur if $S$ has an identity element, since $S^2 = S$ in that event.

The form of $f$ in case (b) is described in detail in [1, Theorem 3.1], but the description is rather complicated and Proposition 2.1 is sufficient for the present needs.

## 3. Main results

We start with a simple lemma.

Lemma 3.1. *Let $R, R'$ be rngs, and suppose $f,g\colon R \to R'$ is derivation pair with $f \neq 0$. If $R'$ is a domain then $g$ is additive.*

PROOF. By (1.2), the additivity of $f$, and the distributive law in $R$, we have

$$\begin{aligned}\big(f(x)g(y)+g(x)f(y)\big)&+\big(f(x)g(z)+g(x)f(z)\big)\\ &=f(xy)+f(xz)=f(xy+xz)\\ &=f(x(y+z))\\ &=f(x)g(y+z)+g(x)f(y+z)\\ &=f(x)g(y+z)+g(x)\big(f(y)+f(z)\big)\end{aligned}$$

for all $x,y,z\in R$. Hence

$$f(x)\big(g(y)+g(z)-g(y+z)\big)=0,\quad x,y,z\in R.$$

Since $f\neq 0$ we see that $g$ is additive. □

Our first main result is the following.

THEOREM 3.2. *Let $R$ be a rng. Any derivation pair $(f,g)$ on $R$ into $\mathbb{C}$ with $f\neq 0$ has one of the forms below, where $\phi\in\widehat{Hom}(R,\mathbb{C})$ and $\phi_1,\phi_2\in Hom(R,\mathbb{C})$ with $\phi_1\neq\phi_2$.*

(i) *There exists $\gamma\in\mathbb{C}^*$ such that $f=\gamma(\phi_1-\phi_2)$ and $g=(\phi_1+\phi_2)/2$.*
(ii) *$g=\phi$ and $f$ is a (nonzero) point derivation at $\phi$.*
(iii) *For $R\neq R^2$ we have $g=0$, $f$ is a point derivation at $0$, and there exists $x_0\in R\setminus R^2$ such that $f(x_0)\neq 0$.*

*Conversely, in each case $(f,g)$ is a derivation pair with $f\neq 0$.*

*Cases* (i), (ii), *and* (iii) *are mutually exclusive.*

*Furthermore, if $R$ is a topological rng and $f\in C(R)$, then $g$, $\phi_1$, $\phi_2$, $\phi\in C(R)$.*

PROOF. Suppose $f,g\colon R\to\mathbb{C}$ is a derivation pair with $f\neq 0$. By Lemma 3.1 we see that $g$ is additive. Applying Proposition 2.1 on the semigroup $(R,\cdot)$, we have three cases to consider.

In case (a) we have $f=\gamma(m_1-m_2)$ and $g=(m_1+m_2)/2$ for some $\gamma\in\mathbb{C}^*$ and multiplicative functions $m_1,m_2\colon R\to\mathbb{C}$ with $m_1\neq m_2$. Since $f$ and $g$ are both additive we see that $m_1=g+f/(2\gamma)$ and $m_2=g-f/(2\gamma)$ are also additive. Therefore $m_1,m_2\in Hom(R,\mathbb{C})$. Defining $\phi_j:=m_j$ we have solution class (i).

In case (b) since $g$ is multiplicative, additive, and nonzero we have $g\in\widehat{Hom}(R,\mathbb{C})$. Thus we are in solution class (ii).

Case (c) immediately gives solution class (iii).

The converse is easily verified, and the mutual exclusivity and topological statements follow from Proposition 2.1. □

For rings we have the following corollary. (Recall that the zero map is not a ring homomorphism.)

Corollary 3.3. *Let $R$ be a ring. Any derivation pair $(f, g)$ on $R$ into $\mathbb{C}$ with $f \neq 0$ has one of the forms below, where $\phi, \phi_1, \phi_2 \in \widehat{Hom}(R, \mathbb{C})$ with $\phi_1 \neq \phi_2$, and $\gamma \in \mathbb{C}^*$.*
(a) *$f = \gamma(\phi_1 - \phi_2)$ and $g = (\phi_1 + \phi_2)/2$.*
(b) *$f = \gamma\phi$ and $g = \phi/2$.*
(c) *$f$ is a (nonzero) point derivation at $\phi$ and $g = \phi$.*
*Conversely, in each case $(f, g)$ is a derivation pair with $f \neq 0$.*
*The cases are mutually exclusive.*
*Moreover, if $R$ is a topological ring and $f \in C(R)$, then $g, \phi_1, \phi_2, \phi \in C(R)$.*

Proof. By Theorem 3.2 we have three solution classes to consider.
In class (i), if $\phi_1, \phi_2 \in \widehat{Hom}(R, \mathbb{C})$ then we are in case (a). If on the other hand one of $\phi_1, \phi_2$ is in $\widehat{Hom}(R, \mathbb{C})$ while the other one is 0, then we are in case (b).
Class (ii) carries over as our case (c).
Class (iii) is eliminated since $R$ has a multiplicative identity, thus $R = R^2$.
The rest follows from Theorem 3.2. □

The results above leave open the question of what forms the point derivations take. The answer to that question depends heavily on the rng or ring. For that reason we give some examples in the next section.

## 4. Examples

In this section we illustrate the application of Theorem 3.2 and Corollary 3.3 to some rngs and rings which are not covered by Proposition 1.1 since they are not algebras over $\mathbb{C}$.

Our first two examples deal with sub-rng $U$ and sub-ring $T$ of $M(2, \mathbb{Z})$, where

$$U := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}, \qquad T := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \tag{4.1}$$

under the usual addition and multiplication. The rng $U$ does not have a two-sided identity but it has left identity $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

First we identify the homomorphisms and point derivations of $U$ and $T$ into $\mathbb{C}$. If $f\colon T \to \mathbb{C}$, let $f|_U$ denote the restriction of $f$ to $U$.

Lemma 4.1. *Define* $h_1, h_2 \colon T \to \mathbb{C}$ *by*

$$h_1 \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := a, \qquad h_2 \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := c$$

*for all* $a, b, c \in \mathbb{Z}$. *We have the following.*

(a) $\widehat{Hom}(U, \mathbb{C}) = \{h_1|_U\}$.

(b) $\widehat{Hom}(T, \mathbb{C}) = \{h_1, h_2\}$.

(c) *If* $f \colon U \to \mathbb{C}$ *is a point derivation at* $h_1|_U$, *then* $f = 0$.

(d) *If* $f \colon T \to \mathbb{C}$ *is a point derivation at* $h_1$ *or* $h_2$, *then* $f = 0$.

(e) *If* $f \colon U \to \mathbb{C}$ *is a point derivation at* $0$, *then* $f = 0$.

Proof. We combine the proofs of (a) and (b). First suppose that $\phi \in \widehat{Hom}(T, \mathbb{C})$. Since $\phi$ is additive we have

$$\begin{aligned} \phi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \phi \left[ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \right] \\ &= a\phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= a\alpha + b\beta + c\gamma \end{aligned}$$

for all $a, b, c \in \mathbb{Z}$, where $\alpha, \beta, \gamma \in \mathbb{C}$. Then by multiplicativity we get for all $a, b, c, a', b', c' \in \mathbb{Z}$ that

$$\begin{aligned} aa'\alpha + (ab' + bc')\beta + cc'\gamma &= \phi \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix} \\ &= \phi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \phi \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \\ &= (a\alpha + b\beta + c\gamma)(a'\alpha + b'\beta + c'\gamma). \end{aligned}$$

It follows that $\alpha = \alpha^2$, $\alpha\gamma = 0$, $\beta = 0$, and $\gamma = \gamma^2$. Since $\phi \neq 0$ we have $(\alpha, \gamma) \in \{(1, 0), (0, 1)\}$, thus $\phi \in \{h_1, h_2\}$ and we have part (b). By restriction to $U$, the same calculations (with $c = c' = 0$ and $\gamma$ non-existent) prove part (a).

We also combine the proofs of (c) and (d). Let $f \colon T \to \mathbb{C}$ be a point derivation at $h_1$. Since $f$ is additive, we find as above that

$$f \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a\delta_1 + b\delta_2 + c\delta_3$$

for all $a, b, c \in \mathbb{Z}$, where $\delta_1, \delta_2, \delta_3 \in \mathbb{C}$. Using this form in (1.2) with $g = h_1$ we have

$$\begin{aligned} aa'\delta_1 + (ab' + bc')\delta_2 + cc'\delta_3 &= f\left[\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}\right] \\ &= f\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} a' + af\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \\ &= (a\delta_1 + b\delta_2 + c\delta_3)a' + a(a'\delta_1 + b'\delta_2 + c'\delta_3) \end{aligned}$$

for all $a, b, c, a', b', c' \in \mathbb{Z}$. It follows that $\delta_1 = \delta_2 = \delta_3 = 0$, so $f = 0$. By restriction to $U$, the appropriately modified calculations prove part (c).

Similar calculations show that the point derivation $f\colon T \to \mathbb{C}$ at $h_2$ is $f = 0$, thus we have part (d).

To prove (e) suppose $f\colon U \to \mathbb{C}$ is a point derivation at 0. By (1.2) with $g = 0$ and $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ we get

$$0 = f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot 0 + 0 \cdot f(y) = f\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} y\right] = f(y)$$

for all $y \in U$. □

With those preliminaries we now have the following.

Example 4.2. With $U$ as defined in (4.1) we get the forms of non-trivial (i.e. $f \neq 0$) derivation pairs on $U$ into $\mathbb{C}$ by using the results of Lemma 4.1(a),(c),(e) in Theorem 3.2. Classes (ii) and (iii) are eliminated since $f = 0$ there. Thus we are left with only class (i), which yields the solutions

$$f\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \gamma a, \quad g\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \frac{a}{2}$$

for all $a, b \in \mathbb{Z}$, where $\gamma \in \mathbb{C}^*$.

Example 4.3. With $T$ as defined in (4.1) we get the forms of non-trivial derivation pairs on $T$ into $\mathbb{C}$ by using the results of Lemma 4.1(b),(d) in Corollary 3.3. Class (c) is eliminated since $f = 0$ there. In class (a) we have the solutions

$$f\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \gamma(a - c), \quad g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \frac{1}{2}(a + c), \quad \text{for all } a, b, c \in \mathbb{Z}.$$

In class (b) the solutions have the form

$$f\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \gamma a, \quad g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \frac{1}{2}a, \quad \text{for all } a, b, c \in \mathbb{Z},$$

or

$$f\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \gamma c, \quad g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \frac{1}{2}c, \quad \text{for all } a, b, c \in \mathbb{Z}.$$

In each case $\gamma \in \mathbb{C}^*$.

Another rng example is the following. (It is not a ring since we exclude the empty word.) Consider the free $\mathbb{Z}$-module with basis $B$ consisting of all non-empty words over the alphabet $L = \{\ell_1, \dots, \ell_n\}$ for some positive integer $n$. This $\mathbb{Z}$-module becomes a $\mathbb{Z}$-algebra by defining the multiplication as follows. The product of two basis elements is defined by concatenation, for example $(\ell_1^2\ell_2)\cdot(\ell_2^2\ell_3\ell_1) = \ell_1^2\ell_2^3\ell_3\ell_1$. The product of two arbitrary $\mathbb{Z}$-module elements is then uniquely determined by the bilinearity of multiplication. Let $\mathbb{Z}\langle\ell_1, \dots, \ell_n\rangle$ denote the $\mathbb{Z}$-algebra (which is a rng) so defined. For any word $w \in B$ and letter $\ell \in L$, let $N_\ell(w)$ denote the number of times $\ell$ appears in $w$ counting multiplicity (so for $w = \ell_1^2\ell_2^3\ell_3\ell_1$ we have $N_{\ell_1}(w) = 3$).

In the example we choose $n = 2$ for simplicity, but it extends to a general positive integer $n$ in the obvious way.

Example 4.4. Let $L = \{p, q\}$ and let $\mathbb{Z}\langle p, q\rangle$ be the $\mathbb{Z}$-algebra defined as above. First we calculate the forms of rng homomorphisms from $\mathbb{Z}\langle p, q\rangle$ into $\mathbb{C}$. Each $\phi \in Hom(\mathbb{Z}\langle p, q\rangle, \mathbb{C})$ is uniquely determined by the values of $\phi(p)$ and $\phi(q)$, which can be chosen to be arbitrary complex numbers. Let $\alpha := \phi(p) \in \mathbb{C}$ and $\beta := \phi(q) \in \mathbb{C}$. Each element $x \in \mathbb{Z}\langle p, q\rangle$ has the form $x = \sum_{j=1}^{n} a_j w_j$ for some $n, a_1, \dots, a_n \in \mathbb{N}$ and basis elements $w_1, \dots, w_n \in B$. Then we have

$$\phi(x) = \sum_{j=1}^{n} a_j \phi(w_j) = \sum_{j=1}^{n} a_j \alpha^{N_p(w_j)} \beta^{N_q(w_j)}.$$

If $f$ is a point derivation at $\phi \in Hom(\mathbb{Z}\langle p, q\rangle, \mathbb{C})$, then $f$ is uniquely determined by the values of $f(p)$ and $f(q)$ (which are again arbitrary complex numbers), together with $\phi(p)$ and $\phi(q)$. This statement follows from (1.2) and the additivity of $f$. Let $\gamma := f(p), \delta := f(q) \in \mathbb{C}$. Then for any basis element $w = p_1 \cdots p_n$ (with $p_j \in \{p, q\}$ for each $j$) we get from (1.2) that

$$\begin{aligned} f(w) &= f(p_1 \cdots p_n) \\ &= f(p_1)\phi(p_2 \cdots p_n) + \phi(p_1) f(p_2 \cdots p_n) \\ &= f(p_1)\phi(p_2) \cdots \phi(p_n) + \phi(p_1)[f(p_2)\phi(p_3 \cdots p_n) + \phi(p_2) f(p_3 \cdots p_n)] \\ &= \cdots \\ &= \sum_{j=1}^{n} f(p_j) \prod_{i \in \{1,\dots,n\}\setminus\{j\}} \phi(p_i) \\ &= N_p(w)\gamma\alpha^{N_p(w)-1}\beta^{N_q(w)} + N_q(w)\delta\alpha^{N_p(w)}\beta^{N_q(w)-1}. \end{aligned}$$

Then for $x = \sum_{j=1}^{k} a_j w_j \in \mathbb{Z}\langle p, q\rangle$ we get by additivity that

$$f(x) = \sum_{j=1}^{k} a_j f(w_j),$$

where each $f(w_j)$ is computed as above.

For $\phi \neq 0$ these formulas for $f$ and $\phi$ yield the forms of derivation pairs from $\mathbb{Z}\langle p, q\rangle$ into $\mathbb{C}$ in classes (i) and (ii) of Theorem 3.2.

For $\phi = 0$ the formulas above yield $f(w) = 0$ for any basis element $w$ of length at least 2. So for a point derivation $f$ at 0 satisfying $f \neq 0$ we have

$$f(w) = \begin{cases} \gamma & \text{if } w = p \\ \delta & \text{if } w = q \\ 0 & \text{otherwise} \end{cases}$$

with $(\gamma, \delta) \neq (0, 0)$. This is the form of point derivations at 0 in class (iii) of Theorem 3.2.

For the next example we start with another lemma.

Lemma 4.5. *Consider the ring* $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$, *and let* $R'$ *be a ring containing* $\mathbb{Z}[\sqrt{2}]$. *Then there are exactly two elements* $h_1, h_2 \in Hom(\mathbb{Z}[\sqrt{2}], R')$, *namely*

$$h_1(a + b\sqrt{2}) := a + b\sqrt{2}, \quad \text{and} \quad h_2(a + b\sqrt{2}) := a - b\sqrt{2}, \qquad a, b \in \mathbb{Z}.$$

*Furthermore, if* $f\colon \mathbb{Z}[\sqrt{2}] \to R'$ *is a point derivation at either* $h_1$ *or* $h_2$, *then* $f = 0$.

Proof. Suppose $\phi \in \widehat{Hom}(\mathbb{Z}[\sqrt{2}], R')$. Defining $\gamma := \phi(\sqrt{2})$, we get by additivity that $\phi(a + b\sqrt{2}) = a\phi(1) + b\phi(\sqrt{2}) = a + b\gamma$ for all $a, b \in \mathbb{Z}$, since $\phi(1) = 1$. Since $\phi$ is multiplicative we have for all $a, b, c, d \in \mathbb{Z}$ that

$$\begin{aligned} ac + 2bd + (bc + ad)\gamma =& \phi((a + b\sqrt{2})(c + d\sqrt{2})) \\ =& \phi(a + b\sqrt{2})\phi(c + d\sqrt{2}) \\ =& (a + b\gamma)(c + d\gamma) \\ =& ac + (bc + ad)\gamma + bd\gamma^2. \end{aligned}$$

Thus $\gamma^2 = 2$, so we have $\phi \in \{h_1, h_2\}$.

Now suppose $f\colon \mathbb{Z}[\sqrt{2}] \to R'$ is a point derivation at $h_1$. By additivity we see that $f(a + b\sqrt{2}) = af(1) + bf(\sqrt{2}) = a\alpha + b\beta$ for all $a, b \in \mathbb{Z}$, where

$\alpha := f(1)$ and $\beta := f(\sqrt{2})$. Thus by (1.2) we have

$$\begin{aligned}(ac+2bd)\alpha+(bc+ad)\beta =& f((a+b\sqrt{2})(c+d\sqrt{2}))\\ =& f(a+b\sqrt{2})(c+d\sqrt{2})+(a+b\sqrt{2})f(c+d\sqrt{2})\\ =& (a\alpha+b\beta)(c+d\sqrt{2})+(a+b\sqrt{2})(c\alpha+d\beta)\\ =& 2ac\alpha+(bc+ad)(\beta+\alpha\sqrt{2})+2bd\beta\sqrt{2},\end{aligned}$$

for all $a,b,c,d \in \mathbb{Z}$. Therefore $\alpha=\beta=0$, so $f=0$. A similar calculation shows that 0 is the only point derivation at $h_2$. □

EXAMPLE 4.6. Let $R=\mathbb{Z}[\sqrt{2}]$. By Corollary 3.3 and Lemma 4.5 we find that the derivation pairs $(f,g)$ on $R$ into $\mathbb{C}$ with $f \neq 0$ have one of the following forms for all $a,b \in \mathbb{Z}$, where $\gamma \in \mathbb{C}^*$.

(a) $f(a+b\sqrt{2})=\gamma b$ and $g(a+b\sqrt{2})=a$.
(b) $f(a+b\sqrt{2})=\gamma(a+b\sqrt{2})$ and $g(a+b\sqrt{2})=\frac{1}{2}(a+b\sqrt{2})$.
(c) $f(a+b\sqrt{2})=\gamma(a-b\sqrt{2})$ and $g(a+b\sqrt{2})=\frac{1}{2}(a-b\sqrt{2})$.

Finally, let $\mathbb{Z}[X_1,\dots,X_n]$ denote the polynomial ring in indeterminates $X_1,\dots,X_n$ over $\mathbb{Z}$. A product of the form $X_1^{p_1}\cdots X_n^{p_n}$ with $p_1,\dots,p_n \in \mathbb{N}\cup\{0\}$ is called a *monomial*. Here we refer to the $n$-tuple $(p_1,\dots,p_n)$ as the *exponent vector*. A *polynomial* is a finite linear combination of monomials with coefficients in $\mathbb{Z}$.

As was the case with Example 4.4 the next result is stated for the case $n=2$, but it is easily extended to any positive integer $n$. (Here $0^0 := 1$ by convention.)

LEMMA 4.7. *Let* $R=\mathbb{Z}[X_1,X_2]$.

(i) $\phi \in \widehat{Hom}(R,\mathbb{C})$ *if and only if there exist* $\alpha,\beta \in \mathbb{C}$ *with* $(\alpha,\beta)\neq(0,0)$ *such that*

$$\phi\Big(\sum_{p\in I} c_p X_1^{p_1}X_2^{p_2}\Big)=\sum_{p\in I} c_p\alpha^{p_1}\beta^{p_2}$$

*for each nonempty finite set* $I$ *of exponent vectors* $p=(p_1,p_2)$ *and* $c_p \in \mathbb{Z}$. *Let* $h_{\alpha,\beta}$ *denote the homomorphism so defined.*

(ii) *If* $f\colon R\to\mathbb{C}$ *is a point derivation at* $h_{\alpha,\beta}$, *then there exist* $\gamma,\delta\in\mathbb{C}$ *such that*

$$f\Big(\sum_{p\in I} c_p X_1^{p_1}X_2^{p_2}\Big)=\sum_{p\in I} c_p\big(p_1\gamma\alpha^{p_1-1}\beta^{p_2}+p_2\delta\alpha^{p_1}\beta^{p_2-1}\big) \tag{4.2}$$

*for each nonempty finite set* $I$ *of exponent vectors* $p=(p_1,p_2)$ *and* $c_p\in\mathbb{Z}$. *Conversely, the function* $f_{\gamma,\delta,\alpha,\beta}$ *defined by* (4.2) *is a point derivation at* $h_{\alpha,\beta}$.

PROOF. Part (i) is straightforward, so we omit the proof.

To prove (ii) suppose $f$ is a point derivation at $g = h_{\alpha,\beta}$. We begin by finding the forms of $f(X_1^j)$ and $f(X_2^k)$. By (1.2) we have

$$\begin{aligned} f(X_1^j) &= f(X_1^{j-1})h_{\alpha,\beta}(X_1) + h_{\alpha,\beta}(X_1^{j-1})f(X_1) \\ &= f(X_1^{j-1})\alpha + \alpha^{j-1}f(X_1) \\ &= \big(f(X_1^{j-2})h_{\alpha,\beta}(X_1) + h_{\alpha,\beta}(X_1^{j-2})f(X_1)\big)\alpha + \alpha^{j-1}f(X_1) \\ &= f(X_1^{j-2})\alpha^2 + 2\alpha^{j-1}f(X_1) \\ &= \cdots \\ &= f(X_1)\alpha^{j-1} + (j-1)\alpha^{j-1}f(X_1) \\ &= \gamma j\alpha^{j-1}, \end{aligned}$$

where we have defined $\gamma := f(X_1) \in \mathbb{C}$. By a similar calculation we get $f(X_2^k) = \delta k\beta^{k-1}$, where $\delta := f(X_2) \in \mathbb{C}$.

Now by (1.2) we have

$$\begin{aligned} f(X_1^jX_2^k) &= f(X_1^j)h_{\alpha,\beta}(X_2^k) + h_{\alpha,\beta}(X_1^j)f(X_2^k) \\ &= j\gamma\alpha^{j-1}\beta^k + k\delta\alpha^j\beta^{k-1} \end{aligned}$$

for all $j, k \in \mathbb{N} \cup \{0\}$. By the additivity of $f$ we arrive at (4.2). □

Thus we have the following.

EXAMPLE 4.8. We get the forms of derivation pairs $(f, g)$ on the ring $\mathbb{Z}[X_1, X_2]$ into $\mathbb{C}$ by substituting the forms of homomorphisms and point derivations given in Lemma 4.7 into the formulas of Corollary 3.3.

It is interesting to note the strong similarity between the results in Examples 4.4 and 4.7, even though the former is a non-commutative rng (and not a ring) while the latter is a commutative ring. (In fact the results become isomorphic if we add the empty word to $\mathbb{Z}\langle p, q\rangle$ so that it becomes a ring.) The reason for this is that if either $f \in \widehat{Hom}(R, \mathbb{C})$ or $f$ is a point derivation at $\phi \in \widehat{Hom}(R, \mathbb{C})$, then $f$ is what is termed an *Abelian function*, meaning that $f(x_1 \cdots x_n) = f(x_{\pi(1)} \cdots x_{\pi(n)})$ for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in R$, and permutations $\pi$ on $\{1, \ldots, n\}$. This follows from (1.2), the definition of multiplicative function, and the commutativity of multiplication in the co-domain.

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