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NEW AND ORIGINAL INTEGRAL INEQUALITIES UNDER MONOTONICITY AND CONVEXITY ASSUMPTIONS

Christophe Chesneau

Abstract. This article examines integral inequalities dealing with functions of the form “a function raised to the power of another function” under varying monotonicity and convexity assumptions. First, we assess the validity of a referenced theorem on the subject. Specifically, we present a counterexample and identify a gap in its proof. We then propose an alternative version of the theorem with more flexible convexity assumptions. In addition, we establish new lower and upper bounds for the same integral using refined Hermite–Hadamard integral inequalities. A complementary variant is also discussed. Thus, our results fill gaps in the literature and extend existing results on integral inequalities under classical assumptions.

1. Introduction

Convex and concave functions are crucial in mathematics. The formal definition of these functions is given below.

Definition 1.1 (Convex and concave functions). Let $a, b \in \mathbb{R}$ with $a < b$ and $f\colon [a, b] \to \mathbb{R}$ be a function.

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- Convex function: We say that f is convex if and only if, for any $\epsilon \in [0,1]$ and $x, y \in [a,b]$, we have

$$f(\epsilon x + (1-\epsilon)y) \leq \epsilon f(x) + (1-\epsilon)f(y).$$

If f is twice differentiable, this inequality is equivalent to $f''(x) \geq 0$ for any $x \in [a,b]$.

- Concave function: We say that f is concave if and only if, for any $\epsilon \in [0,1]$ and $x, y \in [a,b]$, we have

$$\epsilon f(x) + (1-\epsilon)f(y) \leq f(\epsilon x + (1-\epsilon)y).$$

If f is twice differentiable, this inequality is equivalent to $f''(x) \leq 0$ for any $x \in [a,b]$.

Further details on convex and concave functions can be found in [2, 3, 7, 8, 9, 10, 11, 13, 14, 15, 19]. One of their interests is the derivation of sharp integral inequalities, which is the focus of this article. Two examples are the Jensen integral inequalities and the Hermite–Hadamard integral inequalities, as formally presented in the two theorems below.

THEOREM 1.2 (Jensen integral inequalities). *Let $a, b \in \mathbb{R}$ with $a < b$, and $f\colon [a,b] \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two functions.*

- *Convex part: If g is convex, then the following holds:*

$$g\left[\frac{1}{b-a}\int_a^b f(x)dx\right] \leq \frac{1}{b-a}\int_a^b g[f(x)]dx.$$

- *Concave part: If g is concave, then the following holds:*

$$\frac{1}{b-a}\int_a^b g[f(x)]dx \leq g\left[\frac{1}{b-a}\int_a^b f(x)dx\right].$$

THEOREM 1.3 (Hermite–Hadamard integral inequalities). *Let $a, b \in \mathbb{R}$ with $a < b$, and $f\colon [a,b] \to \mathbb{R}$ be a function.*

- *Convex part: If f is convex, then the following holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}[f(a)+f(b)].$$

- *Concave part: If f is concave, then the following holds:*

$$\frac{1}{2}[f(a)+f(b)] \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right).$$

These integral inequalities serve as fundamental tools in approximation theory, numerical analysis and optimization. The Hermite–Hadamard integral inequalities, in particular, have been studied extensively, leading to numerous generalizations, variants and refinements. Some of them can be found in [1, 4, 5, 6, 12, 16, 17, 18, 20, 21, 22, 23, 24]. We emphasize an original variant given by [22, Theorem 2.6], as recalled below.

THEOREM 1.4 ([22, Theorem 2.6]). *Let $a, b \in \mathbb{R}$ with $a < b$, and $f\colon [a,b] \to [0,+\infty)$ and $g\colon [a,b] \to [1,+\infty)$ be two functions. We suppose that f and $\log(g)$ are monotonic with an opposite monotonicity, f is convex and $\log(g)$ is convex. Then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \leq [g(a)g(b)]^{[f(a)+f(b)]/4}.$$

The contributions of this article are inspired by the framework of this theorem, which remains relatively unexplored in the existing literature. In the first part, we critically examine the validity of [22, Theorem 2.6] by presenting a counterexample and identifying a gap in the proof. This gap is closely related to a misapplication of the concave part of the Jensen integral inequalities. We then propose an alternative statement of this theorem under varying monotonicity and convexity assumptions. In the second part, we derive new and sharper lower and upper bounds for the main integral, i.e.,

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx,$$

while still relying on monotonicity and convexity assumptions. Our approach is based on a convex property of “a positive function raised to the power of another positive function”, i.e., g^f, and the use of refined Hermite–Hadamard integral inequalities as given in [21]. A variant considering “a positive function raised to the power of another minus positive function”, i.e., g^{-f}, is also proposed. By revisiting an existing theorem and providing sharper bounds, we fill a gap in the literature on convex-type integral inequalities and extend the scope of previous results.

The remainder of this article is structured as follows: In Section 2, we revisit [22, Theorem 2.6], analyzing its proof and limitations. Section 3 presents refined results and alternative inequalities. Finally, Section 4 concludes the article with a summary and discussion of potential future research directions.

2. Revisit of Theorem 1.4

2.1. A counterexample

A counterexample to [22, Theorem 2.6], as recalled in Theorem 1.4, is now elaborated. For simplicity, we take $a = 0$ and $b = 1$. We consider

$$f(x) = \exp(x), \quad x \in [0, 1],$$

which is obviously non-decreasing and convex. We also define

$$g(x) = 7 + \exp(-x), \quad x \in [0, 1],$$

satisfying $g(x) \geq 1$ for any $x \in [0, 1]$, with $\log[g(x)] = \log[7 + \exp(-x)]$, which is non-increasing because, for any $x \in [0, 1]$,

$$\{\log[g(x)]\}' = -\frac{1}{1 + 7\exp(x)} \leq 0,$$

and convex because, for any $x \in [0, 1]$,

$$\{\log[g(x)]\}'' = \frac{7\exp(x)}{[1 + 7\exp(x)]^2} \geq 0.$$

Note that f and $\log(g)$ are of opposite monotonicity. Let us now calculate the two main terms in the inequality of [22, Theorem 2.6]. By numerical integration, we find that

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx = \int_0^1 [7 + \exp(-x)]^{\exp(x)}dx \approx 51.786726.$$

On the other hand, we have

$$\begin{aligned}[g(a)g(b)]^{[f(a)+f(b)]/4} &= \{[7 + \exp(-0)][7 + \exp(-1)]\}^{[\exp(0)+\exp(1)]/4} \\ &\approx 44.232526.\end{aligned}$$

We thus obtain

$$[g(a)g(b)]^{[f(a)+f(b)]/4} \approx 44.232526 < 51.786726 \approx \frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx,$$

which contradicts the result in [22, Theorem 2.6]. In fact, the constant "7" in the definition of g was tuned for this.

On analysis, the first inequality step in the proof of this theorem states that

$$\log\left[\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right] \leq \frac{1}{b-a}\int_a^b \log\left\{[g(x)]^{f(x)}\right\}dx.$$

However, this is an incorrect application of the concave part of the Jensen integral inequalities to the concave function $\log(x)$, $x > 0$; a correct application of it would give the reverse inequality. While the subsequent inequality steps are derived correctly, they are based on this initial assumption, which is incorrect. This motivates the development of a corrected statement in the section below.

2.2. Corrected statement

A possible corrected and improved version of [22, Theorem 2.6], with more flexibility on the convexity assumptions, is given below. The proof mainly uses the concave part of the Jensen integral inequalities, the Chebyshev integral inequality for functions of the same monotonicity, and the concave and convex parts of the Hermite–Hadamard integral inequalities.

THEOREM 2.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and $f\colon [a,b] \to [0,+\infty)$ and $g\colon [a,b] \to [1,+\infty)$ be two functions. We suppose that f and $\log(g)$ are monotonic with the same monotonicity. Furthermore,*

(1) *if f and $\log(g)$ are concave, then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq [g(a)g(b)]^{[f(a)+f(b)]/4},$$

(2) *if f is concave and $\log(g)$ is convex, then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq \left[g\left(\frac{a+b}{2}\right)\right]^{[f(a)+f(b)]/2},$$

(3) *if f is convex and $\log(g)$ is concave, then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq [g(a)g(b)]^{f[(a+b)/2]/2},$$

(4) *if f and $\log(g)$ are convex, then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq \left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]}.$$

Proof of Theorem 2.1. The four points share the same mathematical foundation. To simplify the developments, we work with the logarithm of the main integral. Applying the concave part of the Jensen integral inequalities to the concave function $\log(x)$, $x > 0$, as recalled in Theorem 1.2, we have

$$\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\} \geq \frac{1}{b-a}\int_a^b \log\left\{[g(x)]^{f(x)}\right\}dx = \frac{1}{b-a}\int_a^b f(x)\log[g(x)]dx. \tag{2.1}$$

Since f and $\log(g)$ are of the same monotonicity, the Chebyshev integral inequality applied to f and $\log(g)$ reads as

$$\frac{1}{b-a}\int_a^b f(x)\log[g(x)]dx \geq \left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b \log[g(x)]dx\right]. \tag{2.2}$$

It follows from inequalities (2.1) and (2.2) that

$$\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\} \geq \left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b \log[g(x)]dx\right]. \tag{2.3}$$

Let us now distinguish the assumptions in the four distinct points.

1. Since f and $\log(g)$ are non-negative and concave, the left-hand side of the concave part of the Hermite–Hadamard integral inequalities applied to f and $\log(g)$, as recalled in Theorem 1.3, gives

$$\left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b \log[g(x)]dx\right] \geq \left[\frac{f(a)+f(b)}{2}\right]\left\{\frac{\log[g(a)]+\log[g(b)]}{2}\right\} = \left[\frac{f(a)+f(b)}{4}\right]\log[g(a)g(b)] = \log\left\{[g(a)g(b)]^{[f(a)+f(b)]/4}\right\}. \tag{2.4}$$

Combining inequalities (2.3) and (2.4), we obtain

$$\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\} \geq \log\left\{[g(a)g(b)]^{[f(a)+f(b)]/4}\right\},$$

so, by the non-decreasing property of the exponential function,

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq [g(a)g(b)]^{[f(a)+f(b)]/4}.$$

The point (1) is established.

2. Since f and $\log(g)$ are non-negative, f is concave and $\log(g)$ is convex, the left-hand sides of the concave and convex parts of the Hermite–Hadamard integral inequalities applied to f and $\log(g)$, respectively, as recalled in Theorem 1.3, give

$$\begin{aligned}(2.5)\quad &\left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b \log[g(x)]dx\right]\\ &\geq \left[\frac{f(a)+f(b)}{2}\right]\log\left[g\left(\frac{a+b}{2}\right)\right]=\log\left\{\left[g\left(\frac{a+b}{2}\right)\right]^{[f(a)+f(b)]/2}\right\}.\end{aligned}$$

It follows from inequalities (2.3) and (2.5) that

$$\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\}\geq \log\left\{\left[g\left(\frac{a+b}{2}\right)\right]^{[f(a)+f(b)]/2}\right\},$$

so that

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\geq \left[g\left(\frac{a+b}{2}\right)\right]^{[f(a)+f(b)]/2}.$$

The point (2) is proved.

3. Since f and $\log(g)$ are non-negative, f is convex and $\log(g)$ is concave, the left-hand sides of the convex and concave parts of the Hermite–Hadamard integral inequalities applied to f and $\log(g)$, respectively, as recalled in Theorem 1.3, give

$$\begin{aligned}&\left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b \log[g(x)]dx\right]\\ &\geq f\left(\frac{a+b}{2}\right)\left\{\frac{\log[g(a)]+\log[g(b)]}{2}\right\}\\ (2.6)\quad &=\frac{1}{2}f\left(\frac{a+b}{2}\right)\log[g(a)g(b)]=\log\left\{[g(a)g(b)]^{f[(a+b)/2]/2}\right\}.\end{aligned}$$

Combining inequalities (2.3) and (2.6), we get

$$\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\}\geq \log\left\{[g(a)g(b)]^{f[(a+b)/2]/2}\right\},$$

so that

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\geq [g(a)g(b)]^{f[(a+b)/2]/2}.$$

The point (3) is proved.

4. Since f and $\log(g)$ are non-negative and convex, the left-hand side of the convex part of the Hermite–Hadamard integral inequalities applied to f and $\log(g)$, as recalled in Theorem 1.3, gives

$$\begin{aligned} \left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b \log[g(x)]dx\right] &\geq f\left(\frac{a+b}{2}\right)\log\left[g\left(\frac{a+b}{2}\right)\right] \\ &= \log\left\{\left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]}\right\}. \end{aligned} \tag{2.7}$$

It follows from inequalities (2.3) and (2.7) that

$$\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\} \geq \log\left\{\left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]}\right\},$$

so that

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq \left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]}.$$

The point (4) is proved.

This ends the proof of Theorem 2.1. □

Note that the point (1) gives the same bound as in [22, Theorem 2.6], as recalled in Theorem 1.4, but is defined as a lower bound, and is subject to different monotonicity and convexity assumptions on f and $\log(g)$. To the best of our knowledge, the other points offer new integral inequalities in the literature. In a sense, these results rectify and complete [22, Theorem 2.6], while maintaining the same mathematical approach.

3. Additional contributions

3.1. New results

The theorem below refinds the lower bound of the point (4) of Theorem 2.1 under a different convexity assumption, with a new statement of a sharp upper bound. The proof is innovated by an intermediate convexity result and the use of the Hermite–Hadamard integral inequalities.

THEOREM 3.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and $f\colon [a, b] \to [1, +\infty)$ and $g\colon [a, b] \to [1, +\infty)$ be two two-times differentiable functions. We suppose that f and g are monotonic with the same monotonicity and convex. Then the following holds:*

$$\left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]} \leq \frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \leq \frac{[g(a)]^{f(a)} + [g(b)]^{f(b)}}{2}.$$

PROOF OF THEOREM 3.1. Using standard differentiation rules, for any $x \in [a, b]$, we have

$$\left\{[g(x)]^{f(x)}\right\}' = g(x)^{f(x)-1}\left\{g(x)f'(x)\log[g(x)] + f(x)g'(x)\right\}.$$

Similarly, with an appropriate factorization, for any $x \in [a, b]$, we obtain

$$\begin{aligned}
&\left\{[g(x)]^{f(x)}\right\}'' = \left[\left\{[g(x)]^{f(x)}\right\}'\right]' \\
&= \left[g(x)^{f(x)-1}\left\{g(x)f'(x)\log[g(x)] + f(x)g'(x)\right\}\right]' \\
&= g(x)^{f(x)-2}\Big\{g(x)\left[2f'(x)g'(x)\left\{f(x)\log[g(x)] + 1\right\} + f(x)g''(x)\right] \\
&\quad + [g(x)]^2\log[g(x)]\left\{f''(x) + [f'(x)]^2\log[g(x)]\right\} + [f(x)-1]f(x)[g'(x)]^2\Big\}.
\end{aligned}$$

Thanks to the assumptions made on f and g, all the terms in the sum are non-negative. Just note that, for any $x \in [a, b]$, $f(x) \geq 1$ implies that $f(x) - 1 \geq 0$, $g(x) \geq 1$ implies that $\log[g(x)] \geq 0$, the fact that f and g are of the same monotonicity implies that $f'(x)g'(x) \geq 0$, and the fact that f and g are convex implies that $f''(x) \geq 0$ and $g''(x) \geq 0$. So we have $\left\{[g(x)]^{f(x)}\right\}'' \geq 0$, which means that g^f is convex. It follows from the convex part of the Hermite–Hadamard integral inequalities applied to g^f that

$$\left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]} \leq \frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \leq \frac{[g(a)]^{f(a)} + [g(b)]^{f(b)}}{2}.$$

This concludes the proof of Theorem 3.1. □

Note that if $\log(g)$ is convex, then $g = \exp[\log(g)]$ is convex as a composite function of a convex function with a non-decreasing convex function. Therefore, the framework of this theorem is more flexible than that in the point (4) of Theorem 2.1. Furthermore, we emphasize the novelty of the upper bound, i.e.,

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \leq \frac{[g(a)]^{f(a)}+[g(b)]^{f(b)}}{2}.$$

The convexity approach used in the proof is also original, and will be reused in some refinements presented in the subsection below.

3.2. Refinements

The result below is a well-known improvement of the right-hand side of the Hermite–Hadamard integral inequalities. We refer to the work in [21], which gives a complete study of this.

THEOREM 3.2 ([21, Theorem 1]). *Let $a,b\in\mathbb{R}$ with $a<b$ and $f\colon [a,b]\to\mathbb{R}$ be a function.*

- *Convex part: If f is convex, then the following holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{4}[f(a)+f(b)]+\frac{1}{2}f\Big(\frac{a+b}{2}\Big).$$

- *Concave part: If f is concave, then the following holds:*

$$\frac{1}{4}[f(a)+f(b)]+\frac{1}{2}f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx.$$

The theorem below uses this result to refine the points (1), (2) and (3) of Theorem 2.1.

THEOREM 3.3. *Let $a,b\in\mathbb{R}$ with $a<b$, and $f\colon [a,b]\to[0,+\infty)$ and $g\colon [a,b]\to[1,+\infty)$ be two functions. We suppose that f and $\log(g)$ are monotonic with the same monotonicity. Furthermore,*

(1) *if f and $\log(g)$ are concave, then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq \left\{\Big[g(a)g(b)\Big]^{1/2} g\Big(\frac{a+b}{2}\Big)\right\}^{[f(a)+f(b)]/8+f[(a+b)/2]/4},$$

(2) *if f is concave and $\log(g)$ is convex, then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq \Big[g\Big(\frac{a+b}{2}\Big)\Big]^{[f(a)+f(b)]/4+f[(a+b)/2]/2},$$

(3) *if* f *is convex and* $\log(g)$ *is concave, then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq \left\{\Big[g(a)g(b)\Big]^{1/2} g\Big(\frac{a+b}{2}\Big)\right\}^{f[(a+b)/2]/2}.$$

Proof of Theorem 3.3. The first steps of the proof follow those of the proof of Theorem 2.1. In particular, inequality (2.3) ensures that

$$\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\} \geq \Big[\frac{1}{b-a}\int_a^b f(x)dx\Big]\Big[\frac{1}{b-a}\int_a^b \log[g(x)]dx\Big]. \tag{3.1}$$

Let us now distinguish the assumptions in the three distinct points.

1. Since f and $\log(g)$ are non-negative and concave, the concave part of Theorem 3.2 applied to f and $\log(g)$ gives

$$\begin{aligned}
&\Big[\frac{1}{b-a}\int_a^b f(x)dx\Big]\Big[\frac{1}{b-a}\int_a^b \log[g(x)]dx\Big]\\
&\quad\geq \Big[\frac{f(a)+f(b)}{4}+\frac{1}{2}f\Big(\frac{a+b}{2}\Big)\Big]\Big\{\frac{\log[g(a)]+\log[g(b)]}{4}\\
&\qquad+\frac{1}{2}\log\Big[g\Big(\frac{a+b}{2}\Big)\Big]\Big\}\\
&\quad= \Big[\frac{f(a)+f(b)}{8}+\frac{1}{4}f\Big(\frac{a+b}{2}\Big)\Big]\log\left\{\Big[g(a)g(b)\Big]^{1/2} g\Big(\frac{a+b}{2}\Big)\right\}\\
&\quad= \log\left[\left\{\Big[g(a)g(b)\Big]^{1/2} g\Big(\frac{a+b}{2}\Big)\right\}^{[f(a)+f(b)]/8+f[(a+b)/2]/4}\right]. \qquad (3.2)
\end{aligned}$$

It follows from inequalities (3.1) and (3.2) that

$$\begin{aligned}
&\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\}\\
&\qquad\geq \log\left[\left\{\Big[g(a)g(b)\Big]^{1/2} g\Big(\frac{a+b}{2}\Big)\right\}^{[f(a)+f(b)]/8+f[(a+b)/2]/4}\right],
\end{aligned}$$

so that

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq \left\{\Big[g(a)g(b)\Big]^{1/2} g\Big(\frac{a+b}{2}\Big)\right\}^{[f(a)+f(b)]/8+f[(a+b)/2]/4}.$$

The point (1) is established.

2. Since f and $\log(g)$ are non-negative, f is concave and $\log(g)$ is convex, the concave part of Theorem 3.2 applied to f and the left-hand sides of the convex part of the Hermite–Hadamard integral inequalities applied to $\log(g)$, as recalled in Theorem 1.3, give

$$\begin{aligned}\left[\frac{1}{b-a}\int_a^b f(x)dx\right]&\left[\frac{1}{b-a}\int_a^b \log[g(x)]dx\right]\\ &\geq \left\{\frac{1}{4}[f(a)+f(b)]+\frac{1}{2}f\left(\frac{a+b}{2}\right)\right\}\log\left[g\left(\frac{a+b}{2}\right)\right]\\ &= \log\left\{\left[g\left(\frac{a+b}{2}\right)\right]^{[f(a)+f(b)]/4+f[(a+b)/2]/2}\right\}. \qquad (3.3)\end{aligned}$$

It follows from inequalities (3.1) and (3.3) that

$$\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\} \geq \log\left\{\left[g\left(\frac{a+b}{2}\right)\right]^{[f(a)+f(b)]/4+f[(a+b)/2]/2}\right\},$$

so that

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq \left[g\left(\frac{a+b}{2}\right)\right]^{[f(a)+f(b)]/4+f[(a+b)/2]/2}.$$

The point (2) is proved.

3. Since f and $\log(g)$ are non-negative, f is convex and $\log(g)$ is concave, the left-hand sides of the convex part of the Hermite–Hadamard integral inequalities applied to f, as recalled in Theorem 1.3, and the concave part of Theorem 3.2 applied to $\log(g)$ give

$$\begin{aligned}\left[\frac{1}{b-a}\int_a^b f(x)dx\right]&\left[\frac{1}{b-a}\int_a^b \log[g(x)]dx\right]\\ &\geq f\left(\frac{a+b}{2}\right)\left\{\frac{\log[g(a)]+\log[g(b)]}{4}+\frac{1}{2}\log\left[g\left(\frac{a+b}{2}\right)\right]\right\}\\ &= \frac{1}{2}f\left(\frac{a+b}{2}\right)\log\left\{\left[g(a)g(b)\right]^{1/2}g\left(\frac{a+b}{2}\right)\right\}\\ &= \log\left[\left\{\left[g(a)g(b)\right]^{1/2}g\left(\frac{a+b}{2}\right)\right\}^{f[(a+b)/2]/2}\right]. \qquad (3.4)\end{aligned}$$

It follows from inequalities (3.1) and (3.4) that

$$\log\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\} \geq \log\left[\left\{\left[g(a)g(b)\right]^{1/2}g\left(\frac{a+b}{2}\right)\right\}^{f[(a+b)/2]/2}\right],$$

so that

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \geq \left\{\Big[g(a)g(b)\Big]^{1/2} g\Big(\frac{a+b}{2}\Big)\right\}^{f[(a+b)/2]/2}.$$

The point (3) is proved.
This ends the proof of Theorem 3.3. □

The theorem below uses Theorem 3.2 to refine the upper bound determined in Theorem 3.1.

THEOREM 3.4. *Let $a, b \in \mathbb{R}$ with $a < b$, and $f\colon [a,b] \to [1,+\infty)$ and $g\colon [a,b] \to [1,+\infty)$ be two two-times differentiable functions. We suppose that f and g are monotonic with the same monotonicity and convex. Then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \leq \frac{[g(a)]^{f(a)}+[g(b)]^{f(b)}}{4} + \frac{1}{2}\Big[g\Big(\frac{a+b}{2}\Big)\Big]^{f[(a+b)/2]}.$$

PROOF OF THEOREM 3.4. The first steps of the proof follow those of the proof of Theorem 3.1. In particular, the assumptions made imply that g^f is convex. It follows from the convex part of Theorem 3.2 applied to g^f that

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \leq \frac{[g(a)]^{f(a)}+[g(b)]^{f(b)}}{4} + \frac{1}{2}\Big[g\Big(\frac{a+b}{2}\Big)\Big]^{f[(a+b)/2]}.$$

This concludes the proof of Theorem 3.4. □

In the case where f and g are monotonic with the same monotonicity and convex, the upper bound obtained in this theorem is preferable to that in Theorem 3.1 because it is sharper, i.e., we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx &\leq \frac{[g(a)]^{f(a)}+[g(b)]^{f(b)}}{4} + \frac{1}{2}\Big[g\Big(\frac{a+b}{2}\Big)\Big]^{f[(a+b)/2]}\\ &\leq \frac{[g(a)]^{f(a)}+[g(b)]^{f(b)}}{2}.\end{aligned}$$

3.3. A variant

The result below presents an integral inequality result dealing with the function g^{-f}. It can be seen as a lower bound variant of Theorem 3.4.

THEOREM 3.5. *Let $a, b \in \mathbb{R}$ with $a < b$, and $f\colon [a,b] \to [1,+\infty)$ and $g\colon [a,b] \to [1,+\infty)$ be two two-times differentiable functions. We suppose that f and g are monotonic with the same monotonicity and convex. Then the following holds:*

$$\frac{1}{b-a}\int_a^b [g(x)]^{-f(x)}dx \geq \left\{\frac{[g(a)]^{f(a)}+[g(b)]^{f(b)}}{4}+\frac{1}{2}\left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]}\right\}^{-1}.$$

PROOF OF THEOREM 3.5. A suitable decomposition and an application of the Cauchy-Schwarz integral inequality give

$$\begin{aligned}1 &= \frac{1}{b-a}\int_a^b dx = \frac{1}{b-a}\int_a^b [g(x)]^{-f(x)/2}[g(x)]^{f(x)/2}dx\\ &\leq \frac{1}{b-a}\left\{\int_a^b [g(x)]^{-f(x)}dx\right\}^{1/2}\left\{\int_a^b [g(x)]^{f(x)}dx\right\}^{1/2}\\ &= \left\{\frac{1}{b-a}\int_a^b [g(x)]^{-f(x)}dx\right\}^{1/2}\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\}^{1/2},\end{aligned}$$

so that

$$\text{(3.5)} \qquad \frac{1}{b-a}\int_a^b [g(x)]^{-f(x)}dx \geq \left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\}^{-1}.$$

Note that this result can also be obtained by applying the convex part of the Jensen integral inequalities to the convex function $1/x$.

On the other hand, Theorem 3.4 ensures that

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx \leq \frac{[g(a)]^{f(a)}+[g(b)]^{f(b)}}{4}+\frac{1}{2}\left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]}.$$

This implies that

$$\begin{aligned}\text{(3.6)} \quad &\left\{\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx\right\}^{-1}\\ &\qquad \geq \left\{\frac{[g(a)]^{f(a)}+[g(b)]^{f(b)}}{4}+\frac{1}{2}\left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]}\right\}^{-1}.\end{aligned}$$

It follows from inequalities (3.5) and (3.6) that

$$\frac{1}{b-a}\int_a^b [g(x)]^{-f(x)}dx \geq \left\{\frac{[g(a)]^{f(a)}+[g(b)]^{f(b)}}{4}+\frac{1}{2}\left[g\left(\frac{a+b}{2}\right)\right]^{f[(a+b)/2]}\right\}^{-1}.$$

This ends the proof of Theorem 3.5. □

The lower bound obtained has an original form that does not correspond to that of any existing general integral inequality. To the best of our knowledge, it is new to the literature.

Other possible variants can be obtained by using different techniques. For example, we can think of using the Bernoulli inequality. More specifically,

- if $f\colon [a,b]\to[1,+\infty)$ and $g\colon [a,b]\to[1,+\infty)$, the Bernoulli inequality gives, for any $x\in[a,b]$,

$$[g(x)]^{f(x)}=[1+g(x)-1]^{f(x)}\geq 1+f(x)[g(x)-1]=1-f(x)+f(x)g(x),$$

- if $f\colon [a,b]\to[0,1]$ and $g\colon [a,b]\to[1,+\infty)$, the Bernoulli inequality gives, for any $x\in[a,b]$,

$$[g(x)]^{f(x)}=[1+g(x)-1]^{f(x)}\leq 1+f(x)[g(x)-1]=1-f(x)+f(x)g(x).$$

However, these bounds are independent of the power nature of the function g^f. While this approach enables us to relax certain assumptions regarding f and g, leading to new integral inequalities, it results in a loss of an important degree of sharpness. For this reason, we have decided not to pursue this direction any further.

4. Conclusion

In this article, we have critically examined the validity of [22, Theorem 2.6], focusing on an upper bound for an integral of the form

$$\frac{1}{b-a}\int_a^b [g(x)]^{f(x)}dx.$$

In particular, we presented a counterexample and identified the gap in its proof, which results from a misapplication of the concave part of the Jensen integral inequalities. We then proposed a corrected version of the theorem under more appropriate convexity assumptions. We also established new lower and upper bounds for the main integral using an original convex property and refined Hermite–Hadamard integral inequalities. A variant based on the function g^{-f} is also demonstrated. With these results, we contribute to the development of integral inequalities with potential applications in various fields, such as approximation theory, numerical analysis and optimization. We also provide some techniques that may be useful for future research on the topic, beyond the scope of the article.

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Department of Mathematics, LMNO
University of Caen-Normandie
14032 Caen
France
e-mail: christophe.chesneau@gmail.com