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ALTERNATIVE FUNCTIONAL EQUATIONS. A SURVEY

Gian Luigi Forti

Dedicated to the memory of Professors Roman Ger, Luigi Paganoni and Jürg Rätz

Abstract. This paper provides a survey of several results on alternative functional equations, related to the Cauchy equation and to the quadratic equation, and involving one or more unknown functions.

1. Introduction

A general functional equation in a single unknown function can be written in a short form as

$$\mathcal{F}(f) = \mathcal{G}(f),$$

where the variables or the domain of the function f are not explicitly indicated. The function f and all other functions appearing in $\mathcal{F}$ and $\mathcal{G}$ are defined on a set X, which can have various algebraic and topological structures, and the range of f and that of $\mathcal{F}(f)$ and $\mathcal{G}(f)$ is another set Y with its possible structures.

Supposing that X is in some sense the *natural* domain of f, in many problems which can be modelled by a functional equation, some constraints on the domain appear. Thus, the functional equation $\mathcal{F}(f) = \mathcal{G}(f)$ on X becomes

$$\mathcal{F}(f) = \mathcal{G}(f) \quad \text{on} \quad S \subset X$$

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for a certain subset S. In this way we obtain what is commonly called a *functional equation on a restricted domain* or a *conditional functional equation.* The condition, that is the set S, can have various forms: S can be given explicitly, without any relation to the unknown function f, or it can be given implicitly, depending on the unknown function itself. An ample treatment of these problems can be found in the two papers [16] and [17] by Jean Dhombres and Roman Ger.

As a simple but important example we show here the Mikusiński functional equation:

$$f(x+y)[f(x+y)-f(x)-f(y)]=0, \tag{1.1}$$

where $x, y \in G$, $(G,+)$ is an abelian group, and $f\colon G \to \mathbb{K}$, $\mathbb{K}$ being the real or complex field. This equation is a conditional Cauchy equation, the condition being $f(x+y) \neq 0$. Instead of writing equation (1.1) as a product, one can write

$$f(x+y) \neq 0 \Rightarrow f(x+y)-f(x)-f(y)=0, \tag{1.2}$$

and this form has a meaning also when the range of f is in a group. The form (1.2) of Mikusiński's equation can be called more expressively *alternative equation*: either the first condition or equation is true (in the following we will use the name *equation* also for conditions like that before) or the second one is true.

This paper deals with the alternative functional equations, which can be written as

$$\mathcal{F}_1(f_1) \neq \mathcal{G}_1(f_1) \Rightarrow \mathcal{F}_2(f_2) = \mathcal{G}_2(f_2).$$

Here we can have $f_1 = f_2$, as for the Mikusiński's equation, or $f_1 \neq f_2$. Obviously in the first case the two functional equations must be different.

Of course there is no reason for considering only two equations. Thus we are led to the following definition:

Consider a finite number N of functional equations $\mathcal{F}_i(f_i) = \mathcal{G}_i(f_i)$, $i = 1, 2, \dots, N$, with the condition that if $f_i = f_j$, then the i-th equation is different from the j-th one. The alternative equation generated by them is

$$\mathcal{F}_1(f_1) \neq \mathcal{G}_1(f_1) \Rightarrow \Big[\mathcal{F}_2(f_2) \neq \mathcal{G}_2(f_2) \Rightarrow [\cdots \Rightarrow [\mathcal{F}_N(f_N) = \mathcal{G}_N(f_N)]\Big].$$

If the set where $\mathcal{F}_i(f_i)$ and $\mathcal{G}_i(f_i)$ are taking values admits the possibility of a difference and consequently has a zero element, we can write $\mathcal{E}_i(f_i) = \mathcal{F}_i(f_i) - \mathcal{G}_i(f_i)$ and the alternative equation becomes

$$\mathcal{E}_1(f_1) \neq 0 \Rightarrow \Big[\mathcal{E}_2(f_2) \neq 0 \Rightarrow [\cdots \Rightarrow [\mathcal{E}_N(f_N) = 0]\Big]. \tag{1.3}$$

Moreover, if the set of values is an integral domain, equation (1.3) can be written in the form

$$\prod_{i=1}^{N} \mathcal{E}_i(f_i) = 0.$$

In the following we mainly consider the simplest case $N = 2$ (some exceptions will appear). The paper is divided in two main parts: $\mathcal{E}_1(f) \cdot \mathcal{E}_2(f) = 0$ and $\mathcal{E}_1(f_1) \cdot \mathcal{E}_2(f_2) = 0$, that is only one unknown function is involved, or two different functions. In the latter case we can have $\mathcal{E}_1 = \mathcal{E}_2$.

In this survey only results are reported, without proofs. Moreover, we will always use the additive notation for groups also when they are not abelian.

The starting point of this paper are the survey paper published by Marek Kuczma in 1978 ([45]) and the books of Marek Kuczma ([46]) and of János Aczél and Jean Dhombres ([1]), they are the source of inspiration of the present survey.

We finish this Introduction with two stability theorems, one for the Cauchy equation and the other for the quadratic equation, which will be quoted several times in the following.

Definition 1.1. Let $(G, +)$ be a group and B a Banach space. We say that the pair (G, B) has the *property of stability of homomorphisms* in the sense of Ulam-Hyers if for every function $f\colon G \to B$ such that

$$\|f(x+y) - f(x) - f(y)\| \le K$$

for every $x, y \in G$ and for some K, there exist $g \in \mathrm{Hom}(G, B)$ and K' depending only on K such that

$$\|f(x) - g(x)\| \le K'$$

for all $x \in G$.

Analogously, we say that the pair (G, B) has the *property of stability of the quadratic functional equation* if for every function $f\colon G \to B$ such that

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le K$$

for every $x, y \in G$ and for some K, there exists a quadratic function $q\colon G \to B$ and K' depending only on K such that

$$\|f(x) - q(x)\| \le K'$$

for all $x \in G$.

Theorem 1.2 ([23], [25]). *Let B and H be two Banach spaces. Then the pair (G, B) has the property of stability of homomorphisms or of the quadratic equation if and only if (G, H) has the same property.*

Note that the pair (G,B) has the property of stability of homomorphisms or of the quadratic equation if G is a commutative group or an amenable group, for other possibilities see [27].

THEOREM 1.3 ([23]). *Suppose that the pair (G,B) has the property of stability of homomorphisms and let*

$$f(x+y)-f(x)-f(y)\in M,$$

where M is a bounded subset of the Banach space B. If g is the only homomorphism of G in B such that $f(x)-g(x)$ is bounded, then $g(x)\in\overline{C(-M)}$, where $C(-M)$ is the convex hull of the set $-M$.

THEOREM 1.4 ([25], [67]). *Suppose that the pair (G,B) has the property of stability of the quadratic equation and let*

$$f(x+y)+f(x-y)-2f(x)-2f(y)\in M,$$

where M is a bounded subset of the Banach space B. If q is the only quadratic function such that $f(x)-q(x)$ is bounded, then $q(x)\in\frac{1}{2}\overline{C(-M)}$. More precisely, $q(x)\in\Big\{-\sum_{i=1}^{\infty}\frac{m_i}{4^i}-\frac{m_0}{6}:m_i\in M, i\geq 1, m_0=-2q(0)\Big\}$.

2. Alternative equations with one unknown function

2.1. Mikusiński's equation

The first equation to be treated is the already mentioned Mikusiński's equation. This equation arises from the problem of determining the self-maps of the plane which preserve collineations (see [45] for a detailed description). This equation in the form

$$f(x+y)\neq 0\Rightarrow f(x+y)-f(x)-f(y)=0 \tag{2.1}$$

when $f\colon G\to H$, $(G,+)$ and $(H,+)$ groups, has been solved by L. Dubikajtis, C. Ferenc, R. Ger and M. Kuczma in [18]. Their result is the following:

THEOREM 2.1. *Let $(G,+)$ and $(H,+)$ be groups (not necessarily commutative). A function $f\colon G\to H$ satisfies the equation (2.1) if and only if either it is additive (i.e. $f(x+y)-f(x)-f(y)=0$ for all $x,y\in G$), or it is of the form*

$$f(x)=\begin{cases}0, & x\in Z,\\ c, & x\in G\setminus Z,\end{cases}$$

where Z is a normal subgroup of G of index 2, and $c\in H$ is an arbitrary constant.

Hence, when the group G has no normal subgroups of index 2, equation (2.1) is equivalent to the additive Cauchy equation. Indeed if $f(x+y)=0$ for all $x,y\in G$, taking $y=0$ we see that f is identically zero, so it is additive, otherwise

$$f(x+y)=f(x)+f(y).$$

We say in this case that (2.1) has only *trivial* solutions.

K. Lajkó and Zs. Páles in [47] investigated the equation

$$f\Big(\frac{x+y}{2}\Big)\neq 0 \Rightarrow 2f\Big(\frac{x+y}{2}\Big)-f(x)-f(y)=0, \tag{2.2}$$

that they called Mikusiński-Jensen equation, where I is a real open interval, $x,y\in I$ and $f\colon I\to\mathbb{R}$. They proved that equation (2.2) is equivalent to Jensen equation

$$2f\Big(\frac{x+y}{2}\Big)=f(x)+f(y),$$

i.e., it has only trivial solutions.

2.2. Alternative equations related to the Cauchy equation

Mikusiński's equation is a type of Cauchy alternative equation. A general alternative Cauchy equation can be expressed in the following form

$$\varphi\big(f(x),f(y),f(x+y)\big)\neq 0 \Rightarrow f(x+y)-f(x)-f(y)=0, \tag{2.3}$$

where $f\colon G\to R$, $(G,+)$ is a group, $(R,+,\cdot)$ is an integral domain, and $\varphi\colon R^3\to R$ is a given function. In the case of Mikusiński's equation we have $\varphi(u,v,w)=w$, and in this case it is sufficient that $\varphi\colon H^3\to H$, where H is a group.

The case $\varphi(u,v,w)=w-au-bv$, that is the equation

$$f(x+y)-af(x)-bf(y)\neq 0 \Rightarrow f(x+y)-f(x)-f(y)=0, \tag{2.4}$$

where $a,b\in R$ are constants, was solved in [42]:

THEOREM 2.2. *Let G be a commutative group and R an integral domain of characteristic zero. A function $f\colon G\to R$ satisfies equation* (2.4) *if and only if either it is additive on G, or $a+b=1$ (1 is the unit if R) and $f=const$, or $a+b=0$ and*

$$f(x)=\begin{cases}0, & x\in Z,\\ c, & x\in G\setminus Z,\end{cases}$$

where Z is a subgroup of G of index 2, or $a+b=-1$ and f has the form

$$f(x)=\begin{cases} 0, & x\in Z,\\ c, & x\in Z_1,\\ -c, & x\in Z_2,\end{cases}$$

where Z is a subgroup of G of index 3, and Z_1, Z_2 are the cosets of Z in G. The constant c is arbitrary.

R. Ger in [40] and [41] solved equation (2.3) when $\varphi\colon R^3 \to R$ is an arbitrary centroaffine function, that is the alternative equation

$$cf(x+y)+af(x)+bf(y)\neq 0 \Rightarrow f(x+y)-f(x)-f(y)=0, \tag{2.5}$$

where $f\colon G\to R$, G and R as before, and $a,b,c\in R$.

Ger's result is the following:

THEOREM 2.3. *Let $(G,+)$ be a commutative group and $(R,+,\cdot)$ an integral domain and let $f\colon G\to R$ be a solution of equation (2.5). The following cases are the only possible ones.*

(i) *$a=b=c=0$ and f arbitrary;*
(ii) *a,b,c arbitrary and f additive;*
(iii) *$c-a-b=0$ and f constant;*
(iv) *$b=-a$, $c=0$ and f has the form*

$$f(x)=\begin{cases} 0, & x\in Z,\\ \alpha, & x\in G\setminus Z,\end{cases}$$

where Z is a subgroup of G of index greater than 2 and $\alpha\in R\setminus\{0\}$;

(v) char $R=2$, *$b\neq -a$, $c=a-b$, and f is as in (iv);*
(vi) *$b=-a$, c arbitrary and f is as in (iv) with Z of index 2;*
(vii) *$a=b=0$, $c\neq 0$ and f is as in (vi);*
(viii) *$c=a+b$ and f has the form*

$$f(x)=\begin{cases} 0, & x\in Z,\\ \alpha, & x\in Z_1,\\ -\alpha, & x\in Z_2,\end{cases}$$

where Z is a subgroup of G of index 3, and Z_1, Z_2 are the cosets of Z in G and $2\alpha\neq 0$.

If the function φ is affine but not centroaffine, the method used by Ger is not working, in particular when $\varphi(u,v,w)=w-u-v-1$ and R is the real line $\mathbb{R}$. Actually the behaviour of the solutions of the equation

$$f(x+y)-f(x)-f(y)-1\neq 0 \Rightarrow f(x+y)-f(x)-f(y)=0, \tag{2.6}$$

when $f\colon \mathbb{R} \to \mathbb{R}$ is different from the previous case: the function $f(x) = [x]$ ($[\cdot]$ is the greatest integer less then or equal to x) is a non-additive solution of (2.6) whose range is infinite, while the non-additive solutions of equation (2.5) have range of cardinality at most 3.

Equation (2.6) has been investigated in the paper [22] in a rather more general formulation. Let (G, B), $(G, +)$ a group and B a Banach space, be a pair with the property of the stability of homomorphisms. The following alternative equation has been investigated:

$$f(x+y) - f(x) - f(y) - b \neq 0 \Rightarrow f(x+y) - f(x) - f(y) = 0, \tag{2.7}$$

where $f\colon G \to B$, $b \in B$, $b \neq 0$, and $\|b\| = 1$. By Theorems 1.2 and 1.3 we can reduce the problem to the case $B = \mathbb{R}$, $b = 1$ and the range of f in the interval $[-1, 0]$. Hence the problem to be solved is

$$f(x+y) - f(x) - f(y) - 1 \neq 0 \Rightarrow f(x+y) - f(x) - f(y) = 0, \tag{2.8}$$

with $f\colon G \to [-1, 0]$. The result is given by the following theorem:

Theorem 2.4. *The function $f\colon G \to [-1, 0]$ is a solution of equation* (2.8) *if and only if*

$$f(x) = \begin{cases} 0, & x \in S_0, \\ \sigma \circ \phi \circ \pi(x), & x \in G \setminus (S_0 \cup S_1), \\ -1, & x \in S_1, \end{cases}$$

where

(i) *S_0 is either empty or is a semigroup, S_1 is either empty or is a semigroup, $S_0 \cap S_1 = \emptyset$, $S_0 \cup S_1 \neq \emptyset$, and $H = S_0 \cup S_1$ is a normal subgroup of G and $\pi\colon G \to G/H$ is the natural homomorphism;*

(ii) *$\phi\colon G/H \to \mathbb{R}/\mathbb{Z}$ is an injective homomorphism;*

(iii) *$\sigma\colon (\mathbb{R}/\mathbb{Z}) \setminus \{0\} \to \mathbb{R}$ is the only lifting such that for $x \notin H$, $-1 < \sigma \circ \phi \circ \pi(x) < 0$.*

Clearly, by adding any additive map from G into $\mathbb{R}$ we have all solutions without the restriction on the range. Among the solutions of equation (2.7) there are those which are constant and taking only the values -1 and 0. Then the following theorem holds:

Theorem 2.5. *Equation* (2.7) *has solutions different from $f(x) \equiv -1$ and $f(x) \equiv 0$ if and only if G has a proper normal subgroup H such that G/H is isomorphic to a subgroup of $\mathbb{R}/\mathbb{Z}$.*

A natural question arises: what happens if the range is not in a Banach space, that is if the stability results cannot be used? An answer was given by L. Paganoni in [51] for the case $f\colon \mathbb{Z} \to \mathbb{Z}$ (and can be obtained from Theorem 2.5).

Theorem 2.6. *A function $f\colon \mathbb{Z} \to \mathbb{Z}$ is a solution of equation* (2.6) *with $f(0) = 0$ if and only if it has one of the following forms:*

(i) $f(n) = [\alpha n]$, *for some $\alpha \in \mathbb{R}$;*
(ii) $f(n) = [\alpha n] - \chi_{(q\mathbb{N})}(n)$, *with $\alpha = p/q$, $(p, q) = 1$, $q > 0$;*
(iii) $f(n) = [\alpha n] - \chi_{(-q\mathbb{N})}(n)$, *with $\alpha = p/q$, $(p, q) = 1$, $q > 0$;*
where χ_A is the characteristic function of the set A.

For the case $f(0) = -1$, consider the function $g(n) = -1 - f(n)$.

A further step is considering equation (2.6) in this form

$$f(x + y) - f(x) - f(y) - a \neq 0 \Rightarrow f(x + y) - f(x) - f(y) = 0, \tag{2.9}$$

where $f\colon G \to H$, $(G, +)$ and $(H, +)$ are commutative groups, $a \in H$, $a \neq 0$, and $f(0) = 0$. Equation (2.9) has been investigated in [31], a fundamental tool is the following theorem of Kulikov (see [39, Corollary III.18.4]):

Theorem 2.7. *Every abelian group A is the union of an ascending chain $A_1 \leq \cdots \leq A_n \leq \cdots$ of subgroups, where every A_n is the direct sum of cyclic groups.*

From the following theorem

Theorem 2.8. *Let G be an abelian group which is the union of an ascending chain $\{A_n\}$. The function $f\colon G \to H$ is a solution of equation* (2.9) *if and only if for every $x \in G$, $f(x) = \lim_{n\to\infty} f_n(x)$, where $f_n\colon A_n \to H$ are solutions of* (2.9) *satisfying the compatibility condition $f_n(x) = f_{n-1}(x)$, $n = 2, \ldots$, for every $x \in A_{n-1}$.*

and Kulikov's theorem, it is possible to assume that f is defined on a group G which is the direct sum of cyclic groups , that is $G = \bigoplus_{j\in J} A_j$, where all A_j are cyclic groups. If $J_0 \subset J$ is the set of indices j for which A_j is a finite cyclic group of order m_j, then G is isomorphic to

$$\Big(\bigoplus_{j\in J\setminus J_0} \mathbb{Z}\Big) \oplus \Big(\bigoplus_{j\in J_0} \mathbb{Z}_{m_j}\Big).$$

The next step proves that it is possible to suppose that all cyclic groups are infinite and H is the subgroup generated by a, thus we obtain the following situation: either

$$f\colon \bigoplus_{j\in J} \mathbb{Z} \to \mathbb{Z}, \tag{2.10}$$

or

$$f\colon \bigoplus_{j\in J} \mathbb{Z} \to \mathbb{Z}_m, \tag{2.11}$$

for a certain $m \in \mathbb{N}$, and $f(\psi_j(1)) = 0$ for every $j \in J$, where $\psi_j : \mathbb{Z} \to \bigoplus_{j\in J} \mathbb{Z}$ is the natural injection from the j-th factor $\mathbb{Z}$.

We denote with $S(J)$ and $S(J;m)$ the classes of solutions in cases (2.10) and (2.11) respectively; note that $S(J;2)$ consists of *all* functions $f\colon \bigoplus_{j\in J}\mathbb{Z}\to \mathbb{Z}_2$, hence from now on we assume $m\geq 3$. Moreover, $f\in S(J;m)$ if and only if there exists a function $g\in S(J)$ such that $f=\pi_m\circ g$, where π_m is the natural homomorphism $\pi_m\colon \mathbb{Z}\to\mathbb{Z}_m$; the function g is uniquely determined.

After these steps, the following theorem has been proved:

THEOREM 2.9. *Let G be an abelian torsion-free group. The function $f\colon G\to \mathbb{Z}$ is a solution of equation* (2.9) *with $a=1$, if and only if $f=[\varphi]-\chi_U$, where:*

(i) $\varphi\in \mathrm{Hom}(G,\mathbb{R})$;
(ii) *either $U=\emptyset$ or $U\subset \varphi^{-1}(\mathbb{Z})\setminus\{0\}$ is a subsemigroup of G such that $x\in\varphi^{-1}(\mathbb{Z})\setminus U$ and $y\in\varphi^{-1}(\mathbb{Z})$ and $x+y\in U$ imply $y\in U$.*

In the same paper explicit solutions are then given , when $G=\mathbb{Z}_r$, $G=\mathbb{Q}$ and $G=\mathbb{Z}_{p^\infty}$. Clearly the problem is to find or describe the semigroups U with the properties stated in Theorem 2.9. This problem was attacked in the paper [9]. Suppose that $|J|=n$, and let $\varphi\in \mathrm{Hom}(\bigoplus_{j\in J}\mathbb{Z},\mathbb{R})$ such that $\varphi^{-1}(\mathbb{Z})\setminus\{0\}\neq\emptyset$, and denote by $P(\varphi)$ the set of all semigroups contained in $\varphi^{-1}(\mathbb{Z})$, satisfying (ii) of Theorem 2.9. The following theorem holds true (see [9]):

THEOREM 2.10. *Assume $|J|=n$. Then $U\in P(\varphi)$ if and only if there exists a finite number $k\leq n$ of non-zero linear functionals $\ell_1,\dots,\ell_k$ with ℓ_1 defined on $\mathbb{R}^n$ and $D(\ell_{i+1})=N(\ell_i)$, $i=1,\dots,k-1$, such that*

$$U=\{u\in\varphi^{-1}(\mathbb{Z})\colon \ell_i(u)>0,\quad \textit{for some}\quad i=1,\dots,k\}$$

($D(\cdot)$ and $N(\cdot)$ are the domain and the kernel of $\cdot$, respectively).

If φ is injective, the following holds (without any restriction on the cardinality of J):

THEOREM 2.11. *If $\varphi\in\mathrm{Hom}(\bigoplus_{j\in J}\mathbb{Z},\mathbb{R})$ is injective, then $P(\varphi)=\{\varphi^{-1}(\mathbb{N}), \varphi^{-1}(-\mathbb{N})\}$.*

To conclude the study of the problem if φ is affine but not centroaffine, the remaining case is $\varphi(u,v,w)=cw-au-bv-d$. To present the result proved in [21] we introduce the following notations:

- S denotes the set of solutions of the equation

$$[g(x+y)-g(x)-g(y)-d][g(x+y)-g(x)-g(y)]=0,$$

where $g\colon G\to D$, $(G,+)$ abelian group and $(D,+,.)$ domain of integrity, $d\in D\setminus\{0\}$ (see [22]);

- M denotes the set of functions of the form

$$f(x) = \begin{cases} 0, & x \in Z, \\ \alpha \neq 0, & x \in G \setminus Z. \end{cases}$$

The following has been proved:

THEOREM 2.12. *Let $f\colon G \to D$ be a solution of*

$$[cf(x+y) - af(x) - bf(y) - d][f(x+y) - f(x) - f(y)] = 0,$$

where $a, b, c, d \in D$. The following are the only possible cases:

(i) *a, b, c, d arbitrary and f additive;*
(ii) *$(c - a - b)$ divides d and $f \equiv \alpha$, with $\alpha(c - a - b) = d$;*
(iii) *$a = b = c \neq 0$, f is such that $af = g$, with $g \in S$;*
(iv) *$(c - a - b)$ divides d,* $\mathrm{Char}D = 2$ *and $f \in M$, where Z is a subgroup of G of index greater than 2, and $\alpha(c - a - b) = d$;*
(v) *$a = b \neq 0$, $c = 0$, $2a$ divides d and $f \in M$, where Z is a subgroup of G and $2a\alpha = d$;*
(vi) *$a = b \neq 0$, $2a$ divides d and $f \in M$, where Z is a subgroup of G of index 2 and $2a\alpha = d$.*

Thus, equation (2.9) has been solved when $f\colon G \to H$, with G and H commutative groups, or $f\colon G \to B$, where B is a Banach space and G is a group such that on the couple (G, B) the Cauchy equation is stable. It is natural to ask to investigate equation (2.9) when $f\colon S \to \mathbb{R}$ and S is a group or semigroup, where the Cauchy equation is not stable.

An answer for a special semigroup S has been given by V.A. Faĭziev, R.C. Powers and P.K. Sahoo in the paper [19]. Here $f\colon S \to \mathbb{R}$, where S is the semigroup

$$S = \langle a, b | 2a = a, 2b = b \rangle$$

First it is proved that on S the Cauchy equation is not stable, then that any solution f can be written as

$$f(x) = \alpha\psi(x) + \delta(x),$$

where $\delta\colon S \to [-1, 0]$, $\alpha \in \mathbb{R}$, and $\psi\colon S \to \mathbb{R}$ is a function such that

$$\psi(nx) = n\psi(x), \quad \psi(a) = \psi(b) = 0,$$
$$\psi(nu_1) = \psi(nu_2) = \psi(nu_3) = \psi(nu_4) = n,$$

where $u_1 = a + b$, $u_2 = b + a$, $u_3 = a + b + a$ and $u_4 = b + a + b$.

They proved the following theorem:

THEOREM 2.13. *Let $f\colon S \to \mathbb{R}$ be a function of the form $f(x) = \alpha\psi(x) + \delta(x)$, where α is a positive real number. If f is a solution of equation* (2.9)*, then either $\alpha = 1/q$, or $\alpha = 2/q$ for some integer $q \geq 1$. For any $q \in \mathbb{N}$, the function $f(x) = \frac{1}{q}\psi(x) + \delta(x)$ is a solution of* (2.9) *if and only if*

$$(-\delta(a), -\delta(b), -\delta(qu_1), -\delta(qu_2), -\delta(qu_3), -\delta(qu_4)) \in \mathcal{B},$$

where

$$\begin{aligned}\mathcal{B} = \{(0,0,1,0,0),&(0,0,0,1,0,0),(0,0,1,1,0,0),(0,0,1,1,1,0),\\&(0,0,1,1,0,1),(1,0,1,1,1,0),(0,1,1,1,0,1)\},\end{aligned}$$

and, for any $i \in \{1,2,3,4\}$, and $n \in \mathbb{N}$,

$$\delta(nu_i) = \begin{cases} \delta(qu_i), & \text{if } n \equiv 0 \pmod q, \\ -\left\{\frac{n}{q}\right\}, & \text{if } n \not\equiv 0 \pmod q. \end{cases}$$

For any odd $q \in \mathbb{N}$, the function $f(x) = \frac{2}{q}\psi(x) + \delta(x)$ is a solution of (2.9)*, if and only if*

$$(-\delta(a), -\delta(b), -\delta(qu_1), -\delta(qu_2), -\delta(qu_3), -\delta(qu_4)) = (0,0,1,1,0,0)$$

and for any $i \in \{1,2,3,4\}$, and $n \in \mathbb{N}$,

$$\delta(nu_i) = \begin{cases} \delta(qu_i), & \text{if } n \equiv 0 \pmod q, \\ -\left\{\frac{2n}{q}\right\}, & \text{if } n \not\equiv 0 \pmod q. \end{cases}$$

A similar result holds for $\alpha < 0$ ($\{t\}$ is the fractional part of t).

Equations (2.6)–(2.9) concern functions whose Cauchy difference

$$\mathcal{C}f(x,y) := f(x+y) - f(x) - f(y)$$

can assume two different values: zero and $d \neq 0$. It is natural to pose the following problem: find the functions f such that $\mathcal{C}f(x,y) \in V$, where V is a given set. In the following we present some results for special finite sets V.

Let $f\colon G \to B$, $(G,+)$ a group and B a Banach space, such that on the pair (G,B) the Cauchy equation is stable, and let

$$V = \{0, b, 2b, \dots, Mb\},$$

where $b \in B$, $b \neq 0$, $M \in \mathbb{N}$ and $\|b\| = 1/M$. This equation can be written in the form (1.3), where

$$\mathcal{E}_i(f) = \mathcal{C}(f) - ib, \quad i = 0, \dots, M.$$

Via Theorems 1.2 and 1.3, the problem can be reduced to the following form (see [22]):

find all solutions $f\colon G \to [-1,0]$ of the equation

$$\prod_{i=0}^{M}[f(x+y)-f(x)-f(y)-\frac{i}{M}]=0. \tag{2.12}$$

As a first step it is proved that it is sufficient to determine the solutions without zeros of (2.12). If f is any such solutions, define

$$A_i=\Big\{x\in G: -\frac{i+1}{M}\le f(x)< -\frac{i}{M}\Big\},\quad i=1,\dots,M-1,$$

and $g(x)=f(x)+\frac{i}{M}$, if $x\in A_i$;

$$S_i^f=\Big\{x\in G: f(x)=-\frac{i}{M}\Big\},\quad i=1,\dots,M,\quad H^f=\bigcup_{i=1}^{M}S_i^f.$$

Then S_M^f is either empty or a subsemigroup of G, and H^f is a subgroup of G.

THEOREM 2.14. *Let $f\colon G\to[-1,0)$ be a solution without zeros of* (2.12). *Then the following properties hold:*

(i) $-\frac{1}{M}\le g(x)<0$, $x\in G$;

(ii) $g(x+y)-g(x)-g(y)\in\{0,\frac{1}{M}\}$;

(iii) $H^f=\{x\in G: g(x)=-\frac{1}{M}\}$;

(iv) *if* $W_0=\{(x,y)\in G\times G: g(x+y)-g(x)-g(y)=0\}$, *then for every* n_1,n_2, $(x,y)\in W_0\cap(A_{n_1}\times A_{n_2})$ *implies* $x+y\in\bigcup_{i=n_1+n_2-M}^{n_1+n_2}A_i$;

(v) *if* $W_1=\{(x,y)\in G\times G: g(x+y)-g(x)-g(y)=\frac{1}{M}\}$, *then for every* n_1,n_2, $(x,y)\in W_1\cap(A_{n_1}\times A_{n_2})$ *implies* $x+y\in\bigcup_{i=n_1+n_2+1-M}^{n_1+n_2+1}A_i$;

where $A_i=\emptyset$, *if* $i<0$ *or* $i>M-1$.

Conversely, if g is a function satisfying (i), (ii) *and* $\{A_i\}$, $i=0,\dots,M-1$, *is a family of pairwise disjoint sets with the properties* (iv) *and* (v) *and such that* $\bigcup_{i=0}^{M-1}A_i=G$, *then the function* $f(x)=g(x)-\frac{i}{M}$, *if* $x\in A_i$, *is a solution without zeros of equation* (2.12) *and* $H^f=\{x\in G: g(x)=-\frac{1}{M}\}$.

This theorem does not really give the solutions of equation (2.12), until a procedure to split G into the sets A_i is produced.

Note that the result presented in Theorem 2.4 is a consequence of Theorem 2.14, where $M=1$ (see [22]).

The next problem is to solve the equation

$$f(x+y)-f(x)-f(y)\in V, \tag{2.13}$$

where $f\colon G \to B$, where as before B is a Banach space and G is a group where the Cauchy equation is stable, and

$$V = \{0, v_1, v_2, \dots, v_n\},$$

where $v_1, v_2, \dots, v_n$ are n independent vectors in B. After identifying the set V with the standard basis in $\mathbb{R}^n$, so $f\colon G \to \mathbb{R}^n$, $f = (f_1, \dots, f_n)$, by Theorem 1.3 we can assume that the range of f is contained in $\overline{C(-V)}$ (see [23]).

The following theorem holds:

Theorem 2.15. *A function $f\colon G \to \overline{C(-V)}$ is a solution of equation* (2.13) *if and only if it has one of the following forms:*

(i) *$f = (0, \dots, 0, f_i, 0, \dots, 0)$, for some $i = 1, \dots, n$, and $f_i\colon G \to [-1, 0]$ is a solution of* (2.13) *for $V = \{0, 1\}$;*

(ii) *$f = (0, \dots, 0, f_i, 0, \dots, 0, f_j, 0, \dots, 0)$, for some $i, j = 1, \dots, n$, $i \neq j$, where $f_i, f_j\colon G \to [-1, 0]$ are solutions of* (2.13) *for $V = \{0, 1\}$, such that $f_i(x) + f_j(x) = -1$ for all $x \in G$.*

(The solutions of (2.13) *for $V = \{0, 1\}$ are described in Theorem* 2.4.*)*

This result can be easily extended to the case V infinite of a special form. Namely, let $V = \{v_j\}_{j \in J}$ be a Hamel basis of B, with $\|v_j\| = 1$ for all $j \in J$. As before it is possible to assume that each solution of (2.13) is bounded and its range is contained in $\overline{C(-V)}$. For each subset I of J, p_I denotes the coordinate projection given by

$$p_I(x) = p_I\Big(\sum_{j \in J} \alpha_j v_j\Big) = \sum_{j \in I} \alpha_j v_j,$$

where $x \in B$, $x = \sum_{j \in J} \alpha_j v_j$. Moreover, define $f_I = p_I \circ f$, if f is a solution of (2.13), then f_I is a solution of (2.13) on the set $p_I(V)$.

From Theorem 2.15 it follows immediately the

Theorem 2.16. *Let $f\colon G \to \overline{C(-V)}$ be a solution of* (2.13)*, with $V = \{v_j\}_{j \in J}$. Then f has one of the following forms:*

(i) *there exists $k \in J$, such that for every $i \in J \setminus \{k\}$ it is $f_i \equiv 0$ and $f_k(x) = \lambda_k(x) v_k$, where $\lambda_k\colon G \to [-1, 0]$ is a solution of* (2.13) *with $V = \{0, 1\}$;*

(ii) *there exist $k, h \in J$, such that for every $i \in J \setminus \{k, h\}$ it is $f_i \equiv 0$ and $f_k(x) = \lambda_k(x) v_k$, $f_h(x) = \lambda_h(x) v_h$, where $\lambda_k, \lambda_h\colon G \to [-1, 0]$ are solutions of* (2.13) *with $V = \{0, 1\}$, such that $\lambda_k(x) + \lambda_h(x) = -1$ for all $x \in G$.*

Another particular case for the set V has been investigated in [11]. The set V is a set of vertices of the unit cube in $\mathbb{R}^3$, If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis of $\mathbb{R}^3$, we consider these sets:

$$\begin{aligned} U_1 &= \{0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3\}, \\ U_2 &= \{0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3\}, \\ U_3 &= \{0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3\}, \\ U_4 &= \{0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3\}. \end{aligned}$$

We assume that $f\colon G \to \mathbb{R}^3$ is a solution of (2.13) in U_i and that the Cauchy equation is stable on G. This leads to consider $f\colon G \to \overline{C(-U_i)}$. Moreover, we assume that $f(0) = 0$. Obviously if $f = (f_1, f_2, f_3)$ is a solution of (2.13) with $V = U_i$, then each f_i, $i = 1, 2, 3$, is a solution of (2.13) with $V = \{0, 1\}$ and with range in $[-1, 0]$.

We have the following:

Theorem 2.17. *A function $f\colon G \to \overline{C(-U_1)}$ is a solution of equation* (2.13) *with $V = U_1$ with $f(0) = 0$, if and only if it has one of the following forms: $f = (f_1, f_2, 0)$ or $f = (f_1, 0, f_3)$, where $f_i\colon G \to [-1, 0]$, $i = 1, 2, 3$, are solutions of* (2.13) *with $V = \{0, 1\}$.*

If $f = (f_1, 0, f_3)$ (or $f = (0, f_2, f_3)$) is a solution of (2.13) with $V = U_2$ or with $V = U_3$, we define the following sets;

$$\begin{aligned} A &= \{x \in G : f_1(x) = f_3(x) = 0\}, \\ B &= \{x \in G : f_1(x) \neq 0, f_3(x) = 0\}, \\ C &= \{x \in G : f_1(x) = f_3(x) = -1\}, \\ D &= \{x \in G : f_1(x) = f_3(x) \notin \{-1, 0\}\}, \end{aligned}$$

(the analogous sets are defined for the other case). It is easy to see that A, C and $A \cup B$ are subsemigroups of G, while $A \cup B \cup C$ is a subgroup of G.

Moreover, these other conditions are satisfied:

$$\begin{aligned} &x \in A, y \in B \text{ (or vice versa) implies } x + y, y + x \in A, \\ &x \in B, y \in C \text{ (or vice versa) implies } x + y, y + x \in B, \\ &x \in A, y \in C \text{ (or vice versa) implies } x + y, y + x \in A \cup C, \\ &x, y \in D \text{ implies } x + y, y + x \in A \cup C \cup D, \\ &x \in A \cup B \cup C, y \in D \text{ (or vice versa) implies } x + y, y + x \in D. \end{aligned} \tag{2.14}$$

These conditions are also sufficient for $f = (f_1, 0, f_3)$ to be a solution. We have

Theorem 2.18. *The function $f\colon G \to \overline{C(-U_2)}$ is a solution of equation* (2.13) *with $V = U_2$ with $f(0) = 0$ if and only if it has one of the following forms:*

(i) *$f = (f_1, f_2, 0)$;*

(ii) $f = (f_1, 0, f_1)$, $f = (0, f_2, f_2)$;

(iii) $f = (f_1, 0, f_3)$, $f = (0, f_2, f_3)$, *with the sets* A, B, C *and* D *(and the analogous where the role of* f_1 *is assumed by* f_2*) satisfying the conditions* (2.14)*, where* $f_i \colon G \to [-1, 0]$, $i = 1, 2, 3$, *are solutions of the equation* (2.13) *with* $V = \{0, 1\}$, *with* $f_i(0) = 0$, $i = 1, 2, 3$.

In the case of the set U_3 *we have the same result without the functions* $f = (0, f_2, f_2)$ *and* $f = (0, f_2, f_3)$.

For the last case U_4, we have

THEOREM 2.19. *The function* $f \colon G \to \overline{C(-U_4)}$ *is a solution of equation* (2.13) *with* $V = U_4$ *with* $f(0) = 0$ *if and only if it has one of the following forms:*

(i) $f = (f_1, 0, 0)$, $f = (0, f_2, 0)$;

(ii) $f = (f_1, 0, f_1)$, $f = (0, f_2, f_2)$;

(iii) $f = (f_1, 0, f_3)$, $f = (0, f_2, f_3)$,
with the sets A, B, C *and* D *(and the analogous where the role of* f_1 *is assumed by* f_2*) satisfying the conditions* (2.14)*, where* $f_i \colon G \to [-1, 0]$, $i = 1, 2, 3$, *are solutions of the equation* (2.13) *with* $V = \{0, 1\}$, *with* $f_i(0) = 0$, $i = 1, 2, 3$.

A natural question arises: do functions and sets described in (iii) of Theorems 2.18 and 2.19 actually exist? This is in general not known and may depend on the structure of the group G. We show here that for $G = \mathbb{Z}$ these sets, assumed all non empty, do not exist. First note that $0 \in A$, then $A \cup B \cup C = \{n\alpha : n \in \mathbb{Z}\}$ for some $\alpha > 1$, otherwise we have $D = \emptyset$. Assume that $\alpha \in A$, then $n\alpha \in A$ for all $n \geq 0$; this implies that $C \subset \{n\alpha : n < 0\}$. Let $-q\alpha = \max C$, $q > 0$. If $q = 1$, i.e., $-\alpha \in C$, then $-n\alpha \in C$ for all $n > 0$ and this implies $B = \emptyset$: a contradiction. If $q > 1$ then $(q-1)\alpha \in A$, $-q\alpha \in C$ and we must have $(q-1)\alpha + (-q\alpha) = -\alpha \in A$ and this implies that $A = \alpha\mathbb{Z}$, i.e., $B = C = \emptyset$: a contradiction. In the case $\alpha \in C$ we proceed in the same way and arrive to a contradiction. The last possibility is $\alpha \in B$. In this case $n\alpha \in A \cup B$ for all $n \geq 0$ and $C \subset \{n\alpha : n < 0\}$. If $-\alpha \in C$ then we must have $-\alpha + \alpha = 0 \in B$: a contradiction. Hence $-\alpha \in A \cup B$ and consequently $-n\alpha \in A \cup B$ for all $n > 0$ and this implies that $C = \emptyset$: a contradiction.

Still with values in $\mathbb{R}^3$, we can consider the following set

$$V_3 = \{0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3\}$$

(it is $V_3 \supset U_i$, $i = 1, 2, 3, 4$), that is the set of all vertices of the unit cube in $\mathbb{R}^3$ except one: $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$. Indicating with $\mathcal{P}_i$, $i = 1, 2, 3$, the coordinate planes in $\mathbb{R}^3$, we obtain the following result:

THEOREM 2.20. *A function A function* $f \colon G \to \overline{C(-V_3)}$ *is a solution of equation* (2.13) *with* $V = V_3$ *and with* $f(0) = 0$*, if and only if it has one of the following forms:*

(i) $f = (f_1, 0, 0)$, $f = (0, f_2, 0) f = (0, 0, f_3)$, *where* $f_i \colon G \to [-1, 0]$, $i = 1, 2, 3$, *are solutions of the equation* (2.13) *with* $V = \{0, 1\}$, *with* $f_i(0) = 0$, $i = 1, 2, 3$;

(ii) $f(G) \subset \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ *(but not in a single coordinate plane), and*

$$H = \bigcup_{i=1}^{3} \{x \in G : f_i(x) \in \{-1, 0\}\}$$

is a normal subgroup og G *of index* 4, *such that* $(G/H) + (G/H) \subset H$, *and if* $H^{(i)}$, $i = 1, 2, 3$, *are the cosets of* H, *we have that* $x \in H$ *implies* $f_i(x) = 0$, $i = 1, 2, 3$, $x \in H^{(i)}$ *implies* $f_j(x) = f_k(x) = -1/2$, $i = 1, 2, 3$, $j \neq i$, $k \neq i$, $j \neq k$, $i, j, k \in \{1, 2, 3\}$.

Note that the condition $f(0) = 0$ cannot be eliminated, otherwise there are solutions whose range is not contained in $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. For instance, if G has a normal subgroup K of index 2, the function

$$f(x) = \begin{cases} -\mathbf{e}_1, & x \in K, \\ -\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), & x \in G \setminus K, \end{cases}$$

is a solution of equation (2.13) with $V = V_3$ with $f(0) \neq 0$. It is an open problem to treat this case. Moreover, if $f \colon G \to \mathbb{R}^n$ with $n \geq 4$ and V_n the analogues of V_3 (the vertices of the unit cube except $\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n$), under certain conditions on the group G, there exist solutions whose range is contained in the union of the coordinate hyperplanes, but it is not known if these are the only possible solutions as for $n = 3$.

The functional equation

$$[f(x+y)]^2 = [f(x) + f(y)]^2$$

is equivalent to the alternative equation

$$f(x+y) + f(x) + f(y) \neq 0 \Rightarrow f(x+y) - f(x) - f(y) = 0. \tag{2.15}$$

We present here some results from [44]; these results generalize whose obtained by E. Vincze in [66] and H. Światak and M. Hosszú in [64].

M. Kuczma proved the following

THEOREM 2.21. *Let* $(S, +)$ *be a semigroup and* $(H, +, \cdot)$ *be a commutative ring without divisors of zero. If, moreover, the group* $(H, +)$ *does not contain any element of order* 3 *or does not contain any element of order* 4, *then the only solutions of equation* (2.15) *are the additive functions.*

It is possible to drop the above limitations on the order of elements of H and to obtain the following result:

THEOREM 2.22. *Let $(S,+)$ be a semigroup and $(H,+)$ be either a commutative group or a group containing no element of order 2. Then a function $f\colon S \to H$ satisfies equation (2.15) if and only if it is additive or, in the first case, has the form*

$$f(x) = \begin{cases} c(x), & x \in T, \\ d + c(x), & x \in S \setminus T, \end{cases}$$

where $T + T \subset T$, $(S \setminus T) + S \subset S \setminus T$, $S + (S \setminus T) \subset S \setminus T$, $d \in H$ is an element of order 3 and $c\colon S \to H$ is additive.

In the second case

$$f(x) = \begin{cases} 0, & x \in T, \\ d, & x \in S \setminus T, \end{cases}$$

where T and $S \setminus T$ fulfil the conditions above, and $d \in H$ is an element of order 3.

Similar equations have been investigated in [7], [57] and [58].

We now consider the following alternative equations of Cauchy type which generalize equations (2.6) and (2.12), we deal with the equation

$$f(x_1 + x_2 + \cdots + x_k) - \sum_{i=1}^{k} f(x_i) \in \{0, 1\} \tag{2.16}$$

and the equation

$$f(x_1 + x_2 + \cdots + x_k) - \sum_{i=1}^{k} f(x_i) \in \Big\{0, \frac{1}{k-1}, \frac{2}{k-1}, \ldots, 1\Big\}, \tag{2.17}$$

with $k \geq 3$ (equation (2.16) has been proposed by Bogdan Choczewski), where $f\colon G \to \mathbb{R}$, $(G,+)$ a group where homomorphisms are stable (see [10]).

As a first step, by Theorem 1.3, for equation (2.16) we need only consider functions $f\colon G \to \mathbb{R}$ with range in the interval $\big[-\frac{1}{k-1}, 0\big]$. Setting in (2.16), $x_1 = x_2 = \cdots = x_k = 0$, we have $-(k-1)f(0) \in \{0,1\}$, hence either $f(0) = 0$ or $f(0) = -\frac{1}{k-1}$. If f is a solution with $f(0) = -\frac{1}{k-1}$, then the function $g(x) = -f(x) - \frac{1}{k-1}$ is a solution of (2.16) with $g(0) = 0$, so we can assume that $f(0) = 0$.

Setting $x_3 = x_4 = \cdots = x_k = 0$ in (2.16), we see that f is a solution of the equation

$$f(x_1 + x_2) - f(x_1) - f(x_2) \in \{0, 1\}.$$

If we identify $\mathbb{R}/\mathbb{Z}$ with $(-1, 0]$, the range of f must be a subgroup of $(-1, 0]$ contained in $\big[-\frac{1}{k-1}, 0\big]$. If $k > 3$ then there are no non-trivial subgroups of this type, so we have the following:

THEOREM 2.23. *The only solution f of equation* (2.16)*, with $f(0)=0$ and $k>3$, is the zero function.*

A function $f\colon G\to\left[-\frac{1}{2},0\right]$ is a solution of (2.16) *with $k=3$, if and only if*

$$f(x)=\begin{cases}0, & x\in Z,\\ -\frac{1}{2}, & x\in G\setminus Z,\end{cases}$$

where either $Z=G$, or Z is a normal subgroup of G of index 2*.*

About equation (2.17) we have the following result:

THEOREM 2.24. *A function $f\colon G\to\left[-\frac{1}{k-1},0\right]$, with $f(0)=0$, is a solution of* (2.17) *if and only if it has the following form:*

$$f(x)=\begin{cases}0, & x\in L_0,\\ \sigma\circ\phi\circ\pi(x), & x\in G\setminus(L_0\cup L_{k-1}),\\ -\frac{1}{k-1}, & x\in L_{k-1},\end{cases}$$

where

(i) *L_0 is a subsemigroup of G with $0\in L_0$;*
(ii) *L_{k-1} is a subset (possibly empty) of G such that $K=L_0\cup L_{k-1}$ is a normal subgroup of and $kL_{k-1}\subset L_{k-1}$;*
(iii) *$\pi\colon G\to G/K$ is the natural homomorphism;*
(iv) *$\phi\colon G/K\to\mathbb{R}/\mathbb{Z}^{(k)}$ is an injective homomorphism;*
(v) *$\sigma\colon\mathbb{R}/\mathbb{Z}^{(k)}\to\mathbb{R}$ is the only lifting such that for $x\notin K$, $-\frac{1}{k-1}\leq\sigma\circ\phi\circ\pi(x)<0$.*

($\mathbb{Z}^{(k)}$ is the group $\frac{1}{k-1}\mathbb{Z}$.)

A different form of equation (2.17), that is

$$f(x_1+x_2+\cdots+x_k)-\sum_{i=1}^{k}f(x_i)\in\Big\{0,(k-1)a,2(k-1)a,\cdot,(k-1)^2a\Big\},\tag{2.18}$$

where $f\colon G\to H$, $(G,+)$ and $(H,+)$ abelian groups, $a\in H$ with infinite order, has been investigated by C. Borelli ([8]). After the reduction to the case $H=\mathbb{Z}$, $a=1$, the following theorem is proved:

THEOREM 2.25. *Let $(G,+)$ be an abelian group whose non-zero elements have infinite order. A function $f\colon G\to\mathbb{Z}$ is a solution of equation* (2.18) *with $a=1$ and with $f(0)=0$, if and only if it has the following form:*

$$f(x)=\beta(x)-(k-1)([\alpha(x)]+1-\chi_{L_0}(x)),$$

where $\beta\in\mathrm{Hom}(G,\mathbb{Z})$, $\alpha\in\mathrm{Hom}(G,\mathbb{R})$, L_0 is a subgroups of G with $L_0\subset\alpha^{-1}(\mathbb{Z})$ and such that $(\alpha^{-1}(\mathbb{Z})\setminus L_0)^k\subset\alpha^{-1}(\mathbb{Z})\setminus L_0$ (T^k is the set of all sums of k elements of T).

A function $f\colon G \to \mathbb{Z}$ *is a solution of equation* (2.18) *with* $a = 1$ *and with* $f(0) = -t$, $0 < t \leq k-1$, *if and only if it has the form*

$$f(x) = \beta(x) - (k-1)([\alpha(x)] + 1 - \chi_A(x) - \chi_{L_0}(x) - \chi_S(x)) - t(1 - \chi_K(x)),$$

where

(i) $\beta \in \mathrm{Hom}(G, \mathbb{Z})$, $\alpha \in \mathrm{Hom}(G, \mathbb{R})$, $A = \alpha^{-1}(\mathbb{Z})$;
(ii) K *is the (possibly empty) coset of* A *given by* $K = \{x \in G : \alpha(x) \in \frac{t}{k-1} + \mathbb{Z}\}$;
(iii) L_0 *is a (possibly empty) subset of* K, *such that* $(L_0)^k \subset L_0$ *and* $(K \setminus L_0)^k \subset K \setminus L_0$;
(iv) $S = \{x \in G : \alpha(x) \in \frac{u}{k-1} + \mathbb{Z}, u < t\}$.

In his paper [52] L. Paganoni obtained the following results. For every $\alpha \in \mathbb{R}$, we denote with ϕ_α the homomorphism of $\mathbb{N}$ or $\mathbb{Z}$ into $\mathbb{R}$ given by $\phi_\alpha(n) = \alpha n$. For $\phi \in \mathrm{Hom}(G, \mathbb{R})$, $(G, +)$ abelian group, and $K = \{0, 1, \ldots, k\}$, $\Psi^K_{[\phi]}$ is the class of functions $\psi\colon G \to K$ such that

$$(x, y) \in \Omega^i_{[\phi]} \cap (G_r \times G_s) \Rightarrow x + y \in \bigcup_{u=-k}^{0} G_{r+s+i+u}, \ i = 0, 1,$$

where

$$\Omega^i_{[\phi]} = \{(x, y) \in G \times G\colon [\phi(x + y)] - [\phi(x)] - [\phi(y)] = i\},$$
$$G_t = \{x \in G : \psi(x) = -t\} \ (G_t = \emptyset \quad \text{if} \quad t < 0 \quad \text{or} \quad t > k).$$

The main result is given by the following

Theorem 2.26. *The function* $f\colon \mathbb{N} \to \mathbb{Z}$ *is a solution of equation*

$$f(x + y) - f(x) - f(y) \in K, \tag{2.19}$$

if and only if there exists $\alpha \in \mathbb{R}$ *such that* $f(n) = [\alpha n] + \psi(n)$, *where* $\psi \in \Psi^K_{[\phi]}$. *Moreover,* α *is unique.*

Let $(G, +)$ *be an abelian group, a function* $f\colon G \to \mathbb{Z}$ *is a solution of equation* (2.19), *if and only if there exists* $\varphi \in \mathrm{Hom}(G, \mathbb{R})$, *such that* $f(x) = [\varphi(x)] + \psi(x)$, *where* $\psi \in \Psi^K_{[\varphi]}$. *This representation is unique.*

We finish this section with some alternative equations related to Jensen equation.

The first equation studied by P. Nakmahachalasint in [50] is

$$f(x) + 2f(x+y) + f(x+2y) \neq 0 \Rightarrow f(x) - 2f(x+y) + f(x+2y) = 0, \tag{2.20}$$

where $f\colon S \to G$, $(S, +)$ is a semigroup and $(G, +)$ is a uniquely divisible abelian group (the equation $f(x) - 2f(x+y) + f(x+2y) = 0$ is obviously a different way to write the classical Jensen equation).

The following results are proved:

THEOREM 2.27. *Let $S = \langle a \rangle$ be an infinite cyclic semigroup. Then $f : S \to G$ is a solution of equation* (2.20) *if and only if it has one of the following forms:*

(i) $f(na) = k_0 + k_1 n$, *for all* $n \in \mathbb{N}$;
(ii) $f(na) = (-1)^n (k_0 + k_1 n)$, *for all* $n \in \mathbb{N}$;
(iii) $f(na) = (1 - 4\delta_{n_0})k_0$, *for all* $n \in \mathbb{N}$;
(iv) $f(na) = (-1)^n (1 - 4\delta_{n0})k_0$, *for all* $n \in \mathbb{N}$;

where k_0 and k_1 are arbitrary elements in G, and δ_{ij} is the Kronecker delta.

If $S = \langle a | a^m = a^{m+p} \rangle$ is a finite cyclic semigroup with index m and period p, then $f : S \to G$ is a solution of equation (2.20), *if and only if it has one of the previous forms, with $k_1 = 0$.*

If G is a 2-divisible group, then f is a solution of (2.20) *if and only if it is a Jensen function, that is $f(x) - 2f(x + y) + f(x + 2y) = 0$ for all $x, y \in S$.*

Another equation has been investigated in [63], namely the equation

$$(2.21)\quad f(x - y) - \lambda f(x) + f(x + y) \neq 0 \Rightarrow f(x - y) - 2f(x) + f(x + y) = 0,$$

where $f : G \to H$, $(G, +)$ a group, $(H, +)$ a uniquely divisible abelian group and λ is an integer different from 2. It is proved that if $\lambda \notin \{0, -1, -2\}$, then equation (2.21) is equivalent to the Jensen equation.

A generalization of equation (2.21), that is

$$(2.22)\quad \alpha f(x-y) + \beta f(x) + \gamma f(x+y) \neq 0 \Rightarrow f(x-y) - 2f(x) + f(x+y) = 0,$$

where $f : G \to H$, $(G, +)$ a group, $(H, +)$ a uniquely divisible abelian group, and $(\alpha, \beta, \gamma) \neq (k, -2k, k)$ for all $k \in \mathbb{Z}$, has been studied in [43]. One of the results proved in that paper is given by the following theorem.

THEOREM 2.28. *Let $G = \langle g \rangle$ be an infinite cyclic group. A function $f : G \to H$ is a solution of equation* (2.22) *if and only if either is a Jensen function, or one of the following properties holds:*

(i) $\beta = \alpha + \gamma$ *and*
  (◇) $f(ng) = (-1)^n a$, *for all* $n \in \mathbb{Z}$ *and some* $a \in H$, *or*
  (◇) $\beta = 0$ *and*
    (I) $\{f(ng)\}_{n\in\mathbb{Z}}$ *is the periodic sequence* $\dots, a, b, a, b, \dots$, *for some* $a, b \in H$, *or*
    (II) $\{f(ng)\}_{n\in\mathbb{Z}}$ *is the periodic sequence* $\dots, 2a - b, a, b, a, 2a - b, \dots$, *for some* $a, b \in H$, *or*
  (◇) $\beta = 2\alpha$ *and* $f(ng) = (-1)^n (a + nb)$, *for all* $n \in \mathbb{Z}$ *and some* $a, b \in H$;
(ii) $(\beta, \gamma) = (0, \alpha)$ *and* $\{f(ng)\}_{n\in\mathbb{Z}}$ *is the periodic sequence* $\dots, a, -a, a, -a, \dots$, *for some* $a \in H$;
(iii) $(\beta, \gamma) = (\alpha, \alpha)$ *and*

($\diamond$) $\{f(ng)\}_{n\in\mathbb{Z}}$ *is the periodic sequence* $\dots, a, b, -a, -b, a, b, -a, -b, \dots$, *some* $a, b \in H$, *or*

($\diamond$) $\{f(ng)\}_{n\in\mathbb{Z}}$ *is the periodic sequence* $\dots, a, -2a, a, -2a, \dots$, *for some* $a \in H$, *or*

($\diamond$) $\{f(ng)\}_{n\in\mathbb{Z}}$ *is the periodic sequence* $\dots, -2a, a, a, \dots, a, -2a, \dots$, *of odd period* $p \geq 5$, *for some* $a \in H$.

2.3. Alternative equations related to the Jordan-von Neumann quadratic equation

This subsection is devoted to alternative equations in a single function, involving the Jordan-von Neumann quadratic equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

The first problem we deal with is the analogue of equation (2.6), adapted to the quadratic equation, that is

$$(2.23)\quad f(x+y) + f(x-y) - 2f(x) - 2f(y) - 1 \neq 0 \\ \Rightarrow f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0,$$

where $f\colon G \to \mathbb{R}$.

As in the case of the Cauchy equation, we assume that G is a group where the quadratic equation is stable in the sense of Ulam-Hyers. The stability and Theorem 1.4 permit to reduce the problem to the case of a bounded function f which we can assume to satisfy $f(0) = 0$ (if $f(0) = -\frac{1}{2}$, then the function $g(x) = -f(x) - \frac{1}{2}$ is a solution of equation (2.23) with $g(0) = 0$) and that $f\colon G \to K$, where

$$K = \Big\{ -\frac{1}{3}\sum_{n=1}^{\infty} \frac{3\alpha_n}{4^n} : \alpha_n \in \{0,1\} \Big\} \subset \Big[-\frac{1}{2}, 0 \Big].$$

The general solution is obtained by adding any quadratic function, that is any solution of the equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$.

The following result is then proved ([24], [25]):

Theorem 2.29. *Non-zero bounded solutions of equation* (2.23) *exist only in these two cases:*

(i) *the group* G *has a normal subgroup* Z *of index* 2, *and*

$$f(x) = \begin{cases} 0, & x \in Z, \\ -\frac{1}{4}, & x \in G \setminus Z, \end{cases}$$

(ii) *the group G has a subgroup Z such that the set $(G \setminus Z) \times (G \setminus Z)$ can be split in two (disjoint) sets L and M with the properties that $(x, y) \in L$ if $x + y \notin Z$, and $(x, y) \in M$ if $x - y \in L$; the solution is*

$$f(x) = \begin{cases} 0, & x \in Z, \\ -\frac{1}{3}, & x \in G \setminus Z. \end{cases}$$

Corollary 2.29.1. *If the group G is commutative, then case* (ii) *of Theorem* 2.29 *becomes the following: Z is a subgroup of G of index* 3 *and f has the form given above.*

The fact that f in (2.23) is a real function is not a restriction. Indeed if $f\colon G \to B$, where B is a Banach space and in equation (2.23) instead of $f(x+y)+f(x-y)-2f(x)-2f(y)-1 \neq 0$ we write $f(x+y)+f(x-y)-2f(x)-2f(y) - \beta \neq 0$, with $\beta \in B$, which can be assumed with norm 1, Theorem 1.2 proves that we can consider only the functions with range in the segment having end points 0 and $-\beta/2$; thus we are reduced to the one-dimensional case.

A more general equation has been investigated in [26], that is

$$f(x + y) + f(x - y) - 2f(x) - 2f(y) \in \{0, 1, 2\}, \tag{2.24}$$

where $f\colon G \to \mathbb{R}$, $(G, +)$ commutative group. Again by using stability we can assume that f is bounded and its range is contained in the set

$$\Big\{ -\sum_{i=1}^{\infty} \frac{m_i}{4^i} - \frac{m_0}{6} : m_i \in \{0, 1, 2\}, m_0 = -2f(0) \Big\}.$$

It is useful to change f into the function $k(x) = -\frac{3}{2} f(x)$, so the range of k is contained in the set

$$K = \Big\{ \frac{3}{2} \sum_{i=1}^{\infty} \frac{\alpha_i}{4^i} + \frac{1}{3} k(0) : \alpha_i \in \{0, 1, 2\} \Big\},$$

and should be noted that the given representation of numbers in K is unique. The equation (2.24) becomes

$$k(x + y) + k(x - y) - 2k(x) - 2k(y) \in \{-3, -3/2, 0\}. \tag{2.25}$$

We have $k(0) \in \{0, 3/4, 3/2\}$; since $j(x) = \frac{3}{2} - k(x)$ is a solution of (2.25) with $j(0) = \frac{3}{2} - k(0)$, we obtain that $k(0) = 3/2$ implies $j(0) = 0$. Thus, we can consider only the two cases $k(0) = 0$ and $k(0) = 3/4$.

We have the following result:

Theorem 2.30. *Let k be a non-trivial solution of equation* (2.25), *with $k(0) = 0$. Then $Z = \{x \in G : k(x) = 0\}$ is a subgroup of G and each element of G/Z has one of the following orders:* 2, 3, 4, 5, 6, 7, 12. *Moreover,*

the function k is constant on each coset of Z. If G/Z is a cyclic group, we have these explicit forms for the solutions.
If $[G:Z]=2$, $G/Z=\{Z,H\}$, there are two non-trivial solution of (2.25)*:*
(i) $k(Z)=0$, $k(H)=\frac{3}{8}$;
(ii) $k(Z)=0$, $k(H)=\frac{3}{4}$.
If $[G:Z]=3$, $G/Z=\{Z,H,2H\}$, there are two non-trivial solution of (2.25)*:*
(i) $k(Z)=0$, $k(H)=k(2H)=\frac{1}{2}$;
(ii) $k(Z)=0$, $k(H)=k(2H)=1$.
If $[G:Z]=4$, $G/Z=\{Z,H,2H,3H\}$, there are three non-trivial solution of (2.25)*:*
(i) $k(Z)=0$, $k(H)=k(3H)=\frac{3}{16}$, $k(2H)=\frac{3}{4}$;
(ii) $k(Z)=0$, $k(H)=k(3H)=\frac{9}{16}$, $k(2H)=\frac{3}{4}$;
(iii) $k(Z)=0$, $k(H)=k(3H)=\frac{15}{16}$, $k(2H)=\frac{3}{4}$.
If $[G:Z]=5$, $G/Z=\{Z,H,2H,3H,4H\}$, there are two non-trivial solution of (2.25)*:*
(i) $k(Z)=0$, $k(H)=k(4H)=\frac{3}{5}$, $k(2H)=k(3H)=\frac{9}{10}$;
(ii) $k(Z)=0$, $k(H)=k(4H)=\frac{9}{10}$, $k(2H)=k(3H)=\frac{3}{5}$.
If $[G:Z]=6$, $G/Z=\{Z,H,2H,3H,4H,5H\}$, there are two non-trivial solution of (2.25)*:*
(i) $k(Z)=0$, $k(H)=k(5H)=\frac{7}{8}$, $k(2H)=k(4H)=\frac{1}{2}$, $k(3H)=\frac{3}{8}$;
(ii) $k(Z)=0$, $k(H)=k(5H)=\frac{1}{4}$, $k(2H)=k(4H)=1$, $k(3H)=\frac{3}{4}$.
If $[G:Z]=7$, $G/Z=\{Z,H,2H,3H,4H,5H,6H\}$, there are three non-trivial solution of (2.25)*:*
(i) $k(Z)=0$, $k(H)=k(6H)=\frac{3}{14}$, $k(2H)=k(5H)=\frac{6}{7}$, $k(3H)=k(4H)=\frac{3}{7}$;
(ii) $k(Z)=0$, $k(H)=k(6H)=\frac{3}{7}$, $k(2H)=k(5H)=\frac{3}{14}$, $k(3H)=k(4H)=\frac{6}{7}$;
(iii) $k(Z)=0$, $k(H)=k(6H)=\frac{6}{7}$, $k(2H)=k(5H)=\frac{3}{7}$, $k(3H)=k(4H)=\frac{3}{14}$.
If $[G:Z]=12$, $G/Z=\{Z,H,2H,3H,4H,5H,6H,7H,8H,9H,10H,11H\}$, there is one non-trivial solution of (2.25)*:*
$k(Z)=0$, $k(H)=k(5H)=k(9H)=\frac{7}{10}$, $k(2H)=k(10H)=\frac{1}{4}$, $k(3H)=k(11H)=\frac{15}{16}$, $k(4H)=k(8H)=1$, $k(6H)=\frac{3}{4}$, $k(7H)=\frac{11}{16}$.

The solutions described in Theorem 2.30 in the special case when G/Z is a cyclic group, can be considered *fundamental solutions* since they are the blocks for constructing the solutions in the general case. The procedure is the following. Given a commutative group G, we choose a subgroup Z of G, which will be the zero-set of the possible solutions. If the quotient G/Z is finite, then it is (isomorphic to) the group

$$G/Z=\prod_{i=1}^{s} C_{p_i^{n_i}},$$

where C_r is the cyclic group of order r, and p_is are prime numbers (see, for instance, [56], Theorem 5.1.13).

By Theorem 2.30, we have fundamental solutions only on the groups $C_2, C_3, C_4, C_5, C_6 = C_2 \times C_3, C_7$ and $C_{12} = C_3 \times C_4$, thus if in $\prod_{i=1}^{s} C_{p_i^{n_i}}$ we have cyclic groups different from the previous ones, it does not exist any non-trivial solution of equation (2.25). Otherwise we have the fundamental solutions on factor groups and we must check their compatibility, some examples can be found in [26].

If the quotient group G/Z is infinite, we can use Kulikov's structure theorem 2.7.

In the case $k(0) = \frac{3}{4}$, we have the following:

THEOREM 2.31. *Let k be a solution of equation (2.25) with $k(0) = \frac{3}{4}$, such that $k(G)$ does not contain $\frac{1}{4}$ or $\frac{5}{4}$. Then the set $Z = \{x \in G : k(x) = \frac{3}{4}\}$ is a subgroup of G. Moreover, k is constant on the cosets of Z.*
If $[G : Z] = 4$ and G/Z is cyclic, $G/Z = \{Z, H, 2H, 3H\}$, split Z as $Z^{(1)} \cup Z^{(2)}$, fix $\overline{x} \in G \setminus Z$ and consider the coset $\overline{x} + Z = (\overline{x} + Z^{(1)}) \cup (\overline{x} + Z^{(2)})$. If $-Z^{(1)} = 4\overline{x} + Z^{(2)}) = Z^{(1)}$, the following function is a solution of (2.25):

$$k(x) = \begin{cases} 3/4, & x \in Z, \\ 15/32, & x \in \overline{x} + Z^{(1)},\ x \in 3\overline{x} + Z^{(2)}, \\ 39/32, & x \in \overline{x} + Z^{(2)},\ x \in 3\overline{x} + Z^{(1)}, \\ 9/8, & x \in 2\overline{x} + Z. \end{cases}$$

If $[G : Z] = 5$ $G/Z = \{Z, H, 2H, 3H, 4H\}$, the following function is a solution of (2.25):
$k(Z) = \frac{3}{4}$, $k(H) = k(4H) = \frac{9}{20}$, $k(2H) = k(3H) = \frac{21}{20}$.
If $[G : Z] = 6$ and G/Z is cyclic, $G/Z = \{Z, H, 2H, 3H, 4H, 5H\}$, the following function is a solution of (2.25):
$k(Z) = \frac{3}{4}$, $k(H) = k(5H) = \frac{5}{8}$, $k(2H) = k(4H) = \frac{1}{4}$, $k(3H) = \frac{9}{8}$.

As in the case $k(0) = 0$, other solutions can be constructed by using the structure of the group G.

Other alternative quadratic equations have been investigated in [65] and [62].

The first of these papers considers the following alternative equation:

$$\begin{aligned} (2.26) \quad & f(x+y) + f(x-y) - 2f(x) - 2f(y) \neq 0 \\ & \qquad \Rightarrow f(x+y) + f(x-y) - 2f(x) + 2f(y) = 0, \end{aligned}$$

where $f \colon G \to Y$, $(G, +)$ is a 2-divisible abelian group and Y is a real (or rational or complex) linear space. It is proved that equation (2.26) is equivalent to the quadratic equation.

In the second paper the alternative equation which is investigated is

$$(2.27)\quad f(x+y)+f(x-y)-\alpha f(x)-2f(y)\neq 0 \Rightarrow f(x+y)+f(x-y)-2f(x)+2f(y)=0,$$

where α is a given rational number different from 2. The mapping f is from an abelian group $(G,+)$ to a uniquely divisible abelian group $(H,+)$. The following result has been proved.

THEOREM 2.32. *Let $f\colon G\to H$ be a solution of equation* (2.27). *Then either f is quadratic or one of the following conditions holds:*

(i) *$\alpha=0$ and f is constant;*

(ii) *$\alpha=-1$ and there exists $a\in G$ such that $f(a)\neq 0$ and*

$$f(na)=\begin{cases}0, & \text{if } 3|n,\\ f(a), & \text{otherwise,}\end{cases}$$

for all integers n;

(iii) *$\alpha=-2$ and there exists $a\in G$ such that $f(a)\neq 0$ and*

$$f(na)=\begin{cases}0, & \text{if } n \text{ is even,}\\ f(a), & \text{otherwise,}\end{cases}$$

for all integers n.

If the group G is cyclic infinite, then a function f is a solution of equation (2.27) *if and only if it is either quadratic or has one of the forms above.*

For more special groups we have the following

THEOREM 2.33. *If G is a 6-divisible commutative group, then f is a solution of* (2.27) *with $\alpha\neq 0$, if and only if it is quadratic.*

If G is a finite cyclic group of order $m\geq 2$, $G=\langle g\rangle$, a function f is a solution of (2.27) *if and only if either it is quadratic or one of the following properties hold:*

(i) *$\alpha=0$ and f is constant;*

(ii) *$\alpha=-1$, $3|m$ and*

$$f(ng)=\begin{cases}0, & \text{if } 3|n,\\ k, & \text{otherwise;}\end{cases}$$

(iii) *$\alpha=-2$, m is even and*

$$f(ng)=\begin{cases}0, & \text{if } n \text{ is even,}\\ k, & \text{otherwise,}\end{cases}$$

for all integers n and some $k\in H\setminus\{0\}$.

A series of alternative quadratic equations have been studied by F. Skof and M. Varrone. The main result of the two papers [59] and [61] is the following:

Theorem 2.34. *Let X be a real linear space. In the class of functions $f\colon X \to \mathbb{R}$, each of the following alternative equations*

$$\begin{aligned}
&|f(x+y)| = |2f(x)+2f(y)-f(x-y)|,\\
&|f(x-y)| = |2f(x)+2f(y)-f(x+y)|,\\
&|f(x+y)+f(x-y)-2f(x)| = |2f(y)|,\\
&|f(x+y)+f(x-y)-2f(y)| = |2f(x)|,\\
&|f(x+y)+f(x-y)| = |2f(x)+2f(y)|,
\end{aligned}\tag{2.28}$$

is equivalent to the quadratic equation.

As it is proved in [60], the situation changes if in the equations (2.28) the absolute value is substituted by a norm in a linear space. We exhibit some examples.

Assume $f\colon \mathbb{R} \to E$, where $(E, \|\cdot\|)$ is a real normed space not strictly convex, then there exists two linearly independent points $a, b \in E$ such that $\|a\| = \|b\| = 1$ and $\|a+b\| = \|a\| + \|b\|$. Define

$$f(x) = \begin{cases} ax^2, & \text{if } x \geq 0,\\ bx^2, & \text{if } x < 0.\end{cases}$$

This function is continuous, it is not quadratic, but it satisfies the equations

$$\|f(x+y)+f(x-y)-2f(x)\| = \|2f(y)\|$$

and

$$\|f(x+y)+f(x-y)\| = \|2f(x)+2f(y)\|.$$

The function

$$f(x) = \begin{cases} 2ax^2, & \text{if } |x| \leq 1,\\ (x^2+1)a + (x^2-1)b, & \text{if } |x| > 1,\end{cases}$$

is continuous, it is not quadratic, and satisfies the equation

$$\|f(x+y)+f(x-y)\| = \|2f(x)+2f(y)\|.$$

When the range is in a real linear inner product space $(H, \|\cdot\|)$, we have the following

Theorem 2.35. *Let* $f\colon X \to H$ *where* X *is a real linear space. Then* f *satisfies the equation*

$$\|f(x+y)\| = \|2f(x) + 2f(y) - f(x-y)\|$$

if and only if it is quadratic.

We finish this subsection with an alternative equation connecting additive and quadratic equations ([38]).

The equation is

$$f(x+y) - f(x) - f(y) \neq 0 \Rightarrow f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0, \tag{2.29}$$

where $f\colon G \to H$, $(G, +)$ and $(H, +)$ abelian groups. Without loss of generality it is possible to assume that G is free abelian.

From (2.29) we have either $f(0) = 0$ or $2f(0) = 0$. The first easy result is that if $f(0) \neq 0$, then f is a quadratic map. So, from now on we assume $f(0) = 0$. The following example is of great relevance:

Let $\tilde{f}\colon \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ be the map $\tilde{f}(k) = \hat{k} + 5\mathbb{Z}$, where $\hat{k}$ is the unique element of $\{-1, 0, 1\}$ such that $k \equiv \hat{k} (\mathrm{mod}\ 3)$. It is not difficult to show that $\tilde{f}$ is a solution of equation (2.29), where $G = \mathbb{Z}$ and $H = \mathbb{Z}/5\mathbb{Z}$, and that it is neither additive nor quadratic.

We say that a map f *involves* $\tilde{f}$ if its restriction to some cyclic subgroup of G is equivalent to $\tilde{f}$ (that is that there are two isomorphisms $\varphi\colon \mathbb{Z}/5\mathbb{Z} \to H$ and $\psi\colon \mathbb{Z} \to K$, where K is a cyclic subgroup of G, and $f = \varphi \circ \tilde{f} \circ \psi^{-1}$).

Theorem 2.36. *If there is no* 2*-torsion in* H*, then a solution of* (2.29) *is additive or quadratic or involves* $\tilde{f}$*. Moreover, if* $< x >$ *is the cyclic subgroup of* G *where* $f_{<x>} \simeq \tilde{f}$*, then* $G = < x > + \ker f$ *and* f *is constant on each coset of* $\ker f$*.*

Consider now the case when H is a 2-group.

Theorem 2.37. *If* $2H = 0$*, then any solution* f *of* (2.29) *is constant on the cosets of* $2G$ *and is a quadratic map.*

A second important example is the following:

Let $\tilde{g}\colon \mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ be the map defined by

$$\tilde{g}(k) = \begin{cases} 0 + 8\mathbb{Z}, & \text{if } k \equiv 0 \pmod 4, \\ 1 + 8\mathbb{Z}, & \text{if } k \equiv 1 \pmod 4, \\ 4 + 8\mathbb{Z}, & \text{if } k \equiv 2 \pmod 4, \\ 5 + 8\mathbb{Z}, & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

The map $\tilde{g}$ is a solution of equation (2.29), where $G = \mathbb{Z}$ and $H = \mathbb{Z}/8\mathbb{Z}$, and that it is neither additive nor quadratic.

The final result is given by the next theorem, where we have $H = < f(G) >$.

THEOREM 2.38. *Let $f\colon G\to H$ be a solution of equation* (2.29). *One of the following holds:*

(i) *f is additive;*
(ii) *f is quadratic;*
(iii) *f involves $\tilde{f}$. Furthermore, $H =< \alpha > \oplus E$, where $< \alpha > \simeq \mathbb{Z}/5\mathbb{Z}$ and $2E = 0$, and f is the sum of a map $j\colon G \to < \alpha >$, $G =< x > +\mathrm{ker} j$, and an additive map $G \to E$;*
(iv) *H is a 2-group and $\mathrm{ker} f$ is a subgroup of G, f is constant on each coset of $\mathrm{ker} f$, $G/\mathrm{ker} f$ is cyclic and the map $G/\mathrm{ker} f \to H$ induced by f is equivalent to $\tilde{g}$.*

2.4. The functional equation of the plurality function

Suppose that each of a group of voters, human or artificial, gives his or her first choice among an ordered set A of n alternatives $a_1, a_2, \ldots, a_n$. In order to reach a consensus, a function assigns to each set of choices a subset of A called the group's consensus. The *plurality function* is the consensus function which chooses as consensus all alternatives which receive the largest number of first choices. So the plurality function (as all consensus functions) is a set valued function. It is possible to consider it as having an n-term sequence of 0s and 1s (not all zero) as values, the k-th term being 1 if the k-th alternative is chosen by group consensus from the sequence $A = a_1, a_2, \ldots, a_n$. Thus the domain may consist of ordered n-tuples, with a c in component i, indicating that the alternative a_i was given as first choice by c voters. If fractional votes are allowed, then the domain would consist of non-zero rational, or even real, vectors. In this case the plurality function is defined on $\mathbb{R}^n_+ \setminus \{\underline{0}\}$, with values in $\{0,1\}^n \setminus \{\underline{0}\}$ ($\underline{0}$ is the null vector), the i-th term being 1 exactly if the i-th alternative is chosen by group consensus. It is reasonable to require the homogeneity condition $f(r\underline{x}) = f(\underline{x})$, for all positive r. F.S. Roberts in [53] and [54], listed several axioms in order to characterize the plurality function among consensus functions and he stated that it is of interest in the theory of social choice to determine all functions $f\colon \mathbb{R}^n_+ \setminus \{\underline{0}\} \to \{0,1\}^n \setminus \{\underline{0}\}$ satisfying the following two equations:

$$f(\underline{x}) \star f(\underline{y}) \neq \underline{0} \Rightarrow f(\underline{x}+\underline{y}) - f(\underline{x}) \star f(\underline{y}) = \underline{0}, \tag{2.30}$$

$$f(r\underline{x}) = f(\underline{x}), \tag{2.31}$$

for all $\underline{x}, \underline{y} \in \mathbb{R}^n_+ \setminus \{\underline{0}\}$ and all $r > 0$, where for all $\underline{u}, \underline{v} \in \{0,1\}^n \setminus \{\underline{0}\} =: I^n_0$, it is $\underline{u} \star \underline{v} = \underline{v} \star \underline{u} = (u_1v_1, u_2v_2, \ldots, u_nv_n) \in \{0,1\}^n$.

Obviously, we can at first consider only the first equation (2.30) and investigate it for $f\colon \mathbb{R}^n_+ \setminus \{\underline{0}\} \to \mathbb{R}^n_+$. The first result in this direction has been proved by Z. Moszner in [48]; the main result is given by the following

THEOREM 2.39. *Let* $f = (f_1, \ldots, f_n)$*, where* $f_i \colon \mathbb{R}_+^n \to \mathbb{R}_+$*, and define*

$$Z_i = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n : f_i(x_1, \ldots, x_n) \neq 0\}, \quad i = 1, \ldots, n,$$

$\underline{i} = (i_1, \ldots, i_n)$, $\underline{j} = (j_1, \ldots, j_n)$, $i_1, \ldots, i_n, j_1, \ldots, j_n \in \{0, 1\}$, $\underline{i} \cdot \underline{j} = (i_1 j_1, \ldots, i_n j_n)$ *(if* $E \subset \mathbb{R}_+^n$*, denote* $E^1 = E$*,* $E^0 = \mathbb{R}_+^n \setminus E$*). The function* f *satisfies equation* (2.30) *if and only if*

$$f_i(x_1, \ldots, x_n) = \begin{cases} \exp[a_{i1}(x_1) + \cdots + a_{in}(x_n)], & \underline{x} \in Z_i, \\ \underline{0}, & \underline{x} \in \mathbb{R}_+^n \setminus Z_i, \end{cases}$$

where $a_{i1}, \ldots, a_{in} \colon \mathbb{R} \to \mathbb{R}$*,* $i = 1, \ldots, n$*, are additive functions and the sets* Z_i *satisfy the following conditions:*

$$Z_1 \cup \ldots \cup Z_n = \mathbb{R}_+^n,$$

and for each $\underline{i}, \underline{j}$ *with* $\underline{i} \cdot \underline{j} \neq \underline{0}$ *it is*

$$Z_1^{i_1} \cap \ldots \cap Z_n^{i_n} + Z_1^{j_1} \cap \ldots \cap Z_n^{j_n} \subset Z_1^{i_1 j_1} \cap \ldots \cap Z_n^{i_n j_n}.$$

In the same paper it is proved that if we assume the homogeneity of f for a rational number $r > 0$, then f has its range in $\{0, 1\}^n$. The same author in [49] proved that this last result is true when $r > 0$ is algebraic. A. Bahyrycz in [2] proved that the above result holds also for transcendental numbers r, for $n = 1, 2$, while for $n \geq 3$, for every transcendental number r there exists a solution f of equation (2.30) with $f(r\underline{x}) = f(\underline{x})$ and having range not contained in $\{0, 1\}^n$. Later, in the paper [4], it has been investigated under which assumptions the function f, solution of (2.30), has its range in $\{0, 1\}^n$. From the expression of solutions given in Theorem 2.39 it follows that this is possible if and only if all additive functions a_{ij} are identically zero.

The homogeneity condition $f(r\underline{x}) = f(\underline{x})$ for some $r \neq 1$ and all $\underline{x} \in \mathbb{R}^n$ implies that for all $i = 1, 2, \ldots, n$, $rZ_i = Z_i$, and $a_{ij}(r\underline{x}) = a_{ij}(\underline{x})$, for $\underline{x} \in Z_i$.

The following theorem holds:

THEOREM 2.40. *A function* f *which is solution of equation* (2.30) *with* $f(r\underline{x}) = f(\underline{x})$ *for some* $r \neq 1$*, has its range in* $\{0, 1\}^n$ *if and only if the sets* Z_i *fulfilling the condition* $rZ_i = Z_i$*, satisfy the condition* $Z_i \subset (r - 1)Z_i$*,* $i = 1, \ldots, n$*.*

For the construction of the sets Z_i, see [3].

The full system of the two equations (2.30) and (2.31) has been studied in the two papers [36] and [37]. In these papers the fundamental sets for the construction of the solutions are the level sets of f. The range of any solution of (2.30)–(2.31) is in $\{0, 1\}^n$, and denoting with a Greek letter $\alpha, \beta, \ldots$ the elements of $\{0,1\}^n$, except $\underline{0} = (0,\dots,0)$ and $\underline{1} = (1,\dots,1)$, a partial order has been defined as follows

$$\alpha \le \beta \quad \Leftrightarrow \quad \alpha \star \beta = \alpha.$$

If $\alpha \star \beta = \underline{0}$, we say that α and β are orthogonal and write $\alpha \perp \beta$.

The following result holds:

THEOREM 2.41. *Given a function $f\colon \mathbb{R}^n_+ \setminus \{\underline{0}\} \to \{0,1\}^n \setminus \{\underline{0}\}$, for every $\alpha \in \{0,1\}^n \setminus \{\underline{0}\}$, let $A_\alpha = \{\underline{x} \in \mathbb{R}^n_+ \setminus \{\underline{0}\} : f(\underline{x}) = \alpha\}$; obviously, the family $\{A_\alpha\}_\alpha$ is a partition of $\mathbb{R}^n_+ \setminus \{\underline{0}\}$. The function f is a solution of the system* (2.30)–(2.31) *if and only if the following properties are satisfied:*

(i) *for every $\alpha \in \{0,1\}^n \setminus \{\underline{0}\}$, A_α if not empty, is a convex cone;*
(ii) *for every $\alpha, \beta \in \{0,1\}^n \setminus \{\underline{0}\}$, with $\alpha \ne \beta$, $\alpha \not\perp \beta$, we have $A_\alpha + A_\beta \subset A_{\alpha\star\beta}$.*

Unfortunately this result does not describe the solutions of the system (2.30)–(2.31), everything is delegated to the construction of sets satisfying the conditions (i) and (ii) of Theorem 2.41. In the paper [36] a full description of these sets is given for $n = 1,2,3$. Just to give an idea we present here in detail the case $n = 1$ and $n = 2$.

For $n = 1$, there is obviously only one solution: $f(x) = 1$, $x \in \mathbb{R}_+ \setminus \{0\}$.

For $n = 2$, there are three indices: $(1,0), (0,1)$ and $(1,1)$. There are solutions assuming one, two or three values.

(a) One value solutions: $f(\underline{x}) = \alpha$, $\underline{x} \in \mathbb{R}^2_+ \setminus \{\underline{0}\}$, where α is one of the previous three indices.
(b) Two values solutions: there are two different cases depending whether $(1,1) = \underline{1}$ belongs to the range of the solution.
  (b_1) $f(\mathbb{R}^2_+ \setminus \{\underline{0}\}) = \{(0,1),(1,0)\}$.
  Let t be a half-line in $\mathbb{R}^2_+ \setminus \{\underline{0}\}$ from the origin and let V_1, V_2 be the two disjoint convex cones (one possibly empty) whose union is $(\mathbb{R}^2_+ \setminus \{\underline{0}\}) \setminus \{t\}$.
  If both V_1 and V_2 are nonempty, the solutions are

$$f(\underline{x}) = \begin{cases} \alpha, & \underline{x} \in V_1, \\ \underline{1} - \alpha, & \underline{x} \in V_2, \\ \alpha \text{ or } \underline{1} - \alpha, & \underline{x} \in t, \end{cases}$$

  if one of the cones, say V_2, is empty, the solutions are

$$f(\underline{x}) = \begin{cases} \alpha, & \underline{x} \in V_1, \\ \underline{1} - \alpha, & \underline{x} \in t, \end{cases}$$

  where $\alpha \in \{(0,1),(1,0)\}$.
  (b_2) $f(\mathbb{R}^2_+ \setminus \{\underline{0}\}) = \{(1,1),(1,0)\}$ or $f(\mathbb{R}^2_+ \setminus \{\underline{0}\}) = \{(1,1),(0,1)\}$.

Let u be one of the semi-axis of $\mathbb{R}^2_+ \setminus \{\underline{0}\}$, then the solutions are

$$f(\underline{x}) = \begin{cases} \underline{1}, & \underline{x} \in u, \\ (1,0) \text{ or } (0,1), & \underline{x} \in (\mathbb{R}^2_+ \setminus \{\underline{0}\}) \setminus \{u\}. \end{cases}$$

(c) Three values solutions: let t, V_1 and V_2 as in (b), with $V_1, V_2 \neq \emptyset$, then

$$f(\underline{x}) = \begin{cases} \alpha, & \underline{x} \in V_1, \\ \underline{1} - \alpha, & \underline{x} \in V_2, \\ \underline{1}, & \underline{x} \in t, \end{cases}$$

where $\alpha \in \{(0,1),(1,0)\}$.

In the general case the possibility of giving an analogous description seems hopeless, so in the paper [37] has been presented a geometric-combinatorial method for the construction of the solutions, that is for decomposing $\mathbb{R}^n_+ \setminus \{\underline{0}\}$ as union of a finite number of disjoint convex cones satisfying the properties (i) and (ii) of Theorem 2.41. The method proceeds by iteration through n steps and it is too long and complex to be presented here.

2.5. Other types of alternative functional equations in a single unknown function

In this subsection some other alternative equations are presented with related results.

The first equation for functions from $\mathbb{R}$ into $\mathbb{R}$ is

$$f\big(x + f(x)y\big)f(x)f(y)\big[f(x) + f(x)y - f(x)f(y)\big] = 0. \tag{2.32}$$

This equation has been investigated by N. Brillouët-Bellout and J. Brzdęk in [13] and they proved the following

Theorem 2.42. *A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is a solution of equation* (2.32) *if and only if it has one of the following forms:*

(i) $f \equiv 0$;

(ii) *there is $c \in \mathbb{R}$ such that either $f(x) = \max\{cx+1, 0\}$, or $f(x) = cx+1$, $x \in \mathbb{R}$;*

(iii) *there is $\alpha \in (0,+\infty)$ such that $f(x) \leq 1 - \frac{x}{\alpha}$ for $x \in (\alpha,+\infty)$, and $f(x) = 0$ for $x \in (-\infty,\alpha]$;*

(iv) *there is $\beta \in (-\infty,0)$ such that $f(x) \leq 1 - \frac{x}{\beta}$ for $x \in (-\infty,\beta)$, and $f(x) = 0$ for $x \in [\beta,+\infty)$.*

In the paper [14] the authors study a generalization of equation (2.32), namely the equation

$$f\big(x + M(f(x))y\big)f(x)f(y)\big[f\big(x + M(f(x))y\big) - f(x)f(y)\big] = 0, \tag{2.33}$$

where $f\colon X \to \mathbb{R}$, X is a real linear topological space, and $M\colon \mathbb{R} \to \mathbb{R}$ is a continuous and multiplicative function.

At first has been considered the case $M = 1$, so equation (2.33) becomes

$$f(x+y)f(x)f(y)[f(x+y) - f(x)f(y)] = 0 \tag{2.34}$$

and it is proved the following result:

Theorem 2.43. *Let $f\colon X \to \mathbb{R}$ be continuous and let $S = \{x \in X : f(x) \neq 0\}$. Then f is a solution of equation (2.34) if and only if one of the following statements is valid:*

(i) *there is a continuous functional $g\colon X \to \mathbb{R}$ such that $f = \exp \circ g$;*
(ii) *$S + S \subset X \setminus S$.*

In the general case, that is $M\colon \mathbb{R} \to \mathbb{R}$ a continuous and multiplicative function, with $M(\mathbb{R}) \neq \{0\}$, the following theorem holds.

Theorem 2.44. *Let $f\colon X \to \mathbb{R}$ be continuous, and $f(u)f(v)f(u + M(f(u))v) \neq 0$ for some $u, v \in X$. Then f is a solution of equation (2.33) if and only if there exists a continuous linear functional $L\colon X \to \mathbb{R}$ such that:*

(i) *in the case $M(\mathbb{R}) = \{1\}$, $f = \exp \circ L$;*
(ii) *in the case M is odd, $f(x) = M^{-1}(L(x) + 1)$ for $x \in X$, or $f(x) = M^{-1}\big(\max\{L(x) + 1, 0\}\big)$ for $x \in X$;*
(iii) *in the case M is even and $M(\mathbb{R}) \neq \{1\}$, $f(x) = M_0^{-1}\big(\max\{L(x)+1, 0\}\big)$ for $x \in X$, where $M_0 = M_{|(0,+\infty)}$.*

Another alternative equation of Gołąb-Schinzel type has been investigated by J. Brzdęk in [15]:

$$f(x)f(y) \neq 0 \Rightarrow f\Big(xf(y)^k + yf(x)^n\Big) = f(x)f(y), \tag{2.35}$$

where $f\colon I \to \mathbb{R}$, is continuous, I is a proper real interval, k and n are fixed positive integers and $f\big(xf(y)^k + yf(x)^n\big) \in I$ if $f(x)f(y) \neq 0$. The following result holds true:

Theorem 2.45. *In the previous conditions, $f\colon I \to \mathbb{R}$ is a non-constant continuous solution of (2.35) if and only if there are $s \in \mathbb{R} \setminus \{0\}$ and a real interval K such that the function $t\colon K \to \mathbb{R}$, defined by*

$$t(x) = \begin{cases} s(x^n - x^k), & \text{if } n \neq k, \\ sx^n \ln|x|, & \text{if } n = k, x \neq 0, \\ 0, & \text{if } 0 \in K, x = 0, \end{cases}$$

for $x \in K$, is one-to-one and one of the following two conditions is valid:

(i) *$xy \in K$, for every $x, y \in K$, $t(K) = I$, and $f = t^{-1}$;*

(ii) $0 \in K$, $K \subset [0,1)$, $t(K) \in \{I_0^+, I_0^-\}$, *and*

$$f(x) = \begin{cases} t^{-1}(x), & \text{if } x \in t(K), \\ 0, & \text{otherwise}, \end{cases}$$

for every $x \in I$ $(I_0^+ = I \cap (0,+\infty),\ I_0^- = I \cap (-\infty, 0))$.

Furthermore, f is a constant solution if and only if $f(x) \equiv 0$ or, only in the case where $x + y \in I$ for every $x, y \in I$, $f(x) \equiv 1$.

In the case $I = \mathbb{R}$, the only continuous solutions are $f(x) \equiv 0$ and $f(x) \equiv 0$.

We conclude this subsection with the functional equation

$$f(x)f(y) = f(xy)f\Big(\frac{x+y}{2}\Big), \quad f\colon \mathbb{R} \to \mathbb{R}, \tag{2.36}$$

introduced by I. Fenyö and completely solved by W. Benz in [5] and [6]. The equation (2.36) is not an alternative equation, however as is shown in [1] a consequence of equation (2.36) is the alternative equation

$$f(x)f(y) \neq 0 \Rightarrow f\Big(\frac{x^2+y^2}{2}\Big) = f\Big(\frac{x+y}{2}\Big), \tag{2.37}$$

whose only solutions are the constant functions. From this the following theorem was proved ([1, Ch. 6]):

Theorem 2.46. *A function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies equation* (2.36) *if and only if there exists a subset F of $\mathbb{R}$ such that if $x, y \in F$ then both $(x+y)/2$ and xy belong to F and two constants α, β such that*

(i) *if $0 \notin F$, or if $0 \in F$ and $F \cap \mathbb{R}_- \neq \emptyset$, then $f(x) = \alpha\chi_F(x)$ for all $x \in \mathbb{R}$;*

(ii) *if $0 \in F$ and $F \cap \mathbb{R}_- = \emptyset$, then $f(x) = \alpha\chi_F(x)$ for all $x \neq 0$ and $f(0) = \beta$.*

3. Alternative equations in more than one unknown function

The first (as far as we know) alternative functional equation in two unknown functions was presented as an open problem in Kuczma's paper [45], with the phrase

It would be of considerable interest to solve the equation

$$f(x+y) - f(x) - f(y) \neq 0 \Rightarrow g(x+y) - g(x) - g(y) = 0. \tag{3.1}$$

The solution is not known even for functions $f, g\colon \mathbb{R} \to \mathbb{R}$.

Equation (3.1) has been investigated and solved, under certain conditions, in [32] and [33].

Some definitions are needed in order to present the results. Here we use the multiplicative notation as in the two quoted papers.

If $(X,\cdot)$ is a topological group and $(S,\cdot)$ is a group, we say that $X \in \mathcal{R}(S)$ if there exists a fundamental system $\mathcal{U}$ of open neighbourhoods of the identity of X, with the following property:

if $a\colon U \to S$, $U \in \mathcal{U}$, satisfies the equation $a(x)a(y) = a(xy)$ for $x, y, xy \in U$, then there exists $b \in \operatorname{Hom}(X,S)$ such that $b_{|U} = a$.

For any function $\varphi\colon X \to S$ denote

$$\begin{aligned}
&\Omega_\varphi = \{(x,y) \in X \times X : \varphi(xy) \neq \varphi(x)\varphi(y)\},\\
&H_\varphi = X \setminus \big(p_1(\Omega_\varphi) \cup p_2(\Omega_\varphi)\big),\\
&K_\varphi = X \setminus \big(p_1(\Omega_\varphi) \cup p_2(\Omega_\varphi) \cup p_3(\Omega_\varphi)\big),
\end{aligned}$$

where $p_i\colon X \times X \to X$, $i = 1,2,3$, are the maps given by

$$p_1(x,y) = x, \quad p_2(x,y) = y, \quad p_3(x,y) = xy.$$

The sets H_φ and K_φ are either empty or are subgroups of X.

The following theorem holds:

Theorem 3.1. *Let $(S,\cdot)$ be a Hausdorff topological group and $(X,\cdot)$ a connected topological group belonging to $\mathcal{R}(S)$. Furthermore, suppose that either*

(i) *X is locally connected, or*
(ii) *X is separable (i.e. X has a countable dense subset).*

Finally, let Y be a topological group which is a continuous homomorphic image of X and (f,g) be a continuous solution of equation (3.1) *on Y. Then either f or g belongs to* $\operatorname{Hom}(Y,S)$.

Note that the continuity of f and g implies that Ω_f and Ω_g are open sets, hence

$$\begin{aligned}
p_1(\Omega_f) = p_1(\Omega_f^\circ), \quad p_2(\Omega_f) = p_2(\Omega_f^\circ),\\
p_1(\Omega_g) = p_1(\Omega_g^\circ), \quad p_2(\Omega_g) = p_2(\Omega_g^\circ),
\end{aligned}$$

so it is natural to investigate the equation (3.1) under these weaker than continuity conditions. This has been done always in [33] in the special case when $X = \mathbb{R}^n$, under the condition that one of the functions, say g, has the property

$$p_1(\Omega_g) = p_1(\Omega_g^\circ), \quad p_2(\Omega_g) = p_2(\Omega_g^\circ). \tag{3.2}$$

Assuming that (f,g) is a nontrivial solution of (3.1), we have that $H_g \neq \emptyset$, $H_g \neq \mathbb{R}^n$ and H_g is a proper closed subgroup of $\mathbb{R}^n$. It is known(see, for instance, [12, Ch. VII, § 1.2, Corollary 2]) that given a proper closed subgroup $\mathcal{H}$ of $\mathbb{R}^n$, there exists a bicontinuous isomorphism j of $\mathbb{R}^n$ to itself such that

$$j(\mathcal{H}) = \mathbb{R}^p \times \mathbb{Z}^q \times \{0\}^{n-p-q}, \quad 0 \le p \le n-1,\ 0 \le q \le n,\ 0 \le p+q \le n.$$

Since (f, g) is a solution of (3.1) if and only if also $(f \circ j, g \circ j)$ is, we may assume without loss of generality that

$$H_g = \mathbb{R}^p \times \mathbb{Z}^q \times \{0\}^{n-p-q}.$$

The following result holds ([33]):

Theorem 3.2. *Let $f, g\colon \mathbb{R}^n \to S$ be a pair of functions. Assume that g satisfies condition* (3.2). *The pair (f, g) is a non-trivial solution of equation* (3.1) *if and only if $f = \overline{f} \circ j$, $g = \overline{g} \circ j$, where j is an isomorphism of $\mathbb{R}^n$ into itself and $\overline{f}$, $\overline{g}$ have one of the following forms:*

(i)
$$\overline{f}(x) = \begin{cases} f_0(x), & x \in \mathbb{R}^p \times \mathbb{Z}^q \times \{0\}^{n-p-q}, \\ a(x), & x \in \mathbb{R}^n \setminus (\mathbb{R}^p \times \mathbb{Z}^q \times \{0\}^{n-p-q}), \end{cases}$$

and $\overline{g} = g_1 \otimes g_2$, where $p < n$, $a \in \mathrm{Hom}(\mathbb{R}^n, S)$, $f_0\colon \mathbb{R}^p \times \mathbb{Z}^q \times \{0\}^{n-p-q} \to S$ is an arbitrary function with $f_0 \ne a$, $g_1 \in \mathrm{Hom}(\mathbb{R}^p, S)$ and $g_2\colon \mathbb{R}^{n-p} \to S$ is an odd function satisfying

$$g_2(x+y) = g_2(x)g_2(y) = g_2(y)g_2(x), \quad x \in \mathbb{R}^{n-p},\ y \in \mathbb{Z}^q \times \{0\}^{n-p-q};$$

(ii)
$$\overline{f}(x) = \begin{cases} a(x), & x \in V_1, \\ \alpha a(x), & x \in \overline{V_2}, \end{cases} \qquad \overline{g}(x) = \begin{cases} \gamma c(x), & x \in V_1, \\ c(x), & x \in \overline{V_2}, \end{cases}$$

or

$$\overline{f}(x) = \begin{cases} \alpha a(x), & x \in \overline{V_1}, \\ a(x), & x \in V_2, \end{cases} \qquad \overline{g}(x) = \begin{cases} c(x), & x \in \overline{V_1}, \\ \gamma c(x), & x \in V_2, \end{cases}$$

where $V_1 = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$. $V_2 = -V_1$, $a, c \in \mathrm{Hom}(\mathbb{R}^n, S)$, $\alpha \ne e$, $\gamma \ne e$ (e is the identity of S), $\alpha a(x) = a(x)\alpha$, $\gamma c(x) = c(x)\gamma$, for all $x \in \mathbb{R}^n$;

(iii)
$$\begin{cases} \overline{f}(x) = \alpha^k a(x), \\ \overline{g}(x) = \gamma^{k+1} c(x), \end{cases} \qquad x \in V_k \cup (\mathbb{R}^{n-1} \times \{(k+1)i_n\})$$

or

$$\begin{cases} \overline{f}(x) = \alpha^{k+1} a(x), \\ \overline{g}(x) = \gamma^k c(x), \end{cases} \qquad x \in V_k \cup (\mathbb{R}^{n-1} \times \{k i_n\})$$

where $V_k = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : k < x_n < k+1\}$, $k \in \mathbb{Z}$, $a, c \in \mathrm{Hom}(\mathbb{R}^n, S)$, $\alpha \ne e$, $\gamma \ne e$, $\alpha a(x) = a(x)\alpha$, $\gamma c(x) = c(x)\gamma$, for all $x \in \mathbb{R}^n$.

Corollary 3.2.1. *Let X be a topological group and $\sigma\colon \mathbb{R}^n \to X$ be a surjective continuous open homomorphism. The pair (f,g) is a solution of equation* (3.1) *on X, satisfying the condition* (3.2) *if and only if*

$$f(x) = \overline{f}(\sigma^{-1}(x)), \quad g(x) = \overline{g}(\sigma^{-1}(x)),$$

where $(\overline{f},\overline{g})$ is a solution of (3.1) *on $\mathbb{R}^n$, $\overline{f}$ and $\overline{g}$ are constant on the cosets of $\sigma^{-1}(0)$ and $\overline{g}$ satisfies the condition* (3.2).

The following example shows that without any topological condition on the sets Ω_f and Ω_g, the class of solutions of equation (3.1) may be extremely large and complex. Let $X = S = \mathbb{R}$, and let H be a Hamel basis of $\mathbb{R}$. Fix $h_0 \in H$, so every $x \in \mathbb{R}$ is uniquely representable in the form

$$x = c_0(x)h_0 + \sum_{h_\alpha \neq h_0} c_\alpha(x)h_\alpha, \quad h_\alpha \in H.$$

Define

$$f(x) = [c_0(x)], \quad g(x) = [c_0(x)] + 1 \quad ([t] \text{ is the integral part of } t);$$

the pair (f,g) is a solution of (3.1) and the sets Ω_f and Ω_g are disjoint and dense in $\mathbb{R}^2$, so $\Omega_f^\circ = \Omega_g^\circ = \emptyset$.

Local forms of equation (3.1) have been investigated in the papers [34], [35] and [55].

Now, restricting the investigations to functions from $\mathbb{R}$ into $\mathbb{R}$, it is rather natural to ask what happens if equation (3.1), which can now be written in the form

$$[f(x+y) - f(x) - f(y)][g(x+y) - g(x) - g(y)] = 0, \quad x, y \in \mathbb{R}, \tag{3.3}$$

is extended to the case of three functions, that is the equation

$$[f(x+y)-f(x)-f(y)][g(x+y)-g(x)-g(y)][h(x+y)-h(x)-h(y)] = 0, \tag{3.4}$$

with $f, g, h\colon \mathbb{R} \to \mathbb{R}$.

In this case the situation is completely different: there exist C^∞ functions solving (3.4) and neither of them is additive on the whole $\mathbb{R}$. An example is given as follows.

Define

$$f_0(x) = \exp\left\{ -\frac{1}{x-1} + \frac{1}{x-2} \right\} + x, \quad 1 < x < 2,$$

$$g_0(x) = \exp\left\{ -\frac{1}{x-4} \right\}, \quad x > 4,$$

$$h_0(x) = \exp\left\{ -\frac{1}{x} + \frac{1}{x-1} \right\} + x, \quad 0 < x < 1,$$

then we construct these other functions:

$$f(x)=\begin{cases} x, & |x|\le 1,\ |x|\ge 2,\\ f_0(x), & 1<x<2,\\ -f_0(x), & -2<x<-1,\end{cases}$$

$$g(x)=\begin{cases} -g_0(x), & x<-4,\\ 0, & -4\le x\le 4,\\ g_0(x), & x>4,\end{cases}$$

$$h(x)=\begin{cases} x, & |x|\ge 1,\\ h_0(x), & 0<x<1,\\ 0, & x=0,\\ -h_0(x), & -1<x<0.\end{cases}$$

These three functions are in $C^\infty(\mathbb{R})$, neither of them is in $\mathrm{Hom}(\mathbb{R},\mathbb{R})$ and the triple (f,g,h) is a solutions of equation (3.4). Thus, while continuity is enough to guarantee that equation (3.3) has only trivial solutions, even being in C^∞ is not enough for (3.4).

We obtain only trivial solutions by further increasing the regularity of al least one of the functions, indeed the following theorem holds (see [28]):

Theorem 3.3. *Assume that the triple* (f,g,h) *is a solution of equation* (3.4), *with* g *and* h *continuous and not additive, and* f *real analytic, then* f *is additive, that is* (3.4) *has only trivial solutions.*

If $f\in C^1(\mathbb{R})$, we can write it in the form

$$f(x)=\int_0^x \phi(t)\ dt+F_1x+F_2,$$

where $F_1=f'(0)$, $F_2=f(0)$, and ϕ is any continuous function with $\phi(0)=0$ (see [20]).

By using this representation for the three functions f,g and h, the Cauchy differences of the C^1 functions assume the following forms:

$$f(x+y)-f(x)-f(y)=\int_0^x[\phi(t+y)-\phi(t)]dt-F_2=\int_0^x\Phi_y(t)dt-F_2,$$
$$g(x+y)-g(x)-g(y)=\int_0^x[\gamma(t+y)-\gamma(t)]dt-G_2=\int_0^x\Gamma_y(t)dt-G_2,$$
$$h(x+y)-h(x)-h(y)=\int_0^x[\psi(t+y)-\psi(t)]dt-H_2=\int_0^x\Psi_y(t)dt-H_2.$$

Since for every solution (f,g,h) of (3.4) we must have $F_2\cdot G_2\cdot H_2=0$, now we consider the case $F_2=G_2=H_2=0$, thus equation (3.4) becomes

$$\int_0^x\Phi_y(t)dt\cdot\int_0^x\Gamma_y(t)dt\cdot\int_0^x\Psi_y(t)dt=0. \tag{3.5}$$

In the paper [29] some classes of solutions of equation (3.5) (hence of equation (3.4)) are presented. In order to describe some of those results some notations are needed.

$$A_\phi = \{x \in \mathbb{R} : \phi(x) \neq 0\}, \quad A_\gamma = \{x \in \mathbb{R} : \gamma(x) \neq 0\},$$
$$A_\psi = \{x \in \mathbb{R} : \psi(x) \neq 0\},$$
$$m_\phi = \inf A_\phi, \quad M_\phi = \sup A_\phi, \quad m_\gamma = \inf A_\gamma, \quad M_\gamma = \sup A_\gamma,$$
$$m_\psi = \inf A_\psi, \quad M_\psi = \sup A_\psi.$$

The first theorem considers the case

$$m_\phi < M_\phi \leq m_\gamma < M_\gamma \leq m_\psi < M_\psi$$

and says the following:

THEOREM 3.4. *Under each of the following conditions the triple* (ϕ, γ, ψ) *is a solution of equation* (3.5)*:*

(i) $m_\phi \geq 0$, $m_\psi \geq M_\phi + M_\gamma$ *and*

$$\int_{m_\phi}^{M_\phi} \phi(t)dt = \int_{m_\gamma}^{M_\gamma} \gamma(t)dt = 0;$$

(ii) $M_\psi \leq 0$, $M_\psi \leq m_\phi + m_\gamma$ *and*

$$\int_{m_\psi}^{M_\psi} \psi(t)dt = \int_{m_\gamma}^{M_\gamma} \gamma(t)dt = 0;$$

(iii) $-\infty < m_\phi < 0 < M_\phi \leq m_\gamma < M_\gamma \leq m_\psi < M_\psi \leq +\infty$, $M_\gamma \leq \min\{m_\psi - M_\phi, m_\phi + m_\psi\}$ *and*

$$\int_0^{M_\phi} \phi(t)dt = \int_{m_\phi}^0 \phi(t)dt = \int_{m_\gamma}^{M_\gamma} \gamma(t)dt = 0;$$

(iv) $-\infty < m_\phi < M_\phi < 0 < m_\gamma < M_\gamma < m_\psi < M_\psi \leq +\infty$, $m_\phi \geq M_\gamma - m_\psi$ *and*

$$\int_{m_\phi}^{M_\phi} \phi(t)dt = \int_{m_\gamma}^{M_\gamma} \gamma(t)dt = 0;$$

(v) $-\infty \leq m_\phi < M_\phi < 0 \leq m_\gamma < M_\gamma \leq m_\psi < M_\psi < +\infty$, $M_\phi \leq m_\gamma - M_\psi$ *and*

$$\int_{m_\psi}^{M_\psi} \psi(t)dt = \int_{m_\gamma}^{M_\gamma} \gamma(t)dt = 0;$$

(vi) $-\infty < m_\phi < M_\phi \le m_\gamma < 0 < M_\gamma \le m_\psi < M_\psi \le +\infty$, $m_\psi \ge M_\gamma - m_\phi$ *and*

$$\int_{m_\phi}^{M_\phi} \phi(t)dt = \int_{m_\gamma}^{0} \gamma(t)dt = \int_{0}^{M_\gamma} \gamma(t)dt = 0;$$

(vii) $-\infty \le m_\phi < M_\phi \le m_\gamma < 0 < M_\gamma \le m_\psi < M_\psi < +\infty$, $M_\phi \le M_\gamma - M_\psi$ *and*

$$\int_{m_\phi}^{M_\phi} \phi(t)dt = \int_{m_\gamma}^{0} \gamma(t)dt = \int_{0}^{M_\gamma} \gamma(t)dt = 0;$$

(viii) $-\infty \le m_\phi < M_\phi \le m_\gamma < M_\gamma < 0 < m_\psi < M_\psi < +\infty$, $M_\phi \le m_\gamma - M_\psi$ *and*

$$\int_{m_\gamma}^{M_\gamma} \gamma(t)dt = \int_{m_\psi}^{M_\psi} \psi(t)dt = 0;$$

(ix) $-\infty < m_\phi < M_\phi \le m_\gamma < M_\gamma < 0 < m_\psi < M_\psi \le +\infty$, $M_\gamma \le m_\phi + m_\psi$ *and*

$$\int_{m_\gamma}^{M_\gamma} \gamma(t)dt = \int_{m_\psi}^{M_\psi} \psi(t)dt = 0;$$

(x) $-\infty < m_\phi < M_\phi \le m_\gamma < M_\gamma \le m_\psi < 0 < M_\psi \le +\infty$, $M_\phi \le \min\{m_\gamma - M_\psi, m_\gamma + m_\psi\}$ *and*

$$\int_{0}^{M_\psi} \psi(t)dt = \int_{m_\psi}^{0} \psi(t)dt = \int_{m_\gamma}^{M_\gamma} \gamma(t)dt = 0;$$

(xi) $-\infty \le m_\phi < M_\phi \le m_\gamma < M_\gamma \le m_\psi < 0 < M_\psi < +\infty$, $M_\phi \le m_\gamma - M_\psi$ *and*

$$\int_{0}^{M_\psi} \psi(t)dt = \int_{m_\psi}^{0} \psi(t)dt = \int_{m_\gamma}^{M_\gamma} \gamma(t)dt = 0.$$

Other solutions are given by the following two theorems.

THEOREM 3.5. *Assume* $m_\psi < m_\phi < M_\phi \le 0 \le m_\gamma < M_\gamma < M_\psi$, $m_\psi < m_\phi - M_\gamma$, $M_\psi > M_\gamma - m_\phi$, $\psi(x) = 0$ *for* $x \in [a, b]$, *where* $m_\psi < a \le m_\phi - M_\gamma$, $M_\gamma - m_\phi \le b < M_\psi$. *Moreover, suppose*

$$\int_{m_\phi}^{M_\phi} \phi(t)dt = \int_{m_\gamma}^{M_\gamma} \gamma(t)dt = 0.$$

Then the triple (ϕ, γ, ψ) *is a solution of equation* (3.5).

THEOREM 3.6. *Assume* $-\infty \le m_\psi < m_\phi < m_\gamma < 0 < M_\gamma < M_\phi < M_\psi \le +\infty$, $\phi(x) = 0$ *for* $x \in [m_\gamma, M_\gamma]$,

$$\int_0^{m_\phi} \phi(t)dt = \int_0^{M_\phi} \phi(t)dt = \int_0^{m_\gamma} \gamma(t)dt = \int_0^{M_\gamma} \gamma(t)dt = 0,$$

and $\psi(x) = 0$ *for* $x \in [a, b]$, *with* $a \le \min\{m_\phi - M_\gamma, m_\gamma - M_\phi\}$ *and* $b \ge \max\{M_\gamma - m_\phi, M_\gamma + M_\phi\}$. *Then the triple* (ϕ, γ, ψ) *is a solution of equation* (3.5).

By using the integral representation of regular functions from $\mathbb{R}$ into $\mathbb{R}$, it is possible to study two other alternative equations in two unknown functions.

The first equation is the following

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) \neq 0 \\ \Rightarrow g(x+y) + g(x-y) - 2g(x) - 2g(y) = 0, \tag{3.6}$$

this is the analogue of equation (3.3), where instead of Cauchy equations we have quadratic equations.

Assuming that $f, g \in C^2(\mathbb{R})$, we can write

$$f(x) = \int_0^x \tau(t)dt + \frac{C_1}{2}x^2 + C_2 x + C_3, \tag{3.7}$$

where $f(0) = C_3$, $\tau \in C^1(\mathbb{R}$ with $\tau(0) = \tau'(0) = 0$, and

$$g(x) = \int_0^x \sigma(t)dt + \frac{D_1}{2}x^2 + D_2 x + D_3, \tag{3.8}$$

where $g(0) = D_3$, $\sigma \in C^1(\mathbb{R}$ with $\sigma(0) = \sigma'(0) = 0$.

From (3.7) and (3.8) we can compute the quadratic differences of f and g, obtaining

$$f(x+y)+f(x-y)-2f(x)-2f(y) = \int_0^y [\tau(x+t)+\tau(x-t)-2\tau(t)]dt - 2C_2 y + 2C_3,$$

$$g(x+y)+g(x-y)-2g(x)-2g(y) = \int_0^y [\sigma(x+t)+\sigma(x-t)-2\sigma(t)]dt - 2D_2 y + 2D_3,$$

thus equation (3.6) becomes

$$\Big[\int_0^y [\tau(x+t) + \tau(x-t) - 2\tau(t)]dt - 2C_2 y + 2C_3\Big] \\ \times \Big[\int_0^y [\sigma(x+t) + \sigma(x-t) - 2\sigma(t)]dt - 2D_2 y + 2D_3\Big] = 0.$$

By analyzing the various possibilities for the coefficients C_2, C_3, D_2, D_3, and by using the following

Lemma 3.7. *Let* $F_n\colon \mathbb{R}^2 \to \mathbb{R}$, $n = 1, \dots, N$, *be continuous functions. If* $\prod_{n=1}^{N} \int_a^x F_n(t, y)dt = 0$, *for all* $(x, y) \in \mathbb{R}^2$ *and some* $a \in \mathbb{R}$, *then* $\prod_{n=1}^{N} F_n(x, y) = 0$, *for all* $(x, y) \in \mathbb{R}^2$.

we can prove the following

Theorem 3.8. *The equation* (3.6) *has only trivial solutions of class* C^2, *that is either* $f(x) = \alpha x^2$, $x \in \mathbb{R}$, *or* $g(x) = \beta x^2$, $x \in \mathbb{R}$, *for some* $\alpha, \beta \in \mathbb{R}$.

The second equation connects additive and quadratic equations:

$$[f(x+y) - f(x) - f(y)] \times [g(x+y) + g(x-y) - 2g(x) - 2g(y)] = 0. \tag{3.9}$$

In this case we assume $f \in C^1(\mathbb{R})$ and $g \in C^2(\mathbb{R})$ and obtain the following

Theorem 3.9. *The equation* (3.9) *has only trivial solutions, that is either* $f(x) = \alpha x$, $x \in \mathbb{R}$, *or* $g(x) = \beta x^2$, $x \in \mathbb{R}$, *for some* $\alpha, \beta \in \mathbb{R}$ *(see* [30]*).*

4. Conclusions and open problems

As is clear from the previous sections, the investigation of alternative (or conditional with the condition given by the unknown function(s)) is alive and well and works have constantly been appearing for more than 60 years. The equations which have been studied are mostly related to the additive Cauchy equation and, more recently, to the quadratic or Jordan-von Neumann equation. It would be interesting to investigate alternative equations where other Cauchy equations are involved.

Reading the previous sections, a series of open problems naturally appears.

The functional equation

$$[c(f(x+y) - af(x) - bf(y) - d][f(x+y) - f(x) - f(y)] = 0$$

has been investigated under the assumption that $f\colon G \to R$ where G is an abelian group and R a domain of integrity (see Theorem 2.12), but we do not have any general result when f is defined on a non commutative group.

The same problem appears for Cauchy differences assuming several values.

Some of the previous alternative equations involving the Cauchy equation were solved by using as fundamental tool the Ulam-Hyers stability. Thus a natural problem is to investigate those equations when the additive equation is not stable. The analogous question can be posed for the quadratic equation.

Going to equation (2.29), again the result is complete for commutative groups, while the problem is open in case of non commutativity.

Looking at the results for the plurality function, we see that the crucial point is a special decomposition of $\mathbb{R}^n$ (see Theorem 2.41) and it would be very interesting to give a description (not only an iterative method) of these decompositions.

Concerning the alternative equations with more than one unknown function, it is shown after Corollary 3.2.1, that equation (3.1) in a purely algebraic setting can have infinitely many solutions: the problem of giving a full description of the solutions on a general group is completely open.

Finally, equations (3.6) and (3.9) have been solved under the assumption of a certain regularity of the two functions f and g: what can be said if f and g are only continuous functions? Furthermore, what happens in a purely algebraic setting? We know that in this last case, depending on the groups involved, there are other solutions, not only the trivial ones. As an example, we have that f and $g = f + \frac{1}{2}$, where f is given in Theorem 2.29, satisfy equation (3.6).

Moreover, if we take $f\colon \mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$, given by

$$f(k) = \begin{cases} 0 + 8\mathbb{Z}, & \text{if } k \equiv 0 \pmod 4, \\ 1 + 8\mathbb{Z}, & \text{if } k \equiv 1 \pmod 4, \\ 4 + 8\mathbb{Z}, & \text{if } k \equiv 2 \pmod 4, \\ 5 + 8\mathbb{Z}, & \text{if } k \equiv 3 \pmod 4, \end{cases}$$

then the pair (f, f) is a non trivial solution of equation (3.9).

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Dipartimento di Matematica
Università degli Studi di Milano
via C. Saldini 50
20133 Milano
Italy
e-mail: gianluigi.forti@unimi.it