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ON SOME INEQUALITIES WITH FIBONACCI NUMBERS VIA WEIGHTED REVERSE HÖLDER INEQUALITIES

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Abstract. In this paper, we introduce new inequalities for weighted sums of powers. By utilizing known Fibonacci identities alongside our generalized inequalities, we derive new sequences of inequalities for Fibonacci numbers.

1. Introduction and preliminaries

Hölder's inequality is a fundamental inequality in mathematical analysis that generalizes the Cauchy-Schwarz inequality to multiple sequences and different exponents. It plays a crucial role in various branches of modern mathematics, such as linear algebra, classical real and complex analysis, probability and statistics, and differential equations. Over the years, numerous research papers have been published on refinements, generalizations, and applications of Hölder inequality and in different areas of mathematics. For example, see [3], [5], [10] and the references therein.

For the reader's convenience, we first introduce the following notation. Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+^n$ be the sets of natural numbers, real numbers, and n-tuples of positive real numbers, respectively.

The classical Hölder's inequality states:

Theorem 1.1 ([5]). *Let* $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ *be* n*-tuples of positive real numbers.*

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(i) *If* $p > 1$ *and* $q = \frac{p}{p-1}$, *then*

$$\sum_{i=1}^{n} x_i y_i \leq \Big(\sum_{i=1}^{n} x_i^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} y_i^q\Big)^{\frac{1}{q}}. \tag{1.1}$$

(ii) *If* $0 < p < 1$ *and* $q = \frac{p}{p-1}$, *then the reverse inequality holds in* (1.1).

If $\mathbf{w} = (w_1, \ldots, w_n)$ is a positive n-tuple of real numbers, then Hölder's inequality can be stated in the following form (see [5]):

$$\sum_{i=1}^{n} w_i x_i y_i \leq \Big(\sum_{i=1}^{n} w_i x_i^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} w_i y_i^q\Big)^{\frac{1}{q}}. \tag{1.2}$$

In 2012, Sulaiman introduced the following reverses of Hölder's integral inequality:

Theorem 1.2 ([10]). *Let* f *and* g *be positive functions satisfying*

$$0 < m \leq f(x)g(x), \quad \forall x \in [a, b].$$

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\frac{\displaystyle\int_a^b f^p(x)dx \int_a^b g^q(x)dx}{\Big(\displaystyle\int_a^b f^{pq}(x)dx\Big)^{1/q} \Big(\displaystyle\int_a^b g^{pq}(x)\,dx\Big)^{1/p}} \leq \frac{1}{m} \int_a^b f(x)g(x)dx.$$

Theorem 1.3 ([10]). *Let* f *and* g *be positive functions satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \in [a, b].$$

Let $p > 0, q > 0$. *Then*

$$\Big(\int_a^b f^p(x)dx\Big)^{1/p} \Big(\int_a^b g^q(x)dx\Big)^{1/q} \leq \frac{M}{m} \Big(\int_a^b (f(x)g(x))^{p/2}dx\Big)^{1/p} \Big(\int_a^b (f(x)g(x))^{q/2}dx\Big)^{1/q}.$$

In [8], discrete analogues of these results were given, leading to new inequalities for power sums, as stated in the following theorems. In these theorems, the author used the notation

$$S_n^{[\alpha]}(\mathbf{x}) = \sum_{i=1}^{n} x_i^{\alpha},$$

for $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$.

Theorem 1.4 ([8]). *Let* $p > 1, q = \frac{p}{p-1}$, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ *and* $m = \min_i \{x_i^\alpha\}$.

(i) *Let* $x_i \geq 1,\ i = 1, \ldots, n$. *If* u *and* v *are real numbers such that* $\alpha = \frac{u}{p} + \frac{v}{q}$ *and* $0 < \alpha < \beta$ *then*

$$\frac{S_n^{[u]}(\mathbf{x})\ S_n^{[v]}(\mathbf{x})}{\left(S_n^{[uq]}(\mathbf{x})\right)^{1/q}\left(S_n^{[vp]}(\mathbf{x})\right)^{1/p}} \leq \frac{1}{m} S_n^{[\alpha]}(\mathbf{x}) \leq \frac{1}{m \cdot n^{\frac{\alpha}{\beta}-1}} \left(S_n^{[\beta]}(\mathbf{x})\right)^{\alpha/\beta}$$
$$\leq \frac{1}{m} S_n^{[\beta]}(\mathbf{x}) \leq \frac{1}{m} \left(S_n^{[\alpha]}(\mathbf{x})\right)^{\beta/\alpha}.$$

(ii) *If* u *and* v *are real numbers such that* $\alpha = \frac{u}{p} + \frac{v}{q}$ *and* $\alpha > \beta > 0$ *then*

$$\frac{S_n^{[u]}(\mathbf{x})\ S_n^{[v]}(\mathbf{x})}{\left(S_n^{[uq]}(\mathbf{x})\right)^{1/q}\left(S_n^{[vp]}(\mathbf{x})\right)^{1/p}} \leq \frac{1}{m} S_n^{[\alpha]}(\mathbf{x}) \leq \frac{1}{m} \left(S_n^{[\beta]}(\mathbf{x})\right)^{\alpha/\beta}.$$

Theorem 1.5 ([8]). *Let* u *and* v *be real numbers such that* $\alpha = \frac{u}{p} + \frac{v}{q}$. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, *and let* $m = \min_i \{x_i^{\frac{u}{p}-\frac{v}{q}}\}$, $M = \max_i \{x_i^{\frac{u}{p}-\frac{v}{q}}\}$.

(i) *If* $0 < p, q < 1$ *then we have*

$$\left(S_n^{[u]}(\mathbf{x})\right)^{\frac{1}{p}} \left(S_n^{[v]}(\mathbf{x})\right)^{\frac{1}{q}} \leq \frac{M}{m} \left(S_n^{[\alpha p/2]}(\mathbf{x})\right)^{\frac{1}{p}} \left(S_n^{[\alpha q/2]}(\mathbf{x})\right)^{\frac{1}{q}}$$
$$\leq \frac{M}{m} n^{\frac{1}{p}+\frac{1}{q}-1} \left(S_n^{[\alpha]}(\mathbf{x})\right).$$

(ii) *Let* $x_i \geq 1,\ i = 1, \ldots, n$. *If* $p, q \geq 1$ *and* $\alpha \geq 0$ *then we have*

$$\left(S_n^{[u]}(\mathbf{x})\right)^{\frac{1}{p}} \left(S_n^{[v]}(\mathbf{x})\right)^{\frac{1}{q}} \leq \frac{M}{m} \left(S_n^{[\alpha p/2]}(\mathbf{x})\right)^{\frac{1}{p}} \left(S_n^{[\alpha q/2]}(\mathbf{x})\right)^{\frac{1}{q}}$$
$$\leq \frac{Mn^2}{m} S_n^{[\alpha p]}(\mathbf{x})\, S_n^{[\alpha q]}(\mathbf{x}).$$

In this paper, we define the following notation for weighted sums of powers:

$$S_n^{[\alpha]}(\mathbf{x}, \mathbf{w}) = \sum_{i=1}^{n} w_i x_i^\alpha,$$

where $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{w} = (w_1, \ldots, w_n)$ are vectors in $\mathbb{R}_+^n$.

We also use the following result:

Proposition 1.1 ([5]). *If* $\alpha > \beta > 0$ *and* $w_i \geq 1$ *then*

$$\left(S_n^{[\alpha]}(\mathbf{x}, \mathbf{w})\right)^{1/\alpha} \leq \left(S_n^{[\beta]}(\mathbf{x}, \mathbf{w})\right)^{1/\beta}. \tag{1.3}$$

The aim of this paper is to further generalize the results presented in [8] by incorporating weights into the sums. The paper is organized as follows. In Section 2, we obtain weighted reverse Hölder's inequalities and present series of inequalities for weighted sums of powers. Further, in Section 3, we apply obtained results to Fibonacci sums.

2. Inequalities for weighted sums of powers

In this section, we generalize the inequalities presented in Theorems 1.4 and 1.5 by introducing positive weights. To achieve this, we first establish the discrete form of weighted reverse Hölder's inequalities as presented in Theorems 1.2 and 1.3.

Theorem 2.1. *Let* $\mathbf{x}=(x_1,\ldots,x_n)$, $\mathbf{y}=(y_1,\ldots,y_n)$ *and* $\mathbf{w}=(w_1,\ldots,w_n)$ *be vectors in* $\mathbb{R}_+^n$ *such that*

$$0<m\le x_iy_i,\quad i=1,\ldots,n.$$

Let $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\frac{\left(\sum_{i=1}^n w_ix_i^p\right)\left(\sum_{i=1}^n w_iy_i^q\right)}{\left(\sum_{i=1}^n w_ix_i^{pq}\right)^{1/q}\left(\sum_{i=1}^n w_iy_i^{pq}\right)^{1/p}}\le\frac{1}{m}\sum_{i=1}^n w_ix_iy_i. \tag{2.1}$$

Proof. Using the weighted Hölder's inequality (1.2) the following is obtained

$$\begin{aligned}\sum_{i=1}^n w_ix_i^p&=\sum_{i=1}^n w_ix_i^{\frac{1}{p}}y_i^{\frac{1}{p}}x_i^{p-\frac{1}{p}}y_i^{-\frac{1}{p}}\le\Big(\sum_{i=1}^n w_ix_iy_i\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n w_ix_i^{q(p-\frac{1}{p})}y_i^{-\frac{q}{p}}\Big)^{\frac{1}{q}}\\&\le\Big(\sum_{i=1}^n w_ix_iy_i\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n w_ix_i^{q(p-\frac{1}{p})}\Big(\frac{x_i}{m}\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\\&=\frac{1}{m^{\frac{1}{p}}}\Big(\sum_{i=1}^n w_ix_iy_i\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n w_ix_i^{pq}\Big)^{\frac{1}{q}}.\end{aligned} \tag{2.2}$$

Also, similarly, one gets

$$\sum_{i=1}^n w_iy_i^q\le\frac{1}{m^{\frac{1}{q}}}\Big(\sum_{i=1}^n w_ix_iy_i\Big)^{\frac{1}{q}}\Big(\sum_{i=1}^n w_iy_i^{pq}\Big)^{\frac{1}{p}}. \tag{2.3}$$

Combining (2.2) and (2.3) yields (2.1) . □

Theorem 2.2. *Let* $\mathbf{x}=(x_1,\dots,x_n)$*,* $\mathbf{y}=(y_1,\dots,y_n)$ *and* $\mathbf{w}=(w_1,\dots,w_n)$ *be vectors in* $\mathbb{R}_+^n$ *such that*

$$0<m\le\frac{x_i}{y_i}\le M,\quad i=1,\dots,n. \tag{2.4}$$

Let $p>0, q>0$*. Then*

$$\Big(\sum_{i=1}^n w_ix_i^p\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n w_iy_i^q\Big)^{\frac{1}{q}}\le\frac{M}{m}\Big(\sum_{i=1}^n w_i(x_iy_i)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n w_i(x_iy_i)^{\frac{q}{2}}\Big)^{\frac{1}{q}}. \tag{2.5}$$

Proof. From the assumption (2.4), it follows

$$m+1\le\frac{x_i+y_i}{y_i}\le M+1,\quad i=1,\dots,n \tag{2.6}$$

and

$$\frac{M+1}{M}\le\frac{x_i+y_i}{x_i}\le\frac{m+1}{m},\quad i=1,\dots,n. \tag{2.7}$$

From the left inequalities in (2.6) and (2.7), it follows

$$y_i\le\frac{1}{m+1}(x_i+y_i),\quad x_i\le\frac{M}{M+1}(x_i+y_i).$$

Multiplying these inequalities by weight w_i and summing over i, one gets:

$$\Big(\sum_{i=1}^n w_ix_i^p\Big)^{\frac{1}{p}}\le\frac{M}{M+1}\Big(\sum_{i=1}^n w_i(x_i+y_i)^p\Big)^{\frac{1}{p}} \tag{2.8}$$

$$\Big(\sum_{i=1}^n w_iy_i^q\Big)^{\frac{1}{q}}\le\frac{1}{m+1}\Big(\sum_{i=1}^n w_i(x_i+y_i)^q\Big)^{\frac{1}{q}}. \tag{2.9}$$

From the right inequalities in (2.6) and (2.7), it follows

$$x_i+y_i\le(M+1)y_i,\quad x_i+y_i\le\frac{m+1}{m}x_i. \tag{2.10}$$

By multiplying the inequalities in (2.10) side by side, one gets

$$(x_i+y_i)^2\le\frac{(m+1)(M+1)}{m}x_iy_i. \tag{2.11}$$

From (2.11) it can be deduced that

$$\Big(\sum_{i=1}^n w_i(x_i+y_i)^p\Big)^{\frac{1}{p}}\le\Big(\frac{(m+1)(M+1)}{m}\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^n w_i(x_iy_i)^{\frac{p}{2}}\Big)^{\frac{1}{p}}$$

$$\Big(\sum_{i=1}^{n} w_i(x_i+y_i)^q\Big)^{\frac{1}{q}} \le \Big(\frac{(m+1)(M+1)}{m}\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{n} w_i(x_iy_i)^{\frac{q}{2}}\Big)^{\frac{1}{q}}.$$

Multiplying the inequalities (2.8) and (2.9) side by side, and using the last two inequalities the desired inequality is obtained.

$$\begin{aligned}
&\Big(\sum_{i=1}^{n} w_i x_i^p\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} w_i y_i^q\Big)^{\frac{1}{q}}\\
&\qquad\le \frac{M}{(M+1)(m+1)}\Big(\sum_{i=1}^{n} w_i(x_i+y_i)^p\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} w_i(x_i+y_i)^q\Big)^{\frac{1}{q}}\\
&\qquad\qquad\le \frac{M}{m}\Big(\sum_{i=1}^{n} w_i(x_iy_i)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} w_i(x_iy_i)^{\frac{q}{2}}\Big)^{\frac{1}{q}}. \quad \square
\end{aligned}$$

Remark 2.1. Taking $\mathbf{w} = (1, \dots, 1)$ in Theorems 2.1 and 2.2 we obtain Lemmas 2.1 and 2.2, from paper [8].

In the following theorems, we derive a series of inequalities for weighted sums of powers by utilizing the inequalities established in Theorems 2.1 and 2.2.

Theorem 2.3. *Let p, q, u, v, α be real numbers such that $p > 1$, $q = \frac{p}{p-1}$ and $\alpha = \frac{u}{p} + \frac{v}{q}$. Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$ be vectors in $\mathbb{R}_+^n$ such that $w_i \ge 1$ for $i = 1, \dots, n$, $W_n = \sum_{i=1}^n w_i$ and let $m = \min_i \{x_i^\alpha\}$.*

(i) *Let $x_i \ge 1$, $i = 1, \dots, n$. If $0 < \alpha < \beta$ then*

$$\begin{aligned}
\frac{S_n^{[u]}(\mathbf{x},\mathbf{w})\ S_n^{[v]}(\mathbf{x},\mathbf{w})}{\Big(S_n^{[uq]}(\mathbf{x},\mathbf{w})\Big)^{1/q}\Big(S_n^{[vp]}(\mathbf{x},\mathbf{w})\Big)^{1/p}} &\le \frac{1}{m}S_n^{[\alpha]}(\mathbf{x},\mathbf{w}) \le \frac{1}{m\cdot W_n^{\frac{\alpha}{\beta}-1}}\Big(S_n^{[\beta]}(\mathbf{x},\mathbf{w})\Big)^{\alpha/\beta}\\
&\le \frac{1}{m}S_n^{[\beta]}(\mathbf{x},\mathbf{w}) \le \frac{1}{m}\Big(S_n^{[\alpha]}(\mathbf{x},\mathbf{w})\Big)^{\beta/\alpha}.
\end{aligned}$$

(ii) *If $\alpha > \beta > 0$ then*

$$\frac{S_n^{[u]}(\mathbf{x},\mathbf{w})\ S_n^{[v]}(\mathbf{x},\mathbf{w})}{\Big(S_n^{[uq]}(\mathbf{x},\mathbf{w})\Big)^{1/q}\Big(S_n^{[vp]}(\mathbf{x},\mathbf{w})\Big)^{1/p}} \le \frac{1}{m}S_n^{[\alpha]}(\mathbf{x},\mathbf{w}) \le \frac{1}{m}\Big(S_n^{[\beta]}(\mathbf{x},\mathbf{w})\Big)^{\alpha/\beta}.$$

Proof. (i) By substituting x_i and y_i with $x_i^{u/p}$ and $x_i^{v/q}$, respectively in (2.1) the following is obtained:

$$\frac{\Big(\sum_{i=1}^n w_i x_i^u\Big)\Big(\sum_{i=1}^n w_i x_i^v\Big)}{\Big(\sum_{i=1}^n w_i x_i^{uq}\Big)^{1/q}\Big(\sum_{i=1}^n w_i x_i^{vp}\Big)^{1/p}} \le \frac{1}{m}\sum_{i=1}^{n} w_i x_i^\alpha.$$

Let us also observe that, under this substitution, the condition $0 < m \leq x_i y_i$ is satisfied for $m = \min_i \{x_i^{\alpha}\}$. Furthermore, the following is obtained by Theorem 2.1

$$
\begin{aligned}
(2.12) \qquad \frac{\Big(\sum_{i=1}^{n} w_i x_i^{u}\Big)\Big(\sum_{i=1}^{n} w_i x_i^{v}\Big)}{\Big(\sum_{i=1}^{n} w_i x_i^{uq}\Big)^{1/q}\Big(\sum_{i=1}^{n} w_i x_i^{vp}\Big)^{1/p}} &\leq \frac{1}{m}\sum_{i=1}^{n} w_i x_i^{\alpha} = \frac{1}{m}\sum_{i=1}^{n} w_i \Big(x_i^{\beta}\Big)^{\frac{\alpha}{\beta}} \\
&\leq \frac{W_n}{m}\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^{\beta}\Big)^{\alpha/\beta} \\
&\leq \frac{1}{m}\sum_{i=1}^{n} w_i x_i^{\beta} \leq \frac{1}{m}\Big(\sum_{i=1}^{n} w_i x_i^{\alpha}\Big)^{\frac{\beta}{\alpha}}.
\end{aligned}
$$

Inequalities in (2.12) are calculated by applying reverse Jensen's inequlity for the function $x \mapsto x^{\alpha/\beta}$ where $\alpha < \beta$, then the monotonicity of the exponential function $x \mapsto b^x$, $b = \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^{\beta} \geq 1$ and finally inequality (1.3).

(ii) Similar to the proof of (i), Theorem 2.2 can be applied with substitutions $x_i \to x_i^{u/p}$ and $y_i \to x_i^{v/q}$, and then inequality (1.3). □

Theorem 2.4. *Let p, q, u, v, α be real numbers such that $\alpha = \frac{u}{p} + \frac{v}{q}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ be vectors in $\mathbb{R}_{+}^{n}$ such that $w_i \geq 1$ for $i = 1, \ldots, n$, $W_n = \sum_{i=1}^{n} w_i$ and let $m = \min_i \{x_i^{\frac{u}{p} - \frac{v}{q}}\}$, $M = \max_i \{x_i^{\frac{u}{p} - \frac{v}{q}}\}$.*

(i) *Let $0 < p, q < 1$. If $\alpha > \beta > 0$ then*

$$
\begin{aligned}
\Big(S_n^{[u]}(\mathbf{x},\mathbf{w})\Big)^{\frac{1}{p}}\Big(S_n^{[v]}(\mathbf{x},\mathbf{w})\Big)^{\frac{1}{q}} &\leq \frac{M}{m}\Big(S_n^{[\alpha p/2]}(\mathbf{x},\mathbf{w})\Big)^{\frac{1}{p}}\Big(S_n^{[\alpha q/2]}(\mathbf{x},\mathbf{w})\Big)^{\frac{1}{q}} \\
&\leq \frac{M}{m} W_n^{\frac{1}{p}+\frac{1}{q}-1}\Big(S_n^{[\alpha]}(\mathbf{x},\mathbf{w})\Big) \\
&\leq \frac{M}{m} W_n^{\frac{1}{p}+\frac{1}{q}-1}\Big(S_n^{[\beta]}(\mathbf{x},\mathbf{w})\Big)^{\frac{\alpha}{\beta}}.
\end{aligned}
$$

If $0 < \alpha < \beta$ and $x_i \geq 1$, $i = 1, \ldots, n$ then

$$
\begin{aligned}
\Big(S_n^{[u]}(\mathbf{x},\mathbf{w})\Big)^{\frac{1}{p}}\Big(S_n^{[v]}(\mathbf{x},\mathbf{w})\Big)^{\frac{1}{q}} &\leq \frac{M}{m}\Big(S_n^{[\alpha p/2]}(\mathbf{x},\mathbf{w})\Big)^{\frac{1}{p}}\Big(S_n^{[\alpha q/2]}(\mathbf{x},\mathbf{w})\Big)^{\frac{1}{q}} \\
&\leq \frac{M}{m} W_n^{\frac{1}{p}+\frac{1}{q}-1}\Big(S_n^{[\alpha]}(\mathbf{x},\mathbf{w})\Big) \\
&\leq \frac{M}{m} W_n^{\frac{1}{p}+\frac{1}{q}-1}\Big(S_n^{[\beta]}(\mathbf{x},\mathbf{w})\Big) \\
&\leq \frac{M}{m} W_n^{\frac{1}{p}+\frac{1}{q}-1}\Big(S_n^{[\alpha]}(\mathbf{x},\mathbf{w})\Big)^{\frac{\beta}{\alpha}}.
\end{aligned}
$$

(ii) *Let* $p, q \geq 1$ *and* $x_i \geq 1$, $i = 1, \dots, n$. *If* $\alpha > \beta > 0$ *then*

$$\begin{aligned}
\left(S_n^{[u]}(\mathbf{x},\mathbf{w})\right)^{\frac{1}{p}} \left(S_n^{[v]}(\mathbf{x},\mathbf{w})\right)^{\frac{1}{q}} &\leq \frac{M}{m} \left(S_n^{[\alpha p/2]}(\mathbf{x},\mathbf{w})\right)^{\frac{1}{p}} \left(S_n^{[\alpha q/2]}(\mathbf{x},\mathbf{w})\right)^{\frac{1}{q}} \\
&\leq \frac{MW_n^2}{m} S_n^{[\alpha p]}(\mathbf{x},\mathbf{w})\, S_n^{[\alpha q]}(\mathbf{x},\mathbf{w}) \\
&\leq \frac{MW_n^2}{m} \left(S_n^{[\beta p]}(\mathbf{x},\mathbf{w})\right)^{\frac{\alpha}{\beta}} \left(S_n^{[\beta q]}(\mathbf{x},\mathbf{w})\right)^{\frac{\alpha}{\beta}} .
\end{aligned}$$

If $0 < \alpha < \beta$ *then*

$$\begin{aligned}
\left(S_n^{[u]}(\mathbf{x},\mathbf{w})\right)^{\frac{1}{p}} \left(S_n^{[v]}(\mathbf{x},\mathbf{w})\right)^{\frac{1}{q}} &\leq \frac{M}{m} \left(S_n^{[\alpha p/2]}(\mathbf{x},\mathbf{w})\right)^{\frac{1}{p}} \left(S_n^{[\alpha q/2]}(\mathbf{x},\mathbf{w})\right)^{\frac{1}{q}} \\
&\leq \frac{MW_n^2}{m} S_n^{[\alpha p]}(\mathbf{x},\mathbf{w})\, S_n^{[\alpha q]}(\mathbf{x},\mathbf{w}) \\
&\leq \frac{MW_n^2}{m} S_n^{[\beta p]}(\mathbf{x},\mathbf{w})\, S_n^{[\beta q]}(\mathbf{x},\mathbf{w}) \\
&\leq \frac{MW_n^2}{m} \left(S_n^{[\beta p]}(\mathbf{x},\mathbf{w})\right)^{\frac{\beta}{\alpha}} \left(S_n^{[\beta q]}(\mathbf{x},\mathbf{w})\right)^{\frac{\beta}{\alpha}}.
\end{aligned} \tag{2.13}$$

Proof. (i) First let us notice that by substituting x_i with $x_i^{u/p}$ and y_i with $x_i^{v/q}$ in Theorem 2.2, the condition (2.4) is satisfied for $m = \min_i \{x_i^{\frac{u}{p} - \frac{v}{q}}\}$ and $M = \max_i \{x_i^{\frac{u}{p} - \frac{v}{q}}\}$. With this substitution, inequality (2.5) becomes (2.14). In (2.15), reverse Jensen's inequality is utilized for the functions $x \mapsto x^{p/2}$, $x \mapsto x^{q/2}$ along with monotonicity of the function $x \mapsto x^{1/p}$:

$$\Big(\sum_{i=1}^{n} w_i x_i^{p}\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} w_i x_i^{q}\Big)^{\frac{1}{q}} \leq \frac{M}{m} \Big(\sum_{i=1}^{n} w_i x_i^{\frac{\alpha p}{2}}\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} w_i x_i^{\frac{\alpha q}{2}}\Big)^{\frac{1}{q}} \tag{2.14}$$

$$\leq \frac{M}{m} W_n^{\frac{1}{p} + \frac{1}{q} - 1} \Big(\sum_{i=1}^{n} w_i x_i^{\alpha}\Big). \tag{2.15}$$

If $\alpha > \beta > 0$, Proposition 1.1 is applied on (2.15) and the following is obtained:

$$\frac{M}{m} W_n^{\frac{1}{p} + \frac{1}{q} - 1} \Big(\sum_{i=1}^{n} w_i x_i^{\alpha}\Big) \leq \frac{M}{m} W_n^{\frac{1}{p} + \frac{1}{q} - 1} \Big(\sum_{i=1}^{n} w_i x_i^{\beta}\Big)^{\frac{\alpha}{\beta}}.$$

If $0 < \alpha < \beta$, reverse Jensen's inequlity for the function $x \mapsto x^{\alpha/\beta}$ where $\alpha < \beta$, is applied to (2.15), then the monotonicity of the exponential function

$x \mapsto b^x,\ b = \frac{1}{W_n}\sum_{i=1}^n w_i x_i^\beta \geq 1$, and finally inequality (1.3):

$$\begin{aligned}
\frac{M}{m}W_n^{\frac{1}{p}+\frac{1}{q}-1}\Big(\sum_{i=1}^n w_i x_i^\alpha\Big) &= \frac{M}{m}W_n^{\frac{1}{p}+\frac{1}{q}-1}\sum_{i=1}^n w_i\Big(x_i^\beta\Big)^{\frac{\alpha}{\beta}}\\
&\leq \frac{M}{m}W_n^{\frac{1}{p}+\frac{1}{q}}\Big(\frac{1}{W_n}\sum_{i=1}^n w_i x_i^\beta\Big)^{\alpha/\beta}\\
&\leq \frac{M}{m}W_n^{\frac{1}{p}+\frac{1}{q}-1}\sum_{i=1}^n w_i x_i^\beta \leq \frac{M}{m}W_n^{\frac{1}{p}+\frac{1}{q}-1}\Big(\sum_{i=1}^n w_i x_i^\alpha\Big)^{\frac{\beta}{\alpha}}.
\end{aligned}$$

(ii) Similar to the proof of (*i*), inequality (2.5) is first applied with substitutions $x_i \to x_i^{u/p}$ and $y_i \to x_i^{v/q}$.

If $\alpha > \beta > 0$ it follows that

$$\Big(\sum_{i=1}^n w_i x_i^p\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n w_i x_i^q\Big)^{\frac{1}{q}} \leq \frac{M}{m}\Big(\sum_{i=1}^n w_i x_i^{\frac{\alpha p}{2}}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n w_i x_i^{\frac{\alpha q}{2}}\Big)^{\frac{1}{q}} \tag{2.16}$$

$$\leq \frac{M}{m}\Big(\sum_{i=1}^n w_i x_i^{\frac{\alpha p}{2}}\Big)^{2}\Big(\sum_{i=1}^n w_i x_i^{\frac{\alpha q}{2}}\Big)^{2} \tag{2.17}$$

$$\leq \frac{MW_n^2}{m}\Big(\sum_{i=1}^n w_i x_i^{\alpha p}\Big)\Big(\sum_{i=1}^n w_i x_i^{\alpha q}\Big) \tag{2.18}$$

$$\leq \frac{MW_n^2}{m}\Big(\sum_{i=1}^n w_i x_i^{\beta p}\Big)^{\frac{\alpha}{\beta}}\Big(\sum_{i=1}^n w_i x_i^{\beta q}\Big)^{\frac{\alpha}{\beta}}. \tag{2.19}$$

In (2.16), inequality (2.5) is applied with substitutions $x_i \to x_i^{u/p}$ and $y_i \to x_i^{v/q}$. In (2.17), the monotonicity of the exponential function $x \mapsto a^x$, $a = \sum_{i=1}^n w_i x_i^{\alpha p/2} \geq 1$ is used. Subsequently, in (2.18) Jensen's inequality is applied for the function $x \mapsto x^2$. Finally, Proposition 1.1 is used in (2.19).

Similarly, if $0 < \alpha < \beta$, to derive (2.13), the monotonicity of the exponential function is applied, then Jensen's inequality, and finally inequality (1.3). □

3. Applications

In this section, results obtained in previous section, will be applied to Fibonacci sums, which plays an important role in various branches of mathematics. These sums naturally arise in the problems related to combinatorics, complexity analysis, and discrete mathematics.

The classical Fibonacci and Lucas numbers are defined by the recurrence relations, respectively,

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-2} + F_{n-1}, \quad n \geq 2$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}, \quad n \geq 2.$$

In the literature, many identities related to the sum of Fibonacci numbers can be found. For example, the following identities are given in [6] and [9]:

$$\sum_{i=1}^{n} F_i^2 F_{i+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}, \tag{3.1}$$

$$\sum_{i=1}^{2n} \binom{2n}{i} F_i^2 = 5^{n-1} L_{2n},$$

where L_n is the Lucas number.

In this section, we will select weights $\mathbf{w}$ which allow direct calculation of the sum $W_n = \sum_{i=1}^{n} w_i$. This approach will allow us to obtain different inequalities for Fibonacci numbers by using various identities for Fibonacci numbers.

For example, if we take $x_i = F_i$, $w_i = F_{i+1}$ in the identity (3.1), then $W_n = F_{n+3} - 2$. Using our notation, identity (3.1) can be rewritten as:

$$S_n^{[2]}(\mathbf{x}, \mathbf{w}) = \frac{1}{2} F_n F_{n+1} F_{n+2}. \tag{3.2}$$

Now, using identity (3.2) along with Theorems 2.3 and 2.4 for $\beta = 2$, one obtains the following theorems, respectively.

THEOREM 3.1. *Let p, q, u, v, α be real numbers such that $p > 1$, $q = \frac{p}{p-1}$ and $\alpha = \frac{u}{p} + \frac{v}{q}$.*

(i) *If $0 < \alpha < 2$ then*

$$\frac{\left(\sum_{i=1}^{n} F_{i+1} F_i^u\right)\left(\sum_{i=1}^{n} F_{i+1} F_i^v\right)}{\left(\sum_{i=1}^{n} F_{i+1} F_i^{qu}\right)^{1/q}\left(\sum_{i=1}^{n} F_{i+1} F_i^{vp}\right)^{1/p}} \leq \sum_{i=1}^{n} F_{i+1} F_i^{\alpha}$$

$$\leq \frac{1}{\left(F_{n+3} - 2\right)^{\alpha/2-1}} \left(\frac{1}{2} F_n F_{n+1} F_{n+2}\right)^{\alpha/2} \leq \frac{1}{2} F_n F_{n+1} F_{n+2} \leq \left(\sum_{i=1}^{n} F_{i+1} F_i^{\alpha}\right)^{\frac{2}{\alpha}}.$$

(ii) *If* $\alpha > 2$ *then*

$$\frac{\Big(\sum_{i=1}^{n} F_{i+1}F_i^{u}\Big)\Big(\sum_{i=1}^{n} F_{i+1}F_i^{v}\Big)}{\Big(\sum_{i=1}^{n} F_{i+1}F_i^{qu}\Big)^{1/q}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{vp}\Big)^{1/p}} \leq \sum_{i=1}^{n} F_{i+1}F_i^{\alpha} \leq \Big(\frac{1}{2}F_nF_{n+1}F_{n+2}\Big)^{\alpha/2}.$$

THEOREM 3.2. *Let* p, q, u, v, α *be real numbers such that* $\alpha = \frac{u}{p} + \frac{v}{q}$. *Let* $m = \min_i \big\{F_i^{\frac{u}{p}-\frac{v}{q}}\big\}$, $M = \max_i \big\{F_i^{\frac{u}{p}-\frac{v}{q}}\big\}$.

(i) *Let* $0 < p, q < 1$. *If* $\alpha > 2$ *then*

$$\Big(\sum_{i=1}^{n} F_{i+1}F_i^{u}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{u}\Big)^{\frac{1}{q}} \leq \frac{M}{m}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha p/2}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha q/2}\Big)^{\frac{1}{q}}$$

$$\leq \frac{M}{m}\big(F_{n+3}-2\big)^{\frac{1}{p}+\frac{1}{q}-1}\sum_{i=1}^{n} F_{i+1}F_i^{\alpha} \leq \frac{M}{m}\big(F_{n+3}-2\big)^{\frac{1}{p}+\frac{1}{q}-1}\Big(\frac{1}{2}F_nF_{n+1}F_{n+2}\Big)^{\frac{\alpha}{2}}.$$

If $0 < \alpha < 2$ *then*

$$\Big(\sum_{i=1}^{n} F_{i+1}F_i^{u}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{v}\Big)^{\frac{1}{q}} \leq \frac{M}{m}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha p/2}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha q/2}\Big)^{\frac{1}{q}}$$

$$\leq \frac{M}{m}\Big(F_{n+3}-2\Big)^{\frac{1}{p}+\frac{1}{q}-1}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha}\Big) \leq \frac{M}{2m}\big(F_{n+3}-2\big)^{\frac{1}{p}+\frac{1}{q}-1}F_nF_{n+1}F_{n+2}$$

$$\leq \frac{M}{m}\big(F_{n+3}-2\big)^{\frac{1}{p}+\frac{1}{q}-1}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha}\Big)^{\frac{2}{\alpha}}.$$

(ii) *Let* $p, q \geq 1$. *If* $\alpha > 2$ *then*

$$\Big(\sum_{i=1}^{n} F_{i+1}F_i^{u}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{v}\Big)^{\frac{1}{q}} \leq \frac{M}{m}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha p/2}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha q/2}\Big)^{\frac{1}{q}}$$

$$\leq \frac{M\big(F_{n+3}-2\big)^2}{m}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha p}\Big)\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha q}\Big)$$

$$\leq \frac{M\big(F_{n+3}-2\big)^2}{m}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{2p}\Big)^{\frac{\alpha}{2}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{2q}\Big)^{\frac{\alpha}{2}}.$$

If $0 < \alpha < 2$ *then*

$$\Big(\sum_{i=1}^{n} F_{i+1}F_i^{u}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{v}\Big)^{\frac{1}{q}} \leq \frac{M}{m}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha p/2}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha q/2}\Big)^{\frac{1}{q}}$$

$$\leq \frac{M\big(F_{n+3}-2\big)^2}{m}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha p}\Big)\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\alpha q}\Big)$$
$$\leq \frac{M\big(F_{n+3}-2\big)^2}{m}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{2p}\Big)^{\frac{2}{\alpha}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{2q}\Big)^{\frac{2}{\alpha}}.$$

In the previous theorems, we demonstrated how various inequalities can be derived using known Fibonacci identity and Theorems 2.3 and 2.4. Similarly, other interesting inequalities can be obtained by applying some of the following identities, which can be found in [2], [4], [6], [7] and [9]:

for $i = 1, \dots, n$,

$$x_i = F_i,\ \ w_i = i,\ \ W_n = \frac{n(n+1)}{2},\ \ \beta = 1,\ \ S_n^{[1]}(\mathbf{x},\mathbf{w}) = nF_{n+2} - F_{n+3} + 2,$$
$$x_i = F_i,\ \ w_i = F_{i+1},\ \ W_n = F_{n+3} - 2,\ \ \beta = 1,$$
$$S_n^{[1]}(\mathbf{x},\mathbf{w}) = F_{n+1}^2 - \frac{1+(-1)^n}{2},$$
$$x_i = F_i,\ \ w_i = \binom{n}{i},\ \ W_n = 2^n - 1,\ \ \beta = 1,\ \ S_n^{[1]}(\mathbf{x},\mathbf{w}) = F_{2n},$$
$$x_i = F_i,\ \ w_i = \binom{n}{i},\ \ W_n = 2^n - 1,\ \ \beta = 3,\ \ S_n^{[3]}(\mathbf{x},\mathbf{w}) = \frac{1}{5}(2^nF_{2n} + 3F_n),$$
$$x_i = F_i,\ \ w_i = \binom{n}{i},\ \ W_n = 2^n - 1,\ \ \beta = 4,$$
$$S_n^{[4]}(\mathbf{x},\mathbf{w}) = \frac{1}{25}(3^nL_{2n} - 4(-1)^nL_n + 6\cdot 2^n),$$
$$x_i = F_iF_{i+2},\ \ w_i = 2^{i-1},\ \ W_n = 2^n - 1,\ \ \beta = 1,\ \ S_n^{[1]}(\mathbf{x},\mathbf{w}) = 2^nF_nF_{n+1},$$
$$x_i = F_i,\ \ w_i = 1,\ \ W_n = n,\ \ \beta = 6,\ \ S_n^{[6]}(\mathbf{x},\mathbf{w}) = \frac{1}{4}(F_n^5F_{n+3} + F_{2n}).$$

In [1], the authors pointed out, that particularly interesting are the cases in which the sum $S_n^{[\alpha]}(\mathbf{x})$ can be computed for different values of the parameter α. In our notation, for example, if we choose $w_i = F_{i+1}$, $x_i = F_i$, then for $\alpha = 1$ we have

$$S_n^{[1]}(\mathbf{x},\mathbf{w}) = F_{n+1}^2 - \frac{1+(-1)^n}{2}. \tag{3.3}$$

Using the identities (3.2) and (3.3) the following result is obtained.

THEOREM 3.3. *Let p be a real number such that $p > 1$.*

(i) *If* $\beta > 2$ *then*

$$\frac{\Big(\sum_{i=1}^{n} F_{i+1}F_i^{p+1}\Big)\Big(F_{n+1}^2 - \frac{1+(-1)^n}{2}\Big)}{\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\frac{p^2+p}{p-1}}\Big)^{\frac{p-1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{p}\Big)^{1/p}} \le \frac{1}{2}F_nF_{n+1}F_{n+2}$$

$$\le \frac{1}{\Big(F_{n+3}-2\Big)^{2/\beta-1}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\beta}\Big)^{\frac{2}{\beta}} \le \sum_{i=1}^{n} F_{i+1}F_i^{\beta} \le \Big(\frac{1}{2}F_nF_{n+1}F_{n+2}\Big)^{\frac{\beta}{2}}.$$

(ii) *If* $0 < \beta < 2$ *then*

$$\frac{\Big(\sum_{i=1}^{n} F_{i+1}F_i^{p+1}\Big)\Big(F_{n+1}^2 - \frac{1+(-1)^n}{2}\Big)}{\Big(\sum_{i=1}^{n} F_{i+1}F_i^{\frac{p^2+p}{p-1}}\Big)^{\frac{p-1}{p}}\Big(\sum_{i=1}^{n} F_{i+1}F_i^{p}\Big)^{1/p}} \le \frac{1}{2}F_nF_{n+1}F_{n+2} \le \Big(\sum_{i=1}^{n} F_{i+1}F_i^{\beta}\Big)^{\frac{2}{\beta}}.$$

PROOF. In Theorem 2.3 we take $x_i = F_i$, $w_i = F_{i+1}$, $\alpha = 2$ and $v = 1$, and then using identities (3.2) and (3.3). ☐

Similarly, if we choose $x_i = F_i$, $w_i = \binom{n}{i}$, then for $\alpha = 1$ it follows that

$$S_n^{[1]}(\mathbf{x}, \mathbf{w}) = F_{2n}, \tag{3.4}$$

while for $\alpha = 3$ one gets

$$S_n^{[3]}(\mathbf{x}, \mathbf{w}) = \frac{1}{5}(2^nF_{2n} + 3F_n). \tag{3.5}$$

Now, by applying Theorem 2.3 with $x_i = F_i$, $w_i = \binom{n}{i}$, $\alpha = 3$ and $v = 1$, and using the identities (3.4) and (3.5) the following result is obtained.

THEOREM 3.4. *Let* p *be a real number such that* $p > 1$.

(i) *If* $\beta > 3$ *then*

$$\frac{\Big(\sum_{i=1}^{n}\binom{n}{i}F_i^{2p+1}\Big)\cdot F_{2n}}{\Big(\sum_{i=1}^{n}\binom{n}{i}F_i^{\frac{2p^2+p}{p-1}}\Big)^{\frac{p-1}{p}}\Big(\sum_{i=1}^{n}\binom{n}{i}F_i^{p}\Big)^{1/p}} \le \frac{1}{5}(2^nF_{2n} + 3F_n)$$

$$\le \frac{1}{(2^i-1)^{3/\beta-1}}\Big(\sum_{i=1}^{n}\binom{n}{i}F_i^{\beta}\Big)^{\frac{3}{\beta}} \le \sum_{i=1}^{n}\binom{n}{i}F_i^{\beta} \le \Big(\frac{1}{5}(2^nF_{2n} + 3F_n)\Big)^{\frac{\beta}{3}}.$$

(ii) *If* $0 < \beta < 3$ *then*

$$\frac{\Big(\sum_{i=1}^{n}\binom{n}{i}F_i^{2p+1}\Big)\cdot F_{2n}}{\Big(\sum_{i=1}^{n}\binom{n}{i}F_i^{\frac{2p^2+p}{p-1}}\Big)^{\frac{p-1}{p}}\Big(\sum_{i=1}^{n}\binom{n}{i}F_i^{p}\Big)^{1/p}} \le \frac{1}{5}(2^nF_{2n}+3F_n) \le \Big(\sum_{i=1}^{n}\binom{n}{i}F_i^{\beta}\Big)^{\frac{3}{\beta}}.$$

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