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ON POWERS, ROOTS AND MOORE–PENROSE INVERSES OF MATRICES VIA GENERALIZED FIBONACCI NUMBERS

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Abstract. The purpose of this work is to introduce some new results about the relations between powers, roots and Moore–Penrose inverses of square matrices satisfying a cubic matrix equation and the generalized Fibonacci numbers. The results can also be used for rectangular matrices. Moreover, we give some numerical examples to verify theoretical results.

1. Introduction and preliminaries

The sequence $\{F_n\}$ defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$, where $F_0 = 0$ and $F_1 = 1$ is the well-known Fibonacci sequence. Similarly, the sequence $\{L_n\}$ defined by $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$, where $L_0 = 2$ and $L_1 = 1$ is the Lucas sequence and the sequence $\{M_n\}$ defined by $M_n = 3M_{n-1} - 2M_{n-2}$, $n \geq 2$ with the initial conditions $M_0 = 0$ and $M_1 = 1$ is the Mersenne sequence; see [2, 3, 9, 16].

Throughout the study $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ will denote, respectively, the set of natural numbers, the set of integers, the field of real numbers and the field of complex numbers. Moreover $\mathcal{M}_{k\times l}(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, will denote the set of all $k \times l$ matrices over the field $\mathbb{F}$, and I will denote the appropriately sized unit matrix.

The *generalized Fibonacci sequence* $\{U_n\}$ and the *generalized Lucas sequence* $\{V_n\}$ are, respectively, defined by $U_n = pU_{n-1} + qU_{n-2}$ and $V_n = pV_{n-1} + qV_{n-2}$, $n \in \mathbb{N}$, $n \geq 2$, where $U_0 = 0$, $U_1 = 1$, $V_0 = 2$ and $V_1 = p$ with $p, q \in \mathbb{R}\backslash\{0\}$. Based on these, *negatively indexed generalized Fibonacci* and *generalized Lucas numbers* are defined by $U_{-n} = -\frac{U_n}{(-q)^n}$ and $V_{-n} = \frac{V_n}{(-q)^n}$,

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respectively, for all $n \in \mathbb{N}$. Meanwhile, under the condition that $p^2 + 4q > 0$ we have $U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n = \alpha^n + \beta^n$, known as *Binet formulas* for all $n \in \mathbb{Z}$, where $\alpha = \left(p + \sqrt{p^2 + 4q}\right)/2$ and $\beta = \left(p - \sqrt{p^2 + 4q}\right)/2$ are the solutions of the characteristic equation $x^2 - px - q = 0$. The Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ are, respectively, the particular cases of the sequences $\{U_n\}$ and $\{V_n\}$ for $p = q = 1$. On the other hand, the Mersenne sequence $\{M_n\}$ is the particular case of the sequence $\{U_n\}$ for $p = 3$ and $q = -2$. Also, it can be easily seen that $M_n = 2^n - 1$ for all $n \in \mathbb{Z}$ by using *Binet formula*; see [2, 3, 8, 13, 14].

There are many identities in the related literature about generalized Fibonacci and Lucas numbers; see [8, 13, 14]. Some important identities that we will use in this work are given below. Under the conditions that $p, q \in \mathbb{R}\backslash\{0\}$ and $p^2 + 4q > 0$, the following identities hold for all $m, n, r \in \mathbb{Z}$ [14].

$$U_m V_n = U_{m+n} + (-q)^n U_{m-n}, \tag{1.1}$$

$$V_n = U_{n+1} + qU_{n-1}, \tag{1.2}$$

$$U_{n+r}U_{n-r} - U_n^2 = -(-q)^{n-r}U_r^2, \tag{1.3}$$

$$U_{m+n} = U_m U_{n+1} + qU_{m-1}U_n. \tag{1.4}$$

A matrix $A \in \mathcal{M}_{m\times m}(\mathbb{R})$ that satisfies the equality $A^n = B$ is called a nth root of the matrix $B \in \mathcal{M}_{m\times m}(\mathbb{R})$, where n is a positive integer [11].

The roots of matrices need to be calculated in solving some problems in different fields such as statistics, economics, and healthcare. For example, the Markov chain model is used in finance and healthcare. The transitions of a Markov chain can be described by a stochastic matrix, and determining the stochastic nth root of a stochastic matrix is an important problem for Markov chain models. For this reason, matrix roots are in the field of interest of those working in the areas mentioned above and different methods of obtaining them have been offered in literature; see [5–7, 15, 17].

It is known that the problems that arise in applied sciences basically include the solutions or the functions of the solutions of the system of linear equations $Ax = g$, where $A \in \mathcal{M}_{m\times l}(\mathbb{R})$, $x \in \mathcal{M}_{l\times 1}(\mathbb{R})$ is a vector of unknowns, and $g \in \mathcal{M}_{m\times 1}(\mathbb{R})$ is a vector of known values. On the other hand, if A is an $n \times n$ nonsingular matrix, then the system has the unique solution $x = A^{-1}b$. However, there are cases where A is singular or not a square matrix. In these cases, the system may have a unique solution or infinitely many solutions. That's why a general theory that lays out all these is desired. One such theory involves the use of generalized and Moore–Penrose inverses of matrices [4]. Moreover, note that there are, of course, different methods to calculate the Moore–Penrose inverse of a matrix; see [1, 4, 12].

A matrix $G \in \mathcal{M}_{l\times m}(\mathbb{R})$ is called a *generalized inverse* of a matrix $A \in \mathcal{M}_{m\times l}(\mathbb{R})$ if $AGA = A$ [1]. For any $A \in \mathcal{M}_{m\times l}(\mathbb{R})$, there is a unique matrix $A^\dagger \in \mathcal{M}_{l\times m}(\mathbb{R})$ satisfying the conditions $(AA^\dagger)^T = AA^\dagger$, $(A^\dagger A)^T = A^\dagger A$,

$AA^{\dagger}A = A$, and $A^{\dagger}AA^{\dagger} = A^{\dagger}$, where the matrix A^T is the transpose of the matrix A and the matrix $A^{\dagger}$ is known as the *Moore–Penrose inverse* of the matrix A. Moreover, the property

$$A^{\dagger} = (A^TA)^{\dagger}A^T = A^T(AA^T)^{\dagger} \tag{1.5}$$

is a well-known fact [1,4].

The aim of the study is to give results which can be used to calculate positive integer powers, roots and Moore–Penrose inverses of some class of matrices, more specifically, the class of matrices $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfying the matrix equation $X^3 - V_nX^2 + (-q)^n X = \mathbf{0}$, via generalized Fibonacci numbers and to support these results with some numerical examples.

Before giving the main results, it will be beneficial to introduce some terminologies which will be used in the subsequent lines. For an $X \in \mathcal{M}_{m\times m}$, $p_X(\lambda) = \det(\lambda I - X)$ is the *characteristic polynomial* of X. The *eigenvalues* of X are the roots of the *characteristic equation* $p_X(\lambda) = 0$ or, equivalently, are the zeros of the characteristic polynomial $p_X(\lambda)$. Well-known *Cayley-Hamilton theorem* states that $p_X(X) = \mathbf{0}$. The *minimal polynomial* of X is the monic polynomial $m_X(\lambda)$ *of least degree* such that $m_X(X) = \mathbf{0}$. If $P(\lambda)$ is a polynomial such that $P(X) = \mathbf{0}$ then the minimal polynomial $m_X(\lambda)$ divides $P(\lambda)$. Moreover, $m_X(\lambda)$ divides $p_X(\lambda)$ and, $m_X(\lambda)$ and $p_X(\lambda)$ have the same zeros. All the eigenvalues of a real symmetric matrix X are real.

Now consider the polynomial $P(\lambda) = \lambda^3 - V_n\lambda^2 + (-q)^n \lambda$, $n \in \mathbb{N}\cup\{0\}$ and let $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfies $P(X) = \mathbf{0}$ for some $n \in \mathbb{N}\cup\{0\}$. The polynomial P can be written as $P(\lambda) = (\lambda - \phi_1)(\lambda - \phi_2)\lambda$, where $\phi_1 = \left(V_n + \sqrt{V_n^2 - 4(-q)^n}\right)/2$ and $\phi_2 = \left(V_n - \sqrt{V_n^2 - 4(-q)^n}\right)/2$. So, $\sigma_X \subseteq \{\phi_1, \phi_2, 0\}$, where σ_X is the *spectrum* of the matrix X.

As we have stated earlier $\alpha = \left(p + \sqrt{p^2+4q}\right)/2$, $\beta = \left(p - \sqrt{p^2+4q}\right)/2 \in \mathbb{R}$, where $p, q \in R$, $pq \neq 0$ and $p^2 + 4q > 0$, are the roots of the characteristic equation $x^2 - px - q = 0$. So, $\alpha + \beta = p$ and $\alpha\beta = -q$. Then using Binet formula $V_n = \alpha^n + \beta^n$ for all $n \in \mathbb{Z}$, we obtain

$$\begin{aligned} V_n^2 - 4(-q)^n &= (\alpha^n + \beta^n)^2 - 4(-q)^n \\ &= \alpha^{2n} + \beta^{2n} + 2(\alpha\beta)^n - 4(-q)^n \\ &= \alpha^{2n} + \beta^{2n} - 2(\alpha\beta)^n \\ &= (\alpha^n - \beta^n)^2 \geq 0. \end{aligned}$$

Since $\alpha \neq \pm\beta$, $V_n^2 - 4(-q)^n = 0$ only if $n = 0$ and $n = 0$ leads to $V_0 = 2$, it follows that $\phi_1 = \phi_2 = \dfrac{V_0}{2} = 1 \in \mathbb{R}$ and hence, $P(\lambda) = \lambda(\lambda - 1)^2$. So, if $V_n^2 - 4(-q)^n = 0$, then $n = 0$ and $m_X(\lambda) \in \mathcal{S}_1$, where

$$\mathcal{S}_1 = \left\{\lambda, (\lambda - 1), \lambda(\lambda - 1), \lambda(\lambda - 1)^2\right\}.$$

On the other hand, for $n \in \mathbb{N}$ we have $V_n^2 - 4(-q)^n > 0$. Then $\phi_1, \phi_2 \in \mathbb{R}\backslash\{0\}$ and $\phi_1 \neq \phi_2$. So, $P(\lambda) = \lambda(\lambda - \phi_1)(\lambda - \phi_2)$. Therefore, $m_X(\lambda) \in \mathcal{S}_2$, where

$$S_2 = \{\lambda, (\lambda - \phi_i), \lambda(\lambda - \phi_i), (\lambda - \phi_i)(\lambda - \phi_j), \lambda(\lambda - \phi_i)(\lambda - \phi_j) : \\ i, j = 1, 2, i \neq j\}.$$

So, for any given $X \in \mathcal{M}_{m\times m}$, if $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$ for some $n \in \mathbb{N}\cup\{0\}$, then $m_X(\lambda) \in \mathcal{S}_1$ or $m_X(\lambda) \in \mathcal{S}_2$. Conversely, if $m_X(\lambda) \in \mathcal{S}_1$ or $m_X(\lambda) \in \mathcal{S}_2$, then $m_X(\lambda)$ divides $P(\lambda)$ therefore $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$.

Having characterized the matrices satisfying the matrix equation $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$, we can now begin to give our main results.

The relationships between some sequences and matrices have been investigated in different studies. In [14], it has been shown that the integer powers of the matrices $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfying the matrix equation $X^2 - pX - qI = \mathbf{0}$ can be found using the generalized Fibonacci sequence $\{U_n\}$ and in this way algebraic properties of the sequence $\{U_n\}$ are obtained. Based on these, in [10] it has been shown that the integer powers of the matrices satisfying $X^2 - V_n X + (-q)^n I = \mathbf{0}$ can be obtained using the sequence $\{U_n\}$ and some applications have been given.

In this study we will deal with powers of matrices $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfying a higher degree matrix equation, namely, $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$. First, we will give a relation between generalized Fibonacci numbers and positive integer powers of the matrices satisfying the matrix equation. Then, we will develop a formula for the nth roots of such matrices. It is clear that the matrices $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfying the matrix equation $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$ may be singular or not. For that reason, in the last part of the study we will give some results which can be used for finding the Moore–Penrose inverses of such matrices. Note that the Moore–Penrose inverse applies not only to square matrices, but also to rectangular matrices. In the rest of the work it will be assumed that $p, q \in \mathbb{R}\backslash\{0\}$ and $p^2 + 4q > 0$.

2. Results

The main result given below shows the relationship between generalized Fibonacci numbers and positive integer powers of matrices $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfying the third-degree matrix equation given above and is key to the rest of the work.

THEOREM 2.1. *If a matrix $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfies the equation $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$ for some $n \in \mathbb{N}$, then*

$$X^k = \frac{1}{U_n}\left(U_{nk-n}X - (-q)^n U_{nk-2n} I\right) X \quad \textit{for all } k \in \mathbb{N}.$$

Proof. Proof is obtained by using induction method with equality (1.1) for $m = nk - n$. □

It is clear that for $n = 1$, we have $V_1 = p$ and Theorem 2.1 provides the following formula for the positive integer powers of the matrix X.

Corollary 2.2. *If $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfies the equation $X^3 - pX^2 - qX = \mathbf{0}$, then $X^k = U_{k-1}X^2 + qU_{k-2}X$ for all $k \in \mathbb{N}$.*

Example 2.3. Let

$$X = \begin{pmatrix} 4 & 2 & 2 & 4 \\ 2 & -1 & -3 & -1 \\ 2 & -3 & -1 & -1 \\ -4 & 1 & 1 & -1 \end{pmatrix}.$$

Then, since $p_X(\lambda) = \lambda(\lambda+3)(\lambda-2)^2$ and hence $m_X(\lambda) = \lambda(\lambda+3)(\lambda-2) = \lambda^3+\lambda^2-6\lambda$, it is easily seen that $X^3+X^2-6X = \mathbf{0}$. Therefore, $X^3 - V_nX^2 + (-q)^n X = \mathbf{0}$ if $V_n = -1$ and $(-q)^n = -6$ for some $n \in \mathbb{N}$. Now, if we let, for example, $n = 1$, then $V_1 = p = -1$ and $q = 6$. Then, $U_k = -U_{k-1} + 6U_{k-2}$ and hence, by Corollary 2.2,

$$X^k = U_{k-1}X^2 + qU_{k-2}X =$$

$$\begin{pmatrix} 8U_{k-1} + 24U_{k-2} & 4U_{k-1} + 12U_{k-2} & 4U_{k-1} + 12U_{k-2} & 8U_{k-1} + 24U_{k-2} \\ 4U_{k-1} + 12U_{k-2} & 13U_{k-1} - 6U_{k-2} & 9U_{k-1} - 18U_{k-2} & 13U_{k-1} - 6U_{k-2} \\ 4U_{k-1} + 12U_{k-2} & 9U_{k-1} - 18U_{k-2} & 13U_{k-1} - 6U_{k-2} & 13U_{k-1} - 6U_{k-2} \\ -8U_{k-1} - 24U_{k-2} & 6U_{k-2} - 13U_{k-1} & 6U_{k-2} - 13U_{k-1} & -17U_{k-1} - 6U_{k-2} \end{pmatrix}$$

for all $k \in \mathbb{N}$.

Having stated the result, which enables us to find the positive integer powers of the matrices under consideration, now we can give the following that gives the nth roots of the matrices satisfying the matrix equation.

Theorem 2.4. *If $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfies $X^3 - V_nX^2 + (-q)^n X = \mathbf{0}$ for some $n \in \mathbb{N}$ then $C^n = X$, where $C = \frac{1}{(-q)^{n-1}U_n}\left(-U_{n-1}X^2 + U_{2n-1}X\right)$.*

Proof. If we use (1.1) for $m = n - 1$, and necessary arrangements are made,

$$CX = \frac{1}{U_n}\left(X^2 - qU_{n-1}X\right) \tag{2.1}$$

is obtained. Now, let $A = \frac{1}{(-q)^{n-1}U_n}(-U_{n-1}X + U_{2n-1}I)$. It is obvious that the matrices C, X and A are mutually commutative. If we use (2.1), then we easily obtain

$$C^2 = CXA = \frac{1}{U_n}\left(X - qU_{n-1}I\right)C \tag{2.2}$$

and

$$C^3 = \frac{1}{U_n^2}\left(X - qU_{n-1}I\right)^2 C, \tag{2.3}$$

since $C = XA$. So, using (1.2), (2.2) and (2.3) we get

$$C^3 - pC^2 = \frac{1}{U_n^2}\left(q(U_{n+1}U_{n-1} + (-q)^{n-1})\right)C. \tag{2.4}$$

Now, to use (1.3) for $r = 1$ in (2.4) leads to $C^3 - pC^2 - qC = \mathbf{0}$. So, we can use Corollary 2.2 for the matrix C, and hence we obtain that $C^n = U_{n-1}C^2 + qU_{n-2}C$ for all $n \in \mathbb{N}$. So, if $n-1$ and 1 are taken instead of n and r in (1.3), respectively, and performed some necessary operations, we get

$$C^n = \frac{1}{U_n}\left((-q)^{n-1}C + U_{n-1}CX\right). \tag{2.5}$$

Using (2.1) in (2.5) then gives

$$C^n = \frac{1}{U_n^2}\left(U_{2n-1} - qU_{n-1}^2\right)X. \tag{2.6}$$

If the equation (1.4) used for $n = m-1$,

$$U_{2m-1} = U_m^2 + qU_{m-1}^2$$

or, equivalently,

$$U_{2n-1} - qU_{n-1}^2 = U_n^2 \tag{2.7}$$

is obtained. Combining the equations (2.6) and (2.7), we get $C^n = X$, and thus the proof is completed. □

Example 2.5. Let

$$A = \begin{pmatrix} 29 & 41 & 29 & 29 \\ 70 & 99 & 70 & 70 \\ 41 & 58 & 41 & 41 \\ 29 & 41 & 29 & 29 \end{pmatrix}.$$

Then, it can be easily seen that $A^3 - 198A^2 + A = \mathbf{0}$. If we use the sequences $\{U_n\}$ and $\{V_n\}$ established with $p = 14$, $q = 1$, then, we get $A^3 - V_2A^2 + (-q)^2 A = \mathbf{0}$.

If we use Theorem 2.4 for $n = 2$,

$$C = \frac{1}{-U_2}\left(-U_1 A^2 + U_3 A\right) = \frac{1}{14}\left(A^2 - 197A\right) = \begin{pmatrix} 2 & 3 & 2 & 2 \\ 5 & 7 & 5 & 5 \\ 3 & 4 & 3 & 3 \\ 2 & 3 & 2 & 2 \end{pmatrix}$$

is obtained, and $C^2 = A$. Again, using $\{U_n\}$ and $\{V_n\}$ for $p = 6$, $q = -1$, we get $A^3 - V_3 A^2 + (-q)^3 A = \mathbf{0}$. Hence, if we use Theorem 2.4 for $n = 3$, then,

$$D = \frac{1}{U_3}\left(-U_2 A^2 + U_5 A\right) = \frac{1}{35}\left(-6A^2 + 1189A\right) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

is obtained, and $D^3 = A$. That is, C and D are square root and cube root of the matrix A, respectively.

In the literature, there are many studies in which some identities related to some sequences of numbers are obtained by using matrices. Similarly, some identities related to generalized Fibonacci numbers can be obtained by using Theorem 2.1 for some class of matrices. But, we will consider different results of Theorem 2.1. Recall that, for a matrix $X \in \mathcal{M}_{m\times m}(\mathbb{R})$, if $P(X) = \mathbf{0}$ for some polynomial $P(\lambda)$, then $m_X(\lambda)$, the minimal polynomial of X, divides $P(\lambda)$. So, the matrices $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfying the equation $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$ may be singular or not. But of course one can consider the Moore–Penrose inverse of such matrices. The next result is about the Moore–Penrose inverses of the matrices satisfying the equation $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$.

THEOREM 2.6. *If $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ satisfies the matrix equation $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$ for some $n \in \mathbb{N}$ and $X^2 - V_n X$ is symmetric, then*

$$G = \frac{1}{(-q)^{2n} U_n}\left(-U_{2n} X^2 + U_{3n} X\right)$$

is the Moore–Penrose inverse of the matrix X.

PROOF. First of all, from (1.1) for $m = 2n$ and $m = n$, we get

$$U_{3n} - U_{2n} V_n = -(-q)^n U_n \tag{2.8}$$

and

$$U_{2n} = U_n V_n, \tag{2.9}$$

respectively. Now, we have to show that four conditions in the definition of the Moore–Penrose inverse are satisfied for the matrices X and G.

1. If we use (2.8) and (2.9),

$$XG = -\frac{1}{(-q)^n}\left(X^2 - V_n X\right) \tag{2.10}$$

is obtained. Since $X^2 - V_n X$ is symmetric, XG is also symmetric.
2. Since $GX = XG$, from (1) GX is also symmetric.
3. We have to show that $XGX = X$. Using (2.10) we get

$$XGX = -\frac{1}{(-q)^n}(X^3 - V_n X^2).$$

Since $X^3 - V_n X^2 = -(-q)^n X$ it follows that $XGX = X$.
4. Finally, we must show that $GXG = G$. $GX = -\frac{1}{(-q)^n}\left(X^2 - V_n X\right)$, since $GX = XG$. If $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$ is taken into consideration and necessary arrangements are made, then $GXG = G$ is easily obtained and the proof is completed. □

Corollary 2.7. *If $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ is a symmetric matrix satisfying $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$ for some $n \in \mathbb{N}$ then the matrix*

$$G = \frac{1}{(-q)^{2n} U_n}\left(-U_{2n} X^2 + U_{3n} X\right)$$

is the Moore–Penrose inverse of the matrix X.

Proof. Since X is a symmetric matrix, $X^2 - V_n X$ is also symmetric. So, the proof directly follows from Theorem 2.6. □

As is seen, Theorem 2.6 provides a new way to find the Moore–Penrose inverses of matrices satisfying the matrix equation $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$ using generalized Fibonacci numbers. It is clear that, Corollary 2.7 can be used easily for symmetric matrix X satisfying the matrix equation. Now, we will find the Moore–Penrose inverses of some rectangular matrices $X \in \mathcal{M}_{m\times l}(\mathbb{R})$ or non-symmetric matrices $X \in \mathcal{M}_{m\times m}(\mathbb{R})$ for which $X^T X$ or XX^T satisfy the cubic matrix equation. Let $X \in \mathcal{M}_{m\times l}(\mathbb{R})$. According to (1.5), it is sufficient to find the matrices $(X^T X)^\dagger$ or $(XX^T)^\dagger$ to find the Moore–Penrose inverse of X. Since $X^T X$ and XX^T are symmetric, they satisfy the first condition of Corollary 2.7. Now, if the matrix $X^T X$ or XX^T satisfies an equation of the form $x^3 - ax^2 + bx = 0$, where $a, b \in \mathbb{R}$, $ab \neq 0$ and $a^2 - 4b > 0$, then we can establish a sequence $\{V_n\}$ that meets the conditions $V_n = a$ and $(-q)^n = b$. Although we can choose any number from the set $\mathbb{N}$ as n, notice that the formula that requires the least number of calculations to find the Moore–Penrose inverse of a matrix X is $G = \frac{1}{(-q)^2 U_1}\left(-U_2 X^2 + U_3 X\right)$, which is obtained for $n = 1$. So taking $n = 1$ and using the conditions $V_0 = 2$ and

$V_1 = p \in \mathbb{R}\backslash\{0\}$, we can establish the sequences $\{V_n\}$ and $\{U_n\}$ for $p = a$ and $q = -b$. Then, we can get the Moore–Penrose inverse of the matrix X.

EXAMPLE 2.8. Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 4 & -3 & 5 \end{pmatrix}$. Then, $AA^T = \begin{pmatrix} 6 & 5 & 17 \\ 5 & 6 & 16 \\ 17 & 16 & 50 \end{pmatrix}$ and the characteristic polynomial is $p_A(\lambda) = \lambda^3 - 62\lambda^2 + 66\lambda$. From Cayley-Hamilton Theorem, we get $\left(AA^T\right)^3 - 62\left(AA^T\right)^2 + 66\left(AA^T\right) = \mathbf{0}$. To use Corollary 2.7, we need to construct a sequence $\{V_n\}$ so that $V_n = 62$ and $(-q)^n = 66$. Of course, such a construction is not unique as we stated earlier. For example, we can establish a sequence $\{V_n\}$ satisfying the conditions $V_2 = 62$ and $(-q)^2 = 66$ or satisfying the conditions $V_1 = 62$ and $(-q)^1 = 66$. Let $\{V_n\}$ be the generalized Lucas sequence established by using $p = 62$, $q = -66$. So, $V_1 = p = 62$. So, we can say that

$$\left(AA^T\right)^3 - V_1\left(AA^T\right)^2 + (-q)^1\left(AA^T\right) = \mathbf{0}.$$

Thus, the matrix AA^T satisfies the condition of Corollary 2.7 for $n = 1$. According to Corollary 2.7,

$$\left(AA^T\right)^\dagger = \frac{1}{(-q)^2 U_1}\left(-U_2\left(AA^T\right)^2 + U_3\left(AA^T\right)\right),$$

where U_1, U_2 and U_3 are elements of sequences $\{U_n\}$ which established by using $p = 62$, $q = -66$. So,

$$(AA^T)^\dagger = \frac{1}{198}\begin{pmatrix} 44 & -77 & 11 \\ -77 & 137 & -17 \\ 11 & -17 & 5 \end{pmatrix}$$

is obtained. Hence, from (1.5), the Moore–Penrose inverse of A is given by

$$A^\dagger = A^T(AA^T)^\dagger = \frac{1}{66}\begin{pmatrix} -22 & 43 & -1 \\ 0 & -3 & -3 \\ 22 & -34 & 10 \end{pmatrix}.$$

Noteworthy that Theorem 2.6 can be used not only for square matrices but also for rectangular matrices as the following two examples show.

EXAMPLE 2.9. Let $C = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}$. Then,

$$X = C^TC = \begin{pmatrix} 5 & 11 & 17 & 23 \\ 11 & 25 & 39 & 53 \\ 17 & 39 & 61 & 83 \\ 23 & 53 & 83 & 113 \end{pmatrix}.$$

So, $p_X(\lambda) = \lambda^4 - 204\lambda^3 + 80\lambda^2$ and $m_X(\lambda) = \lambda^3 - 204\lambda^2 + 80\lambda$. It is clear that $X^3 - 204X^2 + 80X = \mathbf{0}$. Therefore, $X^3 - V_n X^2 + (-q)^n X = \mathbf{0}$ if $V_n = 204$ and $(-q)^n = 80$ for some $n \in \mathbb{N}$. Letting $n = 1$, which requires the least computation yields $V_1 = p = 204$ and $q = -80$. Then, according to Corollary 2.7, we get

$$X^\dagger = \left(C^T C\right)^\dagger = \frac{1}{(-q)^2 U_1}\left(-U_2 X^2 + U_3 X\right).$$

Here, U_1, U_2 and U_3 are the elements of the sequence $\{U_n\}$ where $U_n = 204U_{n-1} - 80U_{n-2}$. So,

$$X^\dagger = \left(C^T C\right)^\dagger = \frac{1}{400}\begin{pmatrix} 689 & 353 & 17 & -319 \\ 353 & 181 & 9 & -163 \\ 17 & 9 & 1 & -7 \\ -319 & -163 & -7 & 149 \end{pmatrix}$$

is obtained. According to (1.5),

$$C^\dagger = \frac{1}{20}\begin{pmatrix} -20 & 17 \\ -10 & 9 \\ 0 & 1 \\ 10 & -7 \end{pmatrix}.$$

Example 2.10. Let

$$D = \frac{1}{2}\begin{pmatrix} 2 & 2 & 1 \\ -2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & -1 \end{pmatrix}.$$

Then,

$$DD^T = \frac{1}{4}\begin{pmatrix} 9 & -7 & 1 & -1 \\ -7 & 9 & 1 & -1 \\ 1 & 1 & 9 & 7 \\ -1 & -1 & 7 & 9 \end{pmatrix}$$

is obtained. Also, it can be easily seen that $(DD^T)^3 - 5(DD^T)^2 + 4(DD^T) = \mathbf{0}$. Now, we can establish $\{V_n\}$ with $p = 3$ and $q = -2$. So, we get $(DD^T)^3 - V_2(DD^T)^2 + (-q)^2(DD^T) = \mathbf{0}$. So, matrix DD^T satisfies the condition of Corollary 2.7 for $n = 2$. Now, we can establish $\{U_n\}$ with $p = 3$ and $q = -2$. In this case, $\{U_n\}$ be Mersenne sequence $\{M_n\}$. If Corollary 2.7 is used, then

$$\begin{aligned}(DD^T)^\dagger &= \frac{1}{(2)^4 M_2}\left(-M_4(DD^T)^2 + M_6(DD^T)\right) \\ &= \frac{1}{(2)^4 3}\left(-15(DD^T)^2 + 63(DD^T)\right)\end{aligned}$$

$$= \frac{1}{8}\begin{pmatrix} 3 & 1 & 2 & -2 \\ 1 & 3 & 2 & -2 \\ 2 & 2 & 3 & -1 \\ -2 & -2 & -1 & 3 \end{pmatrix}$$

is obtained. According to (1.5),

$$D^{\dagger} = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 2 & 2 & 2 & -2 \end{pmatrix}$$

is found.

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