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ON MATRICES WITH BIDIMENSIONAL FIBONACCI NUMBERS

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Abstract. In this paper, the bidimensional extensions of the Fibonacci numbers are explored, along with a detailed examination of their properties, characteristics, and some identities. We introduce and study the matrices with bidimensional Fibonacci numbers, focusing in particular on their recurrence relation, key properties, determinant, and various other identities. It is our purpose to study the matrix version of bidimensional Fibonacci numbers and provide new results and sometimes extensions of some results existing in the literature. We aim to introduce these matrices using the bidimensional Fibonacci numbers and to give the determinant of these matrices.

1. Introduction and background

Several investigators have worked with enthusiasm on numerical sequences. Their examinations cover a wide range of fascinating aspects, including exploring unique properties, revealing previously known identities, and even unlocking the mysteries behind generating functions and matrices. One such interesting sequence is the Fibonacci sequence of numbers. The Fibonacci sequence, $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$, in which each number is the sum of the previous two, is defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2}\,, \tag{1.1}$$

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for all integers $n \geq 2$, with initial values $f_0 = 0$ and $f_1 = 1$, the sequence $A000045$ in the OEIS [37]. This sequence satisfies the characteristic equation $x^2 - x - 1 = 0$.

Many researchers have studied some generalizations of the Fibonacci sequence, either by preserving the original recurrence relation while modifying the initial terms, or by preserving the initial terms while introducing slight modifications to the recursive relation, or again, by representing this sequence in matrix form. Several papers have been published discussing new sequences, their generalizations, extensions, and properties. These generalizations give identities similar to those obtained by the ordinary Fibonacci sequence. In the following works [1, 2, 3, 4, 5, 6, 7, 8, 10, 15, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 39, 41, 42], we can find not only properties of the Fibonacci sequence but also related sequences, such as Lucas, Pell, and Pell-Lucas, along with their various applications.

The matrix associated with the Fibonacci sequence, denoted as F, is defined by $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. In 1960, King [29] examined this matrix in his master's thesis, see also [21] and [31, p.395]. Notably, $|F| = -1$, where $|F|$ represents the determinant of a square matrix F. This matrix is considered as the generating matrix for the Fibonacci sequence. It is well known and well explored that, for any positive integer n, the n-th power of this matrix satisfies the relation

$$F^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}.$$

The above result on the sequence of Fibonacci numbers is stated in Theorem 20.1 in [31], for instance.

Fibonacci sequence satisfies many identities. For instance, consider the following identities for non-negative integers n:

$$f_n^2 - f_{n+1}f_{n-1} = (-1)^{n-1}, \tag{1.2}$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence. In particular, this property can be deduced from the determinant of the matrix F^n. It is the Cassini–Simson identity for the Fibonacci sequence that has already appeared in some previous works, see, for example, [31, Theorem 5.3].

The classical Tagiuri–Vajda identity for Fibonacci sequences has already appeared in some previous works (see, e.g., [39, Eq. (20a) on p. 28]).

LEMMA 1.1 (Tagiuri–Vajda's identity). *Let m, s, k be any non-negative integers. We have the following identity:*

$$f_{m+s}f_{m+k} - f_m f_{m+s+k} = (-1)^m f_s f_k\,, \tag{1.3}$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence.

In the next result, the first equation is stated in [39, Eq. (25)], the second is stated in [31, Eqs. (5.12) and (5.8.85)], and the last identity can be found in [39, Eq. (12)].

Lemma 1.2. *Let* $\{f_m\}_{m\geq 0}$ *be the Fibonacci sequence. For all integers* $m \geq 1$, *we have*

$$\text{(a)} \quad (f_m)^2 + (f_{m+1})^2 = f_{2m+1}, \tag{1.4}$$

$$\text{(b)} \quad (f_{m+1})^2 - (f_{m-1})^2 = f_{2m},$$

$$\text{(c)} \quad f_{m+1}^2 - f_m^2 = f_{m-1} f_{m+2}. \tag{1.5}$$

Following auxiliary result about the Fibonacci sequence, will be used in the development of this, and is similar to the Cassini–Simson identity (1.2), as can be seen.

Lemma 1.3. *For all non-negative integers* n, *we have*

$$f_{n+1} f_{n+2} - f_n f_{n+3} = (-1)^n, \tag{1.6}$$

where $\{f_n\}_{n\geq 0}$ *is the Fibonacci sequence.*

In [24] and [27], there were introduced and explored the Gaussian Fibonacci numbers in an algebraic form, defined by the recurrence relation $Gf_{n+1} = Gf_n + iGf_{n-1}$, with initial conditions $Gf_0 = 0$ and $Gf_1 = 1$, where i is the imaginary unit. This formulation extends the traditional unidimensional recurrence model into a complex number framework, offering a broader mathematical perspective. In [34], the authors consider the complex matrix $C = \begin{bmatrix} 1+i & 1 \\ 1 & i \end{bmatrix}$, and it has been observed that $C \cdot F^n = \begin{bmatrix} Gf_{n+2} & Gf_{n+1} \\ Gf_{n+1} & Gf_n \end{bmatrix}$.

Building upon this foundation, we introduce the concept of matrices involving bidimensional Fibonacci numbers, defined in Section 3, and our research problem is the determination of the determinant of the respective matrices. Using this determinant, some new algebraic properties of these numbers are obtained.

The structure of the present work is divided into five sections, as outlined below. In the Introduction, we provide an overview of the fundamental concepts of Fibonacci numbers in their unidimensional (classical or ordinary) form, emphasizing some notable studies. In Section 2, we briefly present the bidimensional Fibonacci numbers $\{f_{(m,n)}\}_{m,n\geq 0}$ and describe a list of key results (both well-known and new) on the bidimensional Fibonacci sequence that this work is concerned with. Also, we state some identities involving these types of numbers. So, properties and identities that will help to establish the relationship between unidimensional and bidimensional versions of the Fibonacci sequence are presented. In Section 3, we define matrices involving the bidimensional Fibonacci sequence denoted by $\{\mathcal{F}_{(m,n)}\}_{m,n\geq 0}$, explaining some characteristics and properties. We show that the elements of this matrix sequence satisfy a recurrence relation similar to the two-dimensional recurrence equation for Fibonacci numbers, and we provide some other results. In Section 4, we state our main result and show that the matrices for the bidimensional Fibonacci sequence are non-singular. In fact, we derive a formula to express this determinant. Finally, in Section 5, we analyze the results obtained in this study and provide suggestions for future research directions.

The study of k-dimensional forms of Horadam-like sequences for $k \geq 0$ has gained significant attention in the literature. For instance, in [34] the authors investigate the bidimensional and tridimensional identities for Fibonacci numbers in their complex form. They also studied Gaussian Fibonacci numbers along with their bidimensional recurrence relations in [33]. In [1], the authors explore n-dimensional forms associated with the Mersenne sequence, whereas [40] provides an overview of the Leonardo sequence and examines the bidimensional recurrence relations derived from its unidimensional model. Similarly, [5] and [32] extend the Narayana sequence by investigating its bidimensional and tridimensional recurrence relations, generalizing its unidimensional structure. Moreover, in [11, 12, 13, 14] the authors present bidimensional extensions of balancing, Lucas-balancing, cobalancing, and Lucas-cobalancing numbers, offering an in-depth analysis of the properties of these newly introduced sequences. In [38], the n-dimensional structures associated with the classical and Gaussian Jacobshtal sequence are investigated. More recently, in [9, 16] the authors introduce and study the k-dimensional recurrence relations of the Gersenne-like sequence.

These and other related studies continue to inspire further research into bidimensional and multidimensional extensions of numerical sequences, expanding upon their unidimensional foundations and unveiling new and interesting mathematical properties.

2. Bidimensional Fibonacci numbers

The process of complexification of the Fibonacci sequence is associated with the insertion of the imaginary unit, the dimensional increase and as well as its corresponding algebraic representation. This section revisits some aspects inherent to recursive relations and bidimensional identities defined from the unidimensional recursive model. As noted by Harman [22], the numbers represented by (n, m) correspond to Gaussian integers of the form $(n, m) = n + mi$, where n and m are integers. Several researchers have explored the extension of one-dimensional Fibonacci sequence identities to two, three, or even higher dimensions through n-dimensional recurrence relations. For further discussions on this topic, see [4, 24, 27, 33, 34, 35, 40], among others.

For the bidimensional version of Fibonacci numbers, we revisit the definition to contextualize this work, studying some of their properties. Additionally, we derive identities involving the concept of bidimensional and complex Fibonacci numbers.

Definition 2.1 (cf., e.g., [22]). For integers $m \geq 0$ and $n \geq 0$, the bidimensional Fibonacci numbers $\{f_{(m,n)}\}_{m,n\geq 0}$ are defined recursively by

$$f_{(m,n)} := f_{(m-1,n)} + f_{(m-2,n)}, \quad \text{for all } n \text{ and } m \geq 2\,, \tag{2.1}$$

$$f_{(m,n)} := f_{(m,n-1)} + f_{(m,n-2)}, \quad \text{for all } m \text{ and } n \geq 2\,, \tag{2.2}$$

with initial terms $f_{(0,0)} = 0$, $f_{(1,0)} = 1$, $f_{(0,1)} = i$, $f_{(1,1)} = 1 + i$, where i is the imaginary unit and $i^2 = -1$.

Definition 2.1 is correct in the sense that $f_{(m,n)}$ does not depend on the path we use for calculation. For instance, in determining $f_{(2,2)}$, we observe that:

$$f_{(2,2)} = \begin{cases} f_{(1,2)} + f_{(0,2)}, & \text{for a path (a),} \\ f_{(2,1)} + f_{(2,0)}, & \text{for a path (b).} \end{cases}$$

- First, consider the path (a): using Equation (2.1), we have

$$f_{(1,2)} = f_{(1,1)} + f_{(1,0)} = (1 + i) + 1 = 2 + i\ ,$$

and

$$f_{(0,2)} = f_{(0,1)} + f_{(0,0)} = i + 0 = i\ ,$$

hence,

$$f_{(2,2)} = f_{(1,2)} + f_{(0,2)} = (2 + i) + i = 2 + i2\,.$$

- Now, consider the path (b): by Equation (2.2), we obtain

$$f_{(2,1)} = f_{(1,1)} + f_{(0,1)} = (1 + i) + i = 1 + i2\ ,$$

and

$$f_{(2,0)} = f_{(1,0)} + f_{(0,0)} = 1 + 0 = 1\ ,$$

hence,

$$f_{(2,2)} \quad = \quad f_{(2,1)} + f_{(2,0)} = (1 + i2) + 1 = 2 + i2\ ,$$

which coincides with the value found for $f_{(2,2)}$ where we used the path (a).

In general, we have the following result.

Proposition 2.2 ([22]). *Let $\{f_{(m,n)}\}_{m,n\geq 0}$ be the bidimensional Fibonacci sequence. Then $f_{(m,n)}$ is independent of the path chosen for its computation, where m and n are integers such that $m \geq 2$ and $n \geq 2$.*

As an illustrative example, we have calculated a few terms from three branches of the bidimensional Fibonacci sequence:

$$\{f_{(m,0)}\}_{m\geq 0} = \{0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots\},$$
$$\{f_{(0,n)}\}_{n\geq 0} = \{0, i,\ i,\ i2,\ 3i,\ 5i, 8i,\ \ldots\},$$
$$\{f_{(m,m)}\}_{m\geq 0} = \{0, 1+i,\ 1+i,\ 2+i2,\ 6+i6,\ 15+i15,\ \ldots\}.$$

In the next couple of lemmata, we state some known results on bidimensional Fibonacci numbers.

Lemma 2.3 ([34, Lemma 8, (a) and (b)]). *Let m and n denote arbitrary non-negative integers. The following properties hold:*

$$f_{(m,0)} = f_m, \tag{2.3}$$
$$f_{(0,n)} = if_n. \tag{2.4}$$

From (2.3) and (2.4), we see that the bidimensional Fibonacci number $f_{(m,0)}$ is real, while $f_{(0,n)}$ is a pure imaginary complex number. Furthermore, $f_{(m,0)} = 0 = f_{(0,n)}$ only when $m = n = 0$.

Lemma 2.4. [34, Lemma 8, (c) and (d)] *Let m and n denote arbitrary non-negative integers. The following properties hold:*

$$\begin{aligned} f_{(m,1)} &= f_m + if_{m+1}, \\ f_{(1,n)} &= f_{n+1} + if_n. \end{aligned} \tag{2.5}$$

Equation (2.5) provides the Gaussian Fibonacci sequence introduced by Horadam in [24] and studied in [4, 22, 27], among others.

The following result establishes a connection between the bidimensional Fibonacci and the sequence of classical Fibonacci.

Lemma 2.5 ([34, Theorem 9]). *For non-negative integers n and m, the bidimensional Fibonacci numbers are described as follows:*

$$f_{(m,n)} = f_m f_{n+1} + if_{m+1}f_n. \tag{2.6}$$

A direct consequence of Lemma 2.5 is that:

Corollary 2.6. *Let n and m be non-negative integers, then:*

(a) *$f_{(m,n)}$ has a non-zero real part for all $m > 0$,*
(b) *$f_{(m,n)}$ has a non-zero imaginary part for all $n > 0$,*
(c) *$f_{(m,n)} \neq 0$ is a complex number for $m \neq n$.*

The next result is the main result of this section, and it is shown that the complex numbers $f_{(m,n)}$ and $f_{(n,m)}$ also interchange the real and complex parts.

THEOREM 2.7. *For all non-negative integers m and n such that $m \neq n$, if $f_{(m,n)} = a + ib$ then $f_{(n,m)} = b + ia$, for some integers a and b.*

PROOF. By Equation (2.6), we know that

$$f_{(m,n)} = f_m f_{n+1} + i f_{m+1} f_n,$$

making $a = f_m f_{n+1}$ and $b = f_{m+1} f_n$ we get

$$f_{(m,n)} = a + ib.$$

Again, by Equation (2.6), we obtain

$$\begin{aligned} f_{(n,m)} &= f_n f_{m+1} + i f_{n+1} f_m \\ &= b + ia, \end{aligned}$$

as required. □

Combining Lemma 2.5 and Theorem 2.7 we have the following.

PROPOSITION 2.8. *For all non-negative integers m, we have $f_{(m,m)} = a_1 + ia_1$ where $a_1 = f_m f_{m+1}$.*

The classical Tagiuri–Vajda identities of the unidimensional Fibonacci sequence are extended to bidimensional in the following results.

First, consider n, which is fixed in the second index coordinate and varies in the first coordinate.

THEOREM 2.9 (First Tagiuri–Vajda's identity). *Let m, n, r, and s be arbitrary non-negative integers. The following identity holds*

$$f_{(m+r,n)} f_{(m+s,n)} - f_{(m,n)} f_{(m+r+s,n)} = (-1)^m (f_r f_s f_{2n+1} + i f_r f_s f_n f_{n+1}). \tag{2.7}$$

PROOF. By Lemma 2.5 we have

$$\begin{aligned} f_{(m+r,n)} f_{(m+s,n)} &= (f_{m+r} f_{n+1} + i f_{m+r+1} f_n)(f_{m+s} f_{n+1} + i f_{m+s+1} f_n) \\ &= (f_{m+r} f_{m+s} f_{n+1}^2 - f_{m+r+1} f_{m+s+1} f_n^2) \\ &\quad + i(f_{m+r} f_{m+s+1} f_n f_{n+1} + f_{m+r+1} f_{m+s} f_n f_{n+1}). \end{aligned}$$

Also

$$\begin{aligned} f_{(m,n)} f_{(m+r+s,n)} &= (f_m f_{m+r+s} f_{n+1}^2 - f_{m+1} f_{m+1+r+s} f_n^2) \\ &\quad + i(f_m f_{m+r+s+1} f_n f_{n+1} + f_{m+1} f_{m+r+s} f_n f_{n+1}). \end{aligned}$$

Applying Equations (1.3) and (1.4), the real component of $f_{(m+r,n)}f_{(m+s,n)} - f_{(m,n)}f_{(m+r+s,n)}$ is

$$\begin{aligned}
&(f_{m+r}f_{m+s}f_{n+1}^2 - f_{m+r+1}f_{m+s+1}f_n^2) - (f_m f_{m+r+s}f_{n+1}^2 - f_{m+1}f_{m+1+r+s}f_n^2)\\
&= (f_{m+r}f_{m+s} - f_m f_{m+r+s})f_{n+1}^2 - (f_{m+1}f_{m+1+r+s} - f_{m+1+r}f_{m+1+s})f_n^2\\
&= (-1)^m f_r f_s f_{n+1}^2 - (-1)^{m+1} f_r f_s f_n^2 = (-1)^m f_r f_s (f_{n+1}^2 + f_n^2)\\
&= (-1)^m f_r f_s f_{2n+1}.
\end{aligned}$$

Again, applying Equations (1.1) and (1.3), the imaginary component of $f_{(m+r,n)}f_{(m+s,n)} - f_{(m,n)}f_{(m+r+s,n)}$ is

$$\begin{aligned}
f_{m+r}&f_{m+s+1}f_n f_{n+1} + f_{m+r+1}f_{m+s}f_n f_{n+1}\\
&\quad - (f_m f_{m+r+s+1}f_n f_{n+1} + f_{m+1}f_{m+r+s}f_n f_{n+1})\\
&= (f_{m+r}f_{m+s+1} - f_m f_{m+r+s+1})f_n f_{n+1}\\
&\quad + (f_{(m+1)+r}f_{(m+1)+(s-1)} - f_{m+1}f_{(m+1)+r+(s-1)})f_n f_{n+1}\\
&= (-1)^m f_r f_s f_n f_{n+1},
\end{aligned}$$

and we have the validity of the result. □

As usual, from Tagiuri–Vajda's identity, we get the results establishing d'Ocagne's identity, Catalan's identity, and Cassini's identity for the bidimensional Fibonacci sequence $\{f_{(m,n)}\}_{m,n\geq 0}$.

Proposition 2.10 (First d'Ocagne's identity). *The following identity holds for non-negative integers m, n and t with $t \geq m$:*

$$f_{(t,n)}f_{(m+1,n)} - f_{(m,n)}f_{(t+1,n)} = (-1)^m (f_{t-m}f_{2n+1} + i f_{t-m}f_n f_{n+1}).$$

Proof. Taking $r = t - m$ and $s = 1$ in Equation (2.7) gives

$$f_{(t,n)}f_{(m+1,n)} - f_{(m,n)}f_{(t+1,n)} = (-1)^m (f_{t-m}f_1 f_{2n+1} + i f_{t-m}f_1 f_n f_{n+1}),$$

as $f_1 = 1$, and this completes the proof. □

In a similar way to Proposition 2.10 we obtain the Catalan identity.

Proposition 2.11 (First Catalan's identity). *For non-negative integers n, t and s with $t \geq s$, it holds:*

$$f_{(t,n)}^2 - f_{(t-s,n)}f_{(t+s,n)} = (-1)^{t-s} f_s^2 (f_{2n+1} + i f_n f_{n+1}).$$

Proof. Using $r = s$ and $m+s = t$ in Equation (2.7), we get the result. □

At the expense of the above result, we obtain the Cassini identity.

Corollary 2.12 (First Cassini's identity). *Let n and $t \geq 1$ be any integers, then*

$$f_{(t,n)}^2 - f_{(t-1,n)} f_{(t+1,n)} = (-1)^{t-1}(f_{2n+1} + i f_n f_{n+1}).$$

Proof. It is enough to take $s = 1$ in Proposition 2.11. □

A similar result can be obtained by considering the variation of the indexes in the second coordinate. For the sake of brevity, we omit the proof of the following results.

Theorem 2.13 (Second Tagiuri–Vajda's identity). *Let m, n, r, and s be arbitrary non-negative integers. The following identity holds*

$$f_{(m,n+r)} f_{(m,n+s)} - f_{(m,n)} f_{(m,n+r+s)} = (-1)^n(-f_{2m+1} f_r f_s + i f_m f_{m+1} f_r f_s).$$

Proposition 2.14 (Second d'Ocagne's identity). *The following identity holds for non-negative integers m, n and t with $t \geq n$:*

$$f_{(m,t)} f_{(m,n+1)} - f_{(m,n)} f_{(m,t+1)} = (-1)^n(-f_{2m+1} f_{t-n} + i f_m f_{m+1} f_{t-n} f_s).$$

Proposition 2.15 (Second Catalan's identity). *For non-negative integers m, t and s with $t \geq s$, it holds:*

$$f_{(m,t)}^2 - f_{(m,t-s)} f_{(m,t+s)} = (-1)^{t-s} f_s^2 (-f_{2m+1} + i f_m f_{m+1}).$$

Corollary 2.16 (Second Cassini's identity). *Let m and $t \geq 1$ be any integers. Then*

$$f_{(m,t)}^2 - f_{(m,t-1)} f_{(m,t+1)} = (-1)^{t-1}(-f_{2m+1} + i f_m f_{m+1}).$$

3. Matrices involving bidimensional Fibonacci numbers

This section introduces the bidimensional Fibonacci matrices, along with their key properties and some identities.

Definition 3.1. For all non-negative integers m and n, the bidimensional Fibonacci matrices $\{\mathcal{F}_{(m,n)}\}_{m,n\geq 0}$ are given by

$$\mathcal{F}_{(m,n)} := \begin{bmatrix} f_{(m+2,n+2)} & -f_{(m+1,n+1)} \\ f_{(m+1,n+1)} & -f_{(m,n)} \end{bmatrix}, \tag{3.1}$$

where $\{f_{(m,n)}\}_{m,n\geq 0}$ is the bidimensional Fibonacci sequence.

In particular, for $m = n = 0$, we obtain

$$\mathcal{F}_{(0,0)} = \begin{bmatrix} f_{(2,2)} & -f_{(1,1)} \\ f_{(1,1)} & -f_{(0,0)} \end{bmatrix} = \begin{bmatrix} 2+i2 & -(1+i) \\ 1+i & 0 \end{bmatrix}. \tag{3.2}$$

While for $m = 1$ and $n = 0$, we get

$$\mathcal{F}_{(1,0)} = \begin{bmatrix} f_{(3,2)} & -f_{(2,1)} \\ f_{(2,1)} & -f_{(1,0)} \end{bmatrix} = \begin{bmatrix} 4+i3 & -(1+i2) \\ 1+i2 & -1 \end{bmatrix}. \tag{3.3}$$

As for $m = 0$ and $n = 1$, we have

$$\mathcal{F}_{(0,1)} = \begin{bmatrix} f_{(2,3)} & -f_{(1,2)} \\ f_{(1,2)} & -f_{(0,1)} \end{bmatrix} = \begin{bmatrix} 3+i4 & -(2+i) \\ 2+i & -i \end{bmatrix}. \tag{3.4}$$

Finally, for $m = n = 1$, we get

$$\mathcal{F}_{(1,1)} = \begin{bmatrix} f_{(3,3)} & -f_{(2,2)} \\ f_{(2,2)} & -f_{(1,1)} \end{bmatrix} = \begin{bmatrix} 6+i6 & -(2+i2) \\ 2+i2 & -(1+i) \end{bmatrix}. \tag{3.5}$$

In the next result we show the recurrence relation for the sequence of matrices $\{\mathcal{F}_{(m,n)}\}_{m,n\geq 0}$.

Proposition 3.2. *The bidimensional Fibonacci matrices* $\{\mathcal{F}_{(m,n)}\}_{m,n\geq 0}$ *satisfy the following bidimensional recurrence relations*

$$\mathcal{F}_{(m+1,n)} = \mathcal{F}_{(m,n)} + \mathcal{F}_{(m-1,n)} \quad \text{for } m \geq 1 \text{ and for all } n, \tag{3.6}$$
$$\mathcal{F}_{(m,n+1)} = \mathcal{F}_{(m,n)} + \mathcal{F}_{(m,n-1)} \quad \text{for all } m \text{ and for } n \geq 1, \tag{3.7}$$

with initial terms $\mathcal{F}_{(0,0)}$, $\mathcal{F}_{(1,0)}$, $\mathcal{F}_{(0,1)}$, *and* $\mathcal{F}_{(1,1)}$ *given in* (3.2), (3.3), (3.4) *and* (3.5), *respectively.*

Proof. Combining Definition 3.1 and Equations (2.1) and (2.2), we have

$$\begin{aligned}
\mathcal{F}_{(m,n)} + \mathcal{F}_{(m-1,n)} &= \begin{bmatrix} f_{(m+2,n+2)} & -f_{(m+1,n+1)} \\ f_{(m+1,n+1)} & -f_{(m,n)} \end{bmatrix} + \begin{bmatrix} f_{(m+1,n+2)} & -f_{(m,n+1)} \\ f_{(m,n+1)} & -f_{(m-1,n)} \end{bmatrix} \\
&= \begin{bmatrix} f_{(m+2,n+2)} + f_{(m+1,n+2)} & -(f_{(m+1,n+1)} + f_{(m,n+1)}) \\ (f_{(m+1,n+1)} + f_{(m,n+1)}) & -(f_{(m,n)} + f_{(m-1,n)}) \end{bmatrix} \\
&= \begin{bmatrix} f_{(m+3,n+2)} & -f_{(m+2,n+1)} \\ f_{(m+2,n+1)} & -f_{(m+1,n)} \end{bmatrix} \\
&= \mathcal{F}_{(m+1,n)},
\end{aligned}$$

which verifies the first branch result.

In a similar way, the proof of the second branch can be done. □

Proposition 3.2 is valid in the sense that the value of $\mathcal{F}_{(m,n)}$ remains invariant regardless of the calculation path chosen. For example, to determine $\mathcal{F}_{(2,2)}$, we observe that:

$$\mathcal{F}_{(2,2)} = \begin{cases} \mathcal{F}_{(1,2)} + \mathcal{F}_{(0,2)}, & \text{path (a)}, \\ \mathcal{F}_{(2,1)} + \mathcal{F}_{(2,0)}, & \text{path (b)}. \end{cases}$$

Considering path (a) and applying (3.6), we have

$$\begin{aligned}\mathcal{F}_{(1,2)} &= \mathcal{F}_{(1,1)} + \mathcal{F}_{(1,0)} \\ &= \begin{bmatrix} 6+i6 & -(2+i2) \\ 2+i2 & -(1+i) \end{bmatrix} + \begin{bmatrix} 4+i3 & -(1+i2) \\ 1+i2 & -1 \end{bmatrix} = \begin{bmatrix} 10+i9 & -(3+i4) \\ 3+i4 & -(2+i) \end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{(0,2)} &= \mathcal{F}_{(0,1)} + \mathcal{F}_{(0,0)} \\ &= \begin{bmatrix} 3+i4 & -(2+i) \\ 2+i & -i \end{bmatrix} + \begin{bmatrix} 2+i2 & -(1+i) \\ 1+i & 0 \end{bmatrix} = \begin{bmatrix} 5+i6 & -(3+i2) \\ 3+i2 & -i \end{bmatrix}.\end{aligned}$$

So

$$\begin{aligned}\mathcal{F}_{(2,2)} &= \mathcal{F}_{(1,2)} + \mathcal{F}_{(0,2)} \\ &= \begin{bmatrix} 10+i9 & -(3+i4) \\ 3+i4 & -(2+i) \end{bmatrix} + \begin{bmatrix} 5+i6 & -(3+i2) \\ 3+i2 & -i \end{bmatrix} = \begin{bmatrix} 15+i15 & -(6+i6) \\ 6+i6 & -(2+i2) \end{bmatrix}.\end{aligned}$$

Taking path (b) into account and applying (3.7), we obtain

$$\begin{aligned}\mathcal{F}_{(2,1)} &= \mathcal{F}_{(1,1)} + \mathcal{F}_{(0,1)} \\ &= \begin{bmatrix} 6+i6 & -(2+i2) \\ 2+i2 & -(1+i) \end{bmatrix} + \begin{bmatrix} 3+i4 & -(2+i) \\ 2+i & -i \end{bmatrix} = \begin{bmatrix} 9+i10 & -(4+i3) \\ 4+i3 & -(1+i2) \end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{(2,0)} &= \mathcal{F}_{(1,0)} + \mathcal{F}_{(0,0)} \\ &= \begin{bmatrix} 4+i3 & -(1+i2) \\ 1+i2 & -1 \end{bmatrix} + \begin{bmatrix} 2+i2 & -(1+i) \\ 1+i & 0 \end{bmatrix} = \begin{bmatrix} 6+i5 & -(2+i3) \\ 2+i3 & -1 \end{bmatrix}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{F}_{(2,2)} &= \mathcal{F}_{(2,1)} + \mathcal{F}_{(2,0)} \\ &= \begin{bmatrix} 9+i10 & -(4+i3) \\ 4+i3 & -(1+i2) \end{bmatrix} + \begin{bmatrix} 6+i5 & -(2+i3) \\ 2+i3 & -1 \end{bmatrix} = \begin{bmatrix} 15+i15 & -(6+i6) \\ 6+i6 & -(2+i2) \end{bmatrix},\end{aligned}$$

which coincides with the value found for $\mathcal{F}_{(2,2)}$ where we used the path (a).

A matrix $\mathcal{F}_{(m,n)}$ can be obtained by the following formula.

Theorem 3.3. *For all non-negative integers m and n, the bidimensional Fibonacci matrices $\{\mathcal{F}_{(m,n)}\}_{m,n\geq 0}$ are given by*

$$\mathcal{F}_{(m,n)} = \begin{bmatrix} f_{m+2}f_{n+3} + if_{m+3}f_{n+2} & -(f_{m+1}f_{n+2} + if_{m+2}f_{n+1}) \\ f_{m+1}f_{n+2} + if_{m+2}f_{n+1} & -(f_m f_{n+1} + if_{m+1}f_n) \end{bmatrix}.$$

Proof. Combining Lemma 2.5 and identity (3.1) we get the result. □

The following result follows directly from Theorem 3.3, taking into account the initial conditions of the Fibonacci sequence and relation (1.1). We omit the respective proof.

Corollary 3.4. *The following properties hold for the bidimensional Fibonacci matrices $\{\mathcal{F}_{(m,n)}\}_{m,n\geq 0}$:*

(a) $\mathcal{F}_{(m,0)} = \begin{bmatrix} 2f_{m+2} + if_{m+3} & -(f_{m+1} + if_{m+2}) \\ f_{m+1} + if_{m+2} & -f_m \end{bmatrix}$,

(b) $\mathcal{F}_{(0,n)} = \begin{bmatrix} f_{n+3} + i2f_{n+2} & -(f_{n+2} + if_{n+1}) \\ f_{n+2} + if_{n+1} & -if_n \end{bmatrix}$.

In a similar way to Theorem 2.7, the next result shows that in each entry of the matrix $\mathcal{F}_{(m,n)}$ the complex numbers also interchange the real and complex parts when we interchange m by n.

Proposition 3.5. *For all non-negative integers m and n, consider the bidimensional Fibonacci matrices $\{\mathcal{F}_{(m,n)}\}_{m,n\geq 0}$. If*

$$\mathcal{F}_{(m,n)} = \begin{bmatrix} x_1 + iy_1 & -(x_2 + iy_2) \\ x_2 + iy_2 & -(x_3 + iy_3) \end{bmatrix}, \text{ then } \mathcal{F}_{(n,m)} = \begin{bmatrix} y_1 + ix_1 & -(y_2 + ix_2) \\ y_2 + ix_2 & -(y_3 + ix_3) \end{bmatrix},$$

where x_t and y_t are integers for $t = 1, 2$ and 3, and i is the imaginary unit.

Proof. By combining Theorem 2.7 with Theorem 3.3, we obtain the statement. □

The following result follows directly from Proposition 2.8 and Proposition 3.5, and we omit its proof in the interest of brevity.

Proposition 3.6. *For all non-negative integers m, the bidimensional Fibonacci matrix $\mathcal{F}_{(m,m)}$ is given by*

$$\mathcal{F}_{(m,m)} = \begin{bmatrix} 1+i & 0 \\ 0 & 1+i \end{bmatrix} \begin{bmatrix} f_{m+2}f_{m+3} & -f_{m+1}f_{m+2} \\ f_{m+1}f_{m+2} & -f_m f_{m+1} \end{bmatrix}. \tag{3.8}$$

4. Determinant of the bidimensional Fibonacci matrices

For all non-negative integers m and n, the determinant of the bidimensional Fibonacci matrices $\mathcal{F}_{(m,n)}$ is given by

$$\det \mathcal{F}_{(m,n)} = \begin{vmatrix} f_{(m+2,n+2)} & -f_{(m+1,n+1)} \\ f_{(m+1,n+1)} & -f_{(m,n)} \end{vmatrix} = \left(f_{(m+1,n+1)}\right)^2 - f_{(m+2,n+2)} f_{(m,n)}.$$

EXAMPLE 4.1. To illustrate this notion, observe that:

(a) $|\mathcal{F}_{(0,0)}| = \begin{vmatrix} 2+i2 & -(1+i) \\ 1+i & 0 \end{vmatrix} = (2+i2)(0) - [-(1+i)(1+i)] = i2,$

(b) $|\mathcal{F}_{(1,0)}| = \begin{vmatrix} 4+i3 & -(1+i2) \\ 1+i2 & -1 \end{vmatrix} = -7+i,$

(c) $|\mathcal{F}_{(0,1)}| = \begin{vmatrix} 3+i4 & -(2+i) \\ 2+i & -i \end{vmatrix} = 7+i,$

(d) $|\mathcal{F}_{(1,1)}| = \begin{vmatrix} 6+i6 & -(2+i2) \\ 2+i2 & -(1+i) \end{vmatrix} = -i4.$

Note that in Example 4.1 we have $|\mathcal{F}_{(0,0)}| = i2 \neq 0$, and this determinant has a real part null. Similarly, $|\mathcal{F}_{(1,1)}| = -i4 \neq 0$ with a real part null. A straightforward calculation gives that $|\mathcal{F}_{(2,2)}| = i12 \neq 0$. These previous cases are special instances of the next result.

PROPOSITION 4.2. *Let $\{\mathcal{F}_{(m,n)}\}_{m,n\geq 0}$ be the sequence of bidimensional Fibonacci matrices of order 2 defined in* (3.1). *If $m = n$, then the determinant $|\mathcal{F}_{(n,n)}|$ is a pure imaginary complex number.*

PROOF. Firstly, we prove that $|\mathcal{F}_{(n,n)}|$ has a real part null. Indeed, a straightforward calculation with the help of (3.8) gives

$$\begin{aligned} |\mathcal{F}_{(n,n)}| &= \begin{vmatrix} 1+i & 0 \\ 0 & 1+i \end{vmatrix} \begin{vmatrix} f_{n+2}f_{n+3} & -f_{n+1}f_{n+2} \\ f_{n+1}f_{n+2} & -f_n f_{n+1} \end{vmatrix} \\ &= i2\big[(f_{n+1}f_{n+2})^2 - f_n f_{n+1} f_{n+2} f_{n+3})\big]. \end{aligned}$$

Now, we need to show that

$$(f_{n+1}f_{n+2})^2 - f_n f_{n+1} f_{n+2} f_{n+3} \neq 0.$$

Note that

$$(f_{n+1}f_{n+2})^2 - f_n f_{n+1} f_{n+2} f_{n+3} = f_{n+1}f_{n+2}(f_{n+1}f_{n+2} - f_n f_{n+3})\,.$$

Since $f_{n+1}f_{n+2} \neq 0$ for all non-negative integers n, it is enough to show that

$$f_{n+1}f_{n+2} - f_n f_{n+3} \neq 0.$$

According to Equation (1.6), we have $f_{n+1}f_{n+2} - f_n f_{n+3} = (-1)^n$, which verifies the result. $\square$

Our main result of this section follows directly from Proposition 4.2.

THEOREM 4.3. *For all non-negative integers n, the determinant $|\mathcal{F}_{(n,n)}|$ is non-singular.*

PROOF. Suffice it to note that $|\mathcal{F}_{(n,n)}| = (-1)^n i2 f_{n+1}f_{n+2} \neq 0$. $\square$

Theorem 4.3 has the following consequence.

COROLLARY 4.4. *For all non-negative integers n, we have*

$$\mathcal{F}_{(n,n)}{}^{-1} = \frac{1}{(-1)^n i2 f_{n+1}f_{n+2}} \begin{bmatrix} 1+i & 0 \\ 0 & 1+i \end{bmatrix} \begin{bmatrix} -f_n f_{n+1} & f_{n+1}f_{n+2} \\ -f_{n+1}f_{n+2} & f_{n+2}f_{n+3} \end{bmatrix}.$$

Now, we present an interesting result which shows that the determinants of the bidimensional matrices $\mathcal{F}_{(m,n)}$ and $\mathcal{F}_{(n,m)}$ are complex numbers anti-conjugate (opposite of the conjugate). For instance, in Example 4.1 we have $|\mathcal{F}_{(1,0)}| = -7+i$ and $|\mathcal{F}_{(0,1)}| = 7+i$.

PROPOSITION 4.5. *For all non-negative integers m and n, if $|\mathcal{F}_{(m,n)}| = x+iy$ for some integers x and y, then $|\mathcal{F}_{(n,m)}| = -(x-iy)$.*

PROOF. Using Corollary 2.6, we obtain

$$f_{(m+2,n+2)} = a_2 + ib_2, \quad f_{(m+1,n+1)} = a_1 + ib_1 \text{ and } f_{(m,n)} = a_0 + ib_0\,,$$

for some complex numbers $a_t + ib_t$ with $t \in \{0,1,2\}$. So,

$$\begin{aligned} |\mathcal{F}_{(m,n)}| &= \begin{vmatrix} f_{(m+2,n+2)} & -f_{(m+1,n+1)} \\ f_{(m+1,n+1)} & -f_{(m,n)} \end{vmatrix} = \begin{vmatrix} a_2+ib_2 & -(a_1+ib_1) \\ a_1+ib_1 & -(a_0+ib_0) \end{vmatrix} \\ &= (a_1^2 - b_1^2 - a_0a_2 + b_0b_2) + i(2a_1b_1 - a_2b_0 - a_0b_2) \\ &= x+iy, \end{aligned}$$

making $x = a_1^2 - b_1^2 - a_0a_2 + b_0b_2$ and $y = 2a_1b_1 - a_2b_0 - a_0b_2$.

Using again Corollary 2.6, we have

$$\begin{aligned} |\mathcal{F}_{(n,m)}| &= \begin{vmatrix} f_{(n+2,m+2)} & -f_{(n+1,m+1)} \\ f_{(n+1,m+1)} & -f_{(n,m)} \end{vmatrix} = \begin{vmatrix} b_2+ia_2 & -(b_1+ia_1) \\ b_1+ia_1 & -(b_0+ia_0) \end{vmatrix} \\ &= (b_1^2 - a_1^2 + a_0a_2 - b_0b_2) + i(2a_1b_1 - a_2b_0 - a_0b_2) \\ &= -(a_1^2 - b_1^2 - a_0a_2 + b_0b_2) + i(2a_1b_1 - a_2b_0 - a_0b_2) \\ &= -x+iy = -(x-iy), \end{aligned}$$

which completes the proof. $\square$

PROPOSITION 4.6. *The following identities hold for non-negative integers m and n:*
(a) $|\mathcal{F}_{(m,0)}| \neq 0$,
(b) $|\mathcal{F}_{(0,n)}| \neq 0$.

PROOF. (a) By Corollary 3.4 we have

$$\begin{aligned}|\mathcal{F}_{(m,0)}| &= \begin{vmatrix} 2f_{m+2} + if_{m+3} & -(f_{m+1} + if_{m+2}) \\ f_{m+1} + if_{m+2} & -f_m \end{vmatrix} \\ &= (f_{m+1} + if_{m+2})^2 - 2f_m(f_{m+2} + if_{m+3}) \\ &= f_{m+1}^2 - f_{m+2}^2 - 2f_{m+2} + i2(f_{m+1}f_{m+2} - f_m f_{m+3}).\end{aligned}$$

Making use of (1.5) and (1.6) we get

$$|\mathcal{F}_{(m,0)}| = -f_m(f_{m+2} + f_{m+4}) + i2(-1)^m.$$

It is enough to observe now that $f_m(f_{m+2} + f_{m+4}) + i2(-1)^m$ is a non-zero complex number for all m.

(b) By Proposition 3.5 and the previous item (a), we have

$$|\mathcal{F}_{(0,n)}| \quad = f_n(f_{n+2} + f_{n+4}) + i2(-1)^n,$$

and this completes the proof. □

To conclude this section, Theorem 4.3 establishes that all bidimensional Fibonacci matrices of the sequence $\{\mathcal{F}_{(m,m)}\}_{m\geq 0}$ are invertible. Inspired by Theorem 4.3 and Example 4.1, we propose the following conjecture:

CONJECTURE. Let $\{\mathcal{F}_{(m,n)}\}_{m,n\geq 0}$ be the sequence of bidimensional Fibonacci matrices of order 2. For all non-negative integers m and n, the determinant is non-vanishing, that is, the determinant satisfies:

$$|\mathcal{F}_{(m,n)}| \neq 0.$$

5. Conclusion

As King's master's thesis shows ([29]), the matrices associated with the number sequences play a fundamental role in research mathematics. This work introduced the matrices of order 2 involving the bidimensional Fibonacci numbers, establishing bidimensional recurrence relations and deriving several identities in complex form. These contributions offer valuable insights into the structural complexity of these numbers through a rigorous mathematical analysis of their recurrence equations. Using matrix algebra, in this paper, matrix representation of bidimensional Fibonacci sequences was obtained. The results presented not only extend existing findings in the literature but also introduce new perspectives on the versatility of Fibonacci numbers and their generalizations, laying a foundation for further theoretical and applied research.

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