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CONDITIONAL EQUATIONS RELATED TO DRYGAS FUNCTIONAL EQUATIONS

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Abstract. We determine the solutions of the conditional Drygas equation for functions f_1 and f_2 that satisfy $(y^2+y)f_1(x) = (x^2+x)f_2(y)$ for all $(x, y) \in \mathbb{R}^2$ under the additional conditions $y = x^2$, or $y = \log(x), x > 0$ or $y = \exp(x)$.

1. Preliminaries

Recall that a function $A\colon \mathbb{R} \to \mathbb{R}$ is additive if the equation $A(x + y) = A(x) + A(y)$ holds for all $x, y \in \mathbb{R}$.

Kuczma [13] proved that any additive function $A\colon \mathbb{R} \to \mathbb{R}$ is $\mathbb{Q}$-homogeneous, that is,

$$A(sx) = sA(x),$$

for all $x \in \mathbb{R}$ and $s \in \mathbb{Q}$. A function $h\colon \mathbb{R} \to \mathbb{R}$ is called quadratic if the equation

$$h(x + y) + h(x - y) = 2h(x) + 2h(y)$$

holds for all $x, y \in \mathbb{R}$.

For instance, consider the additive functions $A_1, A_2\colon \mathbb{R} \to \mathbb{R}$. It is easy to see that $A_1(x)A_2(x)$ and $A_1(x^2)$, $x \in \mathbb{R}$, are quadratic.

A function $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is named symmetric biadditive if B is additive in each variable and satisfies $B(x, y) = B(y, x)$ for all $x, y \in \mathbb{R}$.

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In 1965, Aczél [1] showed that a quadratic function $h\colon \mathbb{R} \to \mathbb{R}$ can be associated with a symmetric biadditive function $B\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by the following formula

$$B(x,y) = \frac{1}{2}[h(x+y) - h(x) - h(y)], \quad x,y \in \mathbb{R}. \tag{1.1}$$

Aczél and Dhombres [2] proved that the function $h\colon \mathbb{R} \to \mathbb{R}$ is quadratic if and only if, there is a symmetric biadditive function $B\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $h(x) = B(x,x)$ for all $x \in \mathbb{R}$. This B is unique (see [2]). Moreover, the $\mathbb{Q}$-homogeneity of biadditive functions yields

$$B(rx, sy) = rsB(x,y), \quad h(rx) = B(rx,rx) = r^2 h(x),$$

for all $x, y \in \mathbb{R}$ and $r, s \in \mathbb{Q}$. By using (1.1) and induction on n, one can show that

$$h\Big(\sum_{i=0}^{n} \omega_i\Big) = \sum_{i=0}^{n} h(\omega_i) + 2 \sum_{0\le j<k\le n} B(\omega_j, \omega_k),$$

for all $n \in \mathbb{N}$ and $\omega_0, \dots, \omega_n \in \mathbb{R}$.

Some mathematicians have investigated additive functions A that satisfy the conditional equation $yA(x) = xA(y)$ for the pairs $(x,y) \in \mathbb{R}^2$ under the condition $P(x,y) = 0$ for some fixed polynomial P of two variables. For some special polynomials P this assumption implies that A is continuous (see for example [4, 12, 14, 15]).

Recently, Z. Boros and E. Garda-Mátyás [5] and [6], E. Garda-Mátyás [11] studied quadratic functions $h\colon \mathbb{R} \to \mathbb{R}$ that satisfy the additional condition

$$y^2 h(x) = x^2 h(y),$$

where (x,y) are arbitrary points on a specified curve.

J. Brzdęk and A. Mureńko [7] established the Gołąb-Schinzel equation under certain additional conditions.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \tag{1.2}$$

which was considered by Drygas [9] is known as the Drygas equation and its solutions as Drygas functions. It is a generalization of the quadratic functional equation. In [10], Ebanks et al. obtained the general solution of the Drygas functional equation as

$$f(x) = A(x) + B(x,x), \quad x \in \mathbb{R}, \tag{1.3}$$

where $A\colon \mathbb{R} \to \mathbb{R}$ is an additive function and $B\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a symmetric biadditive function. The continuous solutions of functional equation (1.2) on $\mathbb{R}$ are of the form $f(x) = \alpha x + \beta x^2$, where $\alpha, \beta \in \mathbb{R}$ are constants (see [16]).

Consider the sets

$$\begin{aligned}
\Delta_0 &= \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y = x^n\}, \quad n \in \mathbb{Z},\ |n| \geq 2,\\
\Delta_1 &= \{(x,y) \in \mathbb{R}^2 : y = x^2\},\\
\Delta_2 &= \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y = \log(x)\},\\
\Delta_3 &= \{(x,y) \in \mathbb{R}^2 : y = \exp(x)\}.
\end{aligned}$$

Motivated by the results of [5], this paper is devoted to finding Drygas functions $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$(y^2 + y) f_1(x) = (x^2 + x) f_2(y), \tag{1.4}$$

for the pairs $(x, y) \in \Delta_j$, where $j = 0, 1, 2, 3$.

M. Dehghanian et al. [8] investigated Drygas functions $f \colon \mathbb{R} \to \mathbb{R}$ that satisfy the conditional equation (1.4) on the graph of a power function.

Lemma 1.1. [5] *Let $m \in \mathbb{N}$ and $\mathbb{F}$ be a field. Suppose that Ω is a set, $\Gamma \subset \mathbb{F}$ contains at least $m + 1$ elements, and the functions $\Lambda_j \colon \Omega \to \mathbb{F}$ $(j = 0, 1, \ldots, m)$ satisfy*

$$\sum_{j=0}^{m} \Lambda_j(x) s^j = 0,$$

for all $x \in \Omega$ and $s \in \Gamma$. Then $\Lambda_j(x) = 0$ for all $x \in \Omega$ and $0 \leq j \leq m$.

This paper contains results for the Drygas functions that satisfy the equation (1.4) for $(x, y) \in \Delta_j$, where $j = 0, 1, 2, 3$.

2. Main results

In the following theorem, we apply Lemma 1.1 with $\Omega = \mathbb{R}_+$, $\mathbb{F} = \mathbb{R}$ and $\Gamma = \mathbb{Q}_+$, where $\mathbb{R}_+$ and $\mathbb{Q}_+$ are the sets of positive real and positive rational numbers, respectively.

Theorem 2.1. *Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is a Drygas function. Then f fulfills the conditional equation*

$$\left(x^2 + x\right) f(y) = \left(y^2 + y\right) f(x), \tag{2.1}$$

for $(x, y) \in \Delta_0$ if and only if

$$f(x) = \alpha \left(x + x^2\right), \quad x \in \mathbb{R},$$

where α is a real constant.

Proof. First, assume that f fulfills (2.1), $x \in \mathbb{R}_+$ and $n \geq 2$. In this case, the equation (2.1) becomes

$$(x+1) f\left(x^n\right)=\left(x^{2n-1}+x^{n-1}\right) f(x), \quad x \in \mathbb{R}_+. \tag{2.2}$$

Substituting $x+s$, $s \in \mathbb{Q}_+$, for x in (2.2), we get

$$(x+s+1) f\left((x+s)^n\right)=\left((x+s)^{2n-1}+(x+s)^{n-1}\right) f(x+s), \quad x \in \mathbb{R}_+.$$

By expanding the binomial terms, we obtain

$$\begin{aligned}(x+s+1) f\Big(\sum_{m=0}^{n}\binom{n}{m} x^m s^{n-m}\Big) \\ =\Big[\sum_{l=0}^{2n-1}\binom{2n-1}{l} x^l s^{2n-l-1}+\sum_{k=0}^{n-1}\binom{n-1}{k} x^k s^{n-k-1}\Big] f(x+s).\end{aligned} \tag{2.3}$$

By (1.3), there exist an additive function $A\colon \mathbb{R} \rightarrow \mathbb{R}$ and a symmetric biadditive function $B\colon \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x)=A(x)+B(x, x)$ for all $x \in \mathbb{R}$. Thus, the equation (2.3) takes the form

$$\begin{aligned}(x+s+1)\Bigg[\sum_{m=0}^{n}\Bigg(\binom{n}{m} s^{n-m} A\left(x^m\right)+\binom{n}{m}^2 s^{2n-2m} B\left(x^m, x^m\right)\Bigg) \\ +2 \sum_{0 \leq i<j \leq n}\binom{n}{i}\binom{n}{j} s^{2n-i-j} B\left(x^i, x^j\right)\Bigg] \\ -\sum_{k=0}^{n-1}\binom{n-1}{k} x^k s^{n-k-1}\left(f(x)+s A(1)+s^2 B(1,1)+2 s B(x, 1)\right) \\ -\sum_{l=0}^{2n-1}\binom{2n-1}{l} x^l s^{2n-l-1}\left(f(x)+s A(1)+s^2 B(1,1)+2 s B(x, 1)\right)=0.\end{aligned} \tag{2.4}$$

Hence, for any fixed $x \in \mathbb{R}_+$, we obtain the polynomial (2.4) of degree at most $2n+1$ in $\mathbb{R}_+$ which is equal to zero for all $s \in \mathbb{Q}_+$. By Lemma 1.1, every coefficient of (2.4) has to be equal to zero. Since the coefficient of s^{2n+1} is $B(1,1)-B(1,1)=0$, then the degree of the polynomial (2.4) is less than $2n+1$. Furthermore, from the coefficient of s^{2n}, we deduce that

$$A(1)+2B(x, 1)+(2n-1) x B(1,1)-x B(1,1)-B(1,1)-2n B(x, 1)=0,$$

for all $x \in \mathbb{R}_+$. Put $x=1$ in the above equality, we have $A(1)=B(1,1)=\frac{f(1)}{2}$. Therefore, as $|n| \geq 2$, we get

$$B(x, 1)=x B(1,1), \quad x \in \mathbb{R}_+. \tag{2.5}$$

From the coefficient of s^{2n-1}, we arrive at

$$\begin{aligned} 0 &= f(x) + (2n-1)x[A(1) + 2B(x,1)] + \binom{2n-1}{2} x^2 B(1,1) \\ &\quad - \binom{n}{1}^2 B(x,x) - 2n(x+1)B(x,1) - 2\binom{n}{2} B(x^2,1) \\ &= f(x) + (2n-1)(1+2x)xB(1,1) + (2n-1)(n-1)x^2 B(1,1) \\ &\quad - n^2 B(x,x) - 2nx^2(x+1)B(1,1) - n(n-1)x^2 B(1,1). \end{aligned}$$

Now with $A(1) = B(1,1)$, and (2.5), we obtain

$$f(x) = (x^2 + x)B(1,1) + n^2 B(x,x) - n^2 x^2 B(1,1), \tag{2.6}$$

for all $x \in \mathbb{R}_+$. Thus,

$$B(x,x) = \frac{f(x) + f(-x)}{2} = x^2 B(1,1) + n^2 \left[B(x,x) - x^2 B(1,1)\right], \quad x \in \mathbb{R}_+.$$

As $n \geq 2$,

$$B(x,x) = x^2 B(1,1) = \frac{f(1)}{2} x^2, \quad x \in \mathbb{R}_+. \tag{2.7}$$

By equations (2.6) and (2.7), we conclude that

$$f(x) = \frac{f(1)}{2}(x + x^2), \quad x \in \mathbb{R}_+.$$

Hence, $A(x) = f(x) - B(x,x) = \frac{f(1)}{2}x$ for all $x \in \mathbb{R}_+$. Also, for $x = 0$ above equation holds, because $f(0) = 0$.

Now, for $x = -u < 0$,

$$\begin{aligned} f(x) &= A(-u) + B(-u,-u) = -A(u) + (-1)^2 B(u,u) \\ &= \frac{f(1)}{2}\left(-u + (-u)^2\right) = \frac{f(1)}{2}\left(x + x^2\right). \end{aligned}$$

Therefore,

$$f(x) = \alpha(x + x^2), \qquad x \in \mathbb{R},$$

where $\alpha = \frac{f(1)}{2}$.

Finally, for the case $n \leq -2$, take $p = -n \geq 2$ in (2.2) to obtain

$$(x+1)f\left(\frac{1}{x^p}\right) = \left(\frac{1}{x^{2p+1}} + \frac{1}{x^{p+1}}\right) f(x) = \left(\frac{1 + x^p}{x^{2p+1}}\right) f(x), \tag{2.8}$$

for $x \in \mathbb{R}_+$. Substitute x^{-p} for x in (2.8) to gain

$$\left(\frac{1}{x^p}+1\right) f\left(x^{p^2}\right) = \left(x^{2p^2+p} + x^{p^2+p}\right) f\left(\frac{1}{x^p}\right).$$

By (2.8), we obtain

$$\frac{1+x^p}{x^p} f\left(x^{p^2}\right) = \frac{x^{2p^2+p}+x^{p^2+p}}{x+1}\left(\frac{1+x^p}{x^{2p+1}}\right) f(x),$$

or

$$(x+1)f\left(x^{p^2}\right) = \left(x^{2p^2-1}+x^{p^2-1}\right) f(x). \tag{2.9}$$

In (2.9), set $p^2 = k \in \mathbb{N}$, and use a similar proof as in the previous case.

Obviously, the converse holds. □

The additive function $\theta\colon \mathbb{R} \to \mathbb{R}$ is named a derivation if $\theta(xy) = x\theta(y) + y\theta(x)$ for all $x, y \in \mathbb{R}$. Thus, every derivation θ satisfies $\theta(x^2) = 2x\theta(x)$ for all $x \in \mathbb{R}$. Moreover, there exist nontrivial derivations on $\mathbb{R}$ (see [13, Theorem 14.2.2]). Also, $\theta(x^2)$ and $(\theta(x))^2$ are quadratic functions (see [3]).

A functional $\mathcal{H}\colon \mathbb{R}^2 \to \mathbb{R}$ is named a bi-derivation if the mappings

$$s \mapsto \mathcal{H}(s,x) \quad \text{and} \quad s \mapsto \mathcal{H}(x,s), \quad s \in \mathbb{R},$$

are derivations for every $x \in \mathbb{R}$.

The set of derivations of order 2, denoted by $\mathfrak{D}_2(\mathbb{R})$, is the set of the additive functions $\theta\colon \mathbb{R} \to \mathbb{R}$ that can be written as

$$\theta(xy) - x\theta(y) - \theta(x)y = \mathcal{H}(x,y),$$

for some bi-derivation $\mathcal{H}$ on $\mathbb{R}^2$.

In the case $n = 1$, condition (2.1) has the form

$$(x+x^2)f(x) = (x+x^2)f(x),$$

whence f can be discontinuous as well.

Equation (2.1) for pairs of $(x, y) \in \mathbb{R}^2$ that fulfill condition $xy = 1$ is as follows

$$f(x) = x^3 f\left(\frac{1}{x}\right), \quad x \in \mathbb{R}\backslash\{0\}. \tag{2.10}$$

Now, by giving a counterexample, we show that there exists a discontinuous Drygas function that satisfies condition (2.10).

Assume that $\theta\colon \mathbb{R} \to \mathbb{R}$ is a nontrivial derivation. Then

$$\theta\left(\frac{1}{x}\right) = -\frac{1}{x^2}\theta(x), \quad x \in \mathbb{R}\backslash\{0\}.$$

Therefore, $f(x) = -\theta(x) + \frac{1}{2}\theta(x^2)$ is a discontinuous Drygas function that fulfills (2.10) for every $x \in \mathbb{R}$.

Lemma 2.2 ([5]). *Assume that $\delta\colon \mathbb{R} \to \mathbb{R}$ is an additive function. Then $\delta \in \mathfrak{D}_2(\mathbb{R})$ if and only if*

$$\delta\left(x^4\right) = 6x^2\delta(x^2) - 8x^3\delta(x),$$

for every $x \in \mathbb{R}$.

Theorem 2.3. *Drygas functions $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ fulfill the condition* (1.4) *for $(x,y) \in \Delta_1$ if and only if there exists an additive function $\delta\colon \mathbb{R} \to \mathbb{R}$ such that*

$$\begin{aligned}
&\delta\left(x^4\right) = 6x^2\delta(x^2) - 8x^3\delta(x) + 3x^4\delta(1),\\
&f_1(x) = (x+1)\left[2\delta(x) - x\delta(1)\right],\\
&\begin{cases} f_2(x) = \dfrac{1}{4}(x+1)\left[6\delta(x) - \dfrac{1}{x}\delta\left(x^2\right) - x\delta(1)\right], & x \in \mathbb{R}\backslash\{0\}\\ f_2(0) = 0. \end{cases}
\end{aligned}$$

In particular, $f_1(1) = 0$ if and only if $\delta \in \mathfrak{D}_2(\mathbb{R})$.

Proof. Since $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ are Drygas functions, by (1.3), there exist additive functions $A_1, A_2\colon \mathbb{R} \to \mathbb{R}$ and symmetric biadditive functions $B_1, B_2\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that

$$f_1(x) = A_1(x) + B_1(x,x) \quad \text{and} \quad f_2(x) = A_2(x) + B_2(x,x),$$

for all $x \in \mathbb{R}$. Put $y = x^2$ in (1.4), to obtain

$$\left(x^2 + x^4\right) f_1(x) = \left(x + x^2\right) f_2\left(x^2\right), \quad x \in \mathbb{R}.$$

By dividing both sides by $x \neq 0$ (since $f_1(0) = f_2(0) = 0$), we have

$$(x+1)f_2\left(x^2\right) = \left(x^3 + x\right) f_1(x), \quad x \in \mathbb{R}. \tag{2.11}$$

Set $x = -1$ in (2.11), then $f_1(-1) = -A_1(1) + B_1(1,1) = 0$. Thus,

$$A_1(1) = B_1(1,1) = \frac{f_1(1)}{2}.$$

Let $s \in \mathbb{Q}$. Substituting $x+s$ for x in (2.11), we get

$$(x+s+1)f_2\left((x+s)^2\right) = \left((x+s)^3 + x + s\right) f_1(x+s), \quad x \in \mathbb{R}. \tag{2.12}$$

By expanding the powers of sums on both side of this equation and by the $\mathbb{Q}$-homogeneity of A_1, A_2, B_1 and B_2, equation (2.12) becomes

$$\begin{aligned}
&xA_2\left(x^2\right) + sA_2\left(x^2\right) + A_2\left(x^2\right) + 2sxA_2(x) + 2s^2A_2(x) + 2sA_2(x) \\
&\quad + s^2xA_2(1) + s^3A_2(1) + s^2A_2(1) + xB_2\left(x^2,x^2\right) + sB_2\left(x^2,x^2\right) \\
&\quad + B_2\left(x^2,x^2\right) + 4s^2xB_2(x,x) + 4s^3B_2(x,x) + 4s^2B_2(x,x) + s^4xB_2(1,1) \\
&\quad + s^5B_2(1,1) + s^4B_2(1,1) + 4sxB_2\left(x^2,x\right) + 4s^2B_2\left(x^2,x\right) + 4sB_2\left(x^2,x\right) \\
&\quad + 2s^2xB_2\left(x^2,1\right) + 2s^3B_2\left(x^2,1\right) + 2s^2B_2\left(x^2,1\right) + 4s^3xB_2(x,1) \\
&\quad + 4s^4B_2(x,1) + 4s^3B_2(x,1) \\
=&x^3A_1(x) + 3sx^2A_1(x) + 3s^2xA_1(x) + s^3A_1(x) + xA_1(x) + sA_1(x) \\
&\quad + sx^3A_1(1) + 3s^2x^2A_1(1) + 3s^3xA_1(1) + s^4A_1(1) + sxA_1(1) + s^2A_1(1) \\
&\quad + x^3B_1(x,x) + 3sx^2B_1(x,x) + 3s^2xB_1(x,x) + s^3B_1(x,x) + xB_1(x,x) \\
&\quad + sB_1(x,x) + s^2x^3B_1(1,1) + 3s^3x^2B_1(1,1) + 3s^4xB_1(1,1) + s^5B_1(1,1) \\
&\quad + s^2xB_1(1,1) + s^3B_1(1,1) + 2sx^3B_1(x,1) + 6s^2x^2B_1(x,1) + 6s^3xB_1(x,1) \\
&\quad + 2s^4B_1(x,1) + 2sxB_1(x,1) + 2s^2B_1(x,1),
\end{aligned}$$

for all $x \in \mathbb{R}$. Hence,

$$\begin{aligned}
0 =&[B_1(1,1) - B_2(1,1)]s^5 \\
&+ [A_1(1) + 3xB_1(1,1) + 2B_1(x,1) - xB_2(1,1) + -4B_2(x,1) - B_2(1,1)]s^4 \\
&+ \big[f_1(x) + 3xA_1(1) + 6xB_1(x,1) + B_1(1,1) + 3x^2B_1(1,1) \\
&\quad - 4xB_2(x,1) - A_2(1) - 4B_2(x,x) - 2B_2\left(x^2,1\right) - 4B_2(x,1)\big]s^3 \\
&+ \big[3xf_1(x) + 3x^2A_1(1) + A_1(1) + x^3B_1(1,1) + xB_1(1,1) \\
&\quad + 6x^2B_1(x,1) + 2B_1(x,1) - A_2(1) - 2A_2(x) - xA_2(1) \\
&\quad - 4xB_2(x,x) - 4B_2(x,x) - 4B_2\left(x^2,x\right) - 2xB_2\left(x^2,1\right) - 2B_2\left(x^2,1\right)\big]s^2 \\
&+ \big[3x^2f_1(x) + f_1(x) + x^3A_1(1) + xA_1(1) + 2x^3B_1(x,1) + 2xB_1(x,1) \\
&\quad - f_2\left(x^2\right) - 2xA_2(x) - 2A_2(x) - 4xB_2\left(x^2,x\right) - 4B_2\left(x^2,x\right)\big]s \\
&+ \big[x^3f_1(x) + xf_1(x) - xf_2\left(x^2\right) - f_2\left(x^2\right)\big].
\end{aligned}$$

By Lemma 1.1, the coefficients of s^n for $n = 0,1,2,3,4,5$ are equal to zero. The coefficient of s^5 implies $B_1(1,1) = B_2(1,1)$. So, by taking $x = 1$ in (2.11), we obtain

$$A_1(1) = B_1(1,1) = A_2(1) = B_2(1,1) = \frac{f_1(1)}{2}.$$

According to the coefficient of s^4 we see that

$$2B_2(x,1) = xB_1(1,1) + B_1(x,1), \quad x \in \mathbb{R}. \tag{2.13}$$

From the coefficient of s^3 and (2.13), we conclude that

$$f_1(x) = 2B_1(x,1) - xB_1(1,1) - 4xB_1(x,1) + B_1\left(x^2,1\right) + 4B_2(x,x), \tag{2.14}$$

for all $x \in \mathbb{R}$. Hence, by (2.14),

$$A_1(x) = \frac{f_1(x) - f_1(-x)}{2} = 2B_1(x,1) - xB_1(1,1), \tag{2.15}$$

and

$$B_1(x,x) = \frac{f_1(x) + f_1(-x)}{2} = 4B_2(x,x) + B_1\left(x^2,1\right) - 4xB_1(x,1), \tag{2.16}$$

for all $x \in \mathbb{R}$.

Replacing x with $-x$ in (2.11) yields

$$(-x+1)f_2\left(x^2\right) = -\left(x^3 + x\right) f_1(-x), \quad x \in \mathbb{R}. \tag{2.17}$$

Adding both sides of (2.11) and (2.17) gives us

$$f_2\left(x^2\right) = \left(x^3 + x\right) A_1(x), \quad x \in \mathbb{R},$$

and hence,

$$f_1(x) = (x+1)A_1(x) = A_1(x) + B_1(x,x), \quad B_1(x,x) = xA_1(x), \tag{2.18}$$

for all $x \in \mathbb{R}$. Thus,

$$f_2\left(x^2\right) = \left(x^2 + 1\right) B_1(x,x), \quad x \in \mathbb{R}. \tag{2.19}$$

From (2.15) and (2.18), we have

$$B_1(x,x) = 2xB_1(x,1) - x^2 B_1(1,1), \quad x \in \mathbb{R}. \tag{2.20}$$

Combining (2.16) and (2.20) yields

$$B_2(x,x) = \frac{3}{2}xB_1(x,1) - \frac{1}{4}B_1\left(x^2,1\right) - \frac{1}{4}x^2 B_1(1,1), \tag{2.21}$$

for all $x \in \mathbb{R}$. For $x \in \mathbb{R}$ and $s \in \mathbb{Q}$, if we write sx instead of x in equation (2.19), then

$$f_2\left(s^2x^2\right) = \left(s^2x^2 + 1\right) B_1(sx,sx), \quad x \in \mathbb{R}.$$

Thus,

$$\left[A_2\left(x^2\right) - B_1(x,x)\right] s^2 + \left[B_2\left(x^2,x^2\right) - x^2 B_1(x,x)\right] s^4 = 0.$$

From Lemma 1.1 we have

$$A_2\left(x^2\right) = B_1(x,x), \quad B_2\left(x^2,x^2\right) = x^2 B_1(x,x).$$

So, $B_2\left(x^2,x^2\right) = x^2 A_2\left(x^2\right)$ for all $x \in \mathbb{R}$. Setting $x^2 = t$, we have $t > 0$ and $B_2(t,t) = tA_2(t)$. It follows that $B_2(x,x) = xA_2(x)$ for all $x > 0$.

Now, for $x = -t < 0$,

$$B_2(x,x) = B_2(-t,-t) = B_2(t,t) = tA_2(t) = -tA_2(-t) = xA_2(x).$$

Therefore, $A_2\left(x\right) = \frac{1}{x}B_2(x,x)$ for all $x \in \mathbb{R}\backslash\{0\}$.

From the above equality and (2.21), we obtain

$$A_2\left(x\right) = \frac{3}{2}B_1(x,1) - \frac{1}{4x}B_1\left(x^2,1\right) - \frac{1}{4}xB_1(1,1).$$

Define the additive function $\delta\colon \mathbb{R} \to \mathbb{R}$ by

$$\delta(x) = B_1(x,1), \quad x \in \mathbb{R}.$$

Therefore,

$$f_1(x) = (x+1)\left[2\delta(x) - x\delta(1)\right],$$

and

$$f_2(x) = \frac{1}{4}(x+1)\left[6\delta(x) - \frac{1}{x}\delta\left(x^2\right) - x\delta(1)\right],$$

for all $x \in \mathbb{R}\backslash\{0\}$.

Next, $f_1(1) = 0$ if and only if $\delta(1) = 0$, or equivalently, if and only if

$$\delta\left(x^4\right) = 6x^2\delta(x^2) - 8x^3\delta(x),$$

for all $x \in \mathbb{R}$. By Lemma 2.2, this is equivalent to $\delta \in \mathfrak{D}_2(\mathbb{R})$.

The only if part is trivial. □

In Theorem 2.3, if we suppose that δ is a derivation, then $f_1(x) = 2f_2(x)$ for all $x \in \mathbb{R}$.

Example 2.4. Let $0 \neq a \in \mathbb{R}$. Define $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ by

$$f_1(x) = 2a(x+1)\theta(x), \quad f_2(x) = a(x+1)\theta(x),$$

for all $x \in \mathbb{R}$, where $\theta\colon \mathbb{R} \to \mathbb{R}$ is a nontrivial derivation. Then f_1, f_2 are discontinuous Drygas functions and satisfy the conditions of Theorem 2.3 with $\delta(x) = a\theta(x)$ for all $x \in \mathbb{R}$.

THEOREM 2.5. *Drygas functions $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ satisfy the conditional equation* (1.4) *on $\mathbb{R}_+$ for $(x, y) \in \Delta_2$ and $f_1(x) = x^3 f_1\left(\frac{1}{x}\right)$ for all $x \in \mathbb{R}_+$ if and only if*

$$f_1(x) = f_2(x) = \alpha\left(x + x^2\right), \quad x \in \mathbb{R}, \tag{2.22}$$

where α is a real constant.

PROOF. The conditional equation (1.4) for $y = \log(x)$ is

$$\left[(\log(x))^2 + \log(x)\right] f_1(x) = (x^2 + x) f_2(\log(x)), \quad x \in \mathbb{R}_+. \tag{2.23}$$

Replacing x with $\frac{1}{x}$ in (2.23), we arrive, by using the fact that $f_1(x) = x^3 f_1\left(\frac{1}{x}\right)$, at

$$\left[(\log(x))^2 - \log(x)\right] f_1(x) = (x^2 + x) f_2(-\log(x)), \quad x \in \mathbb{R}_+. \tag{2.24}$$

Substituting x^2 for x in (2.23) and applying properties of logarithmic and Drygas functions, we see that

$$\begin{aligned}\left[4(\log(x))^2 + 2\log(x)\right] f_1(x^2) &= \left(x^4 + x^2\right) f_2(2\log(x)) \\ &= \left(x^4 + x^2\right)\left[3 f_2(\log(x)) + f_2(-\log(x))\right],\end{aligned} \tag{2.25}$$

for all $x \in \mathbb{R}_+$.

From (2.23), (2.24) and (2.25) we deduce that

$$\left[4(\log(x))^2 + 2\log(x)\right] f_1(x^2) = \frac{x^4 + x^2}{x^2 + x}\left[4(\log(x))^2 + 2\log(x)\right] f_1(x),$$

which implies

$$(x + 1) f_1(x^2) = \left(x^3 + x\right) f_1(x), \tag{2.26}$$

for all $x \in \mathbb{R}_+ \setminus \{1, \exp\left(-\frac{1}{2}\right)\}$.

Obviously, (2.26) holds for $x = 1$.

Putting $x = \exp(1)$ in (2.24), we have $f_2(-1) = 0$. So, $A_2(1) = B_2(1, 1)$, where $A_2 \colon \mathbb{R} \to \mathbb{R}$ is an additive function and $B_2 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a symmetric biadditive function and $f_2(x) = A_2(x) + B_2(x, x)$.

Taking $x = \exp\left(-\frac{1}{2}\right)$ in (2.24), we get

$$\begin{aligned}\frac{3}{4} f_1\left(\exp\left(-\frac{1}{2}\right)\right) &= \left(\exp(-1) + \exp\left(-\frac{1}{2}\right)\right) f_2\left(\frac{1}{2}\right) \\ &= \left(\exp(-1) + \exp\left(-\frac{1}{2}\right)\right)\left[A_2\left(\frac{1}{2}\right) + B_2\left(\frac{1}{2}, \frac{1}{2}\right)\right]\end{aligned}$$

$$= \frac{3}{4}\left(\exp(-1) + \exp\left(-\frac{1}{2}\right)\right) A_2(1)$$
$$= \frac{3}{4}\left(\exp(-1) + \exp\left(-\frac{1}{2}\right)\right) \frac{f_2(1)}{2}.$$

Hence,

$$f_2(1) = \frac{2}{\left(\exp(-1) + \exp\left(-\frac{1}{2}\right)\right)} f_1\left(\exp\left(-\frac{1}{2}\right)\right). \tag{2.27}$$

Setting $x = \exp(-1)$ in (2.24), we obtain

$$2f_1(\exp(-1)) = (\exp(-2) + \exp(-1))f_2(1). \tag{2.28}$$

It follows from (2.27) and (2.28) that

$$\left(\exp\left(-\frac{1}{2}\right) + 1\right) f_1(\exp(-1))$$
$$= \left(\exp\left(-\frac{3}{2}\right) + \exp\left(-\frac{1}{2}\right)\right) f_1\left(\exp\left(-\frac{1}{2}\right)\right).$$

Therefore,

$$(x+1)f_1(x^2) = \left(x^3 + x\right) f_1(x),$$

for all $x \in \mathbb{R}_+$. By Theorem 2.1,

$$f_1(x) = \alpha\left(x + x^2\right), \quad x \in \mathbb{R},$$

where $\alpha = \frac{f_1(1)}{2}$. By replacing $f_1(x)$ in (2.23), we have

$$f_2(\log(x)) = \alpha\left[(\log(x))^2 + \log(x)\right], \qquad x \in \mathbb{R}_+,$$

where $\alpha = \frac{f_1(1)}{2}$. Consequently

$$f_2(x) = \alpha\left(x + x^2\right) = f_1(x), \qquad x \in \mathbb{R},$$

where $\alpha = \frac{f_1(1)}{2}$.

One can easily verify the sufficiency of (2.22). □

As a consequence, Theorem 2.5 can be generalized to the case of exponential functions, that is $(x, y) \in \Delta_3$, because the logarithmic and exponential functions of the same basis are inverses of each other.

Corollary 2.6. Drygas functions $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ satisfy the conditional equation (1.4) for $(x,y) \in \Delta_3$ and $f_2(x) = x^3 f_2\left(\frac{1}{x}\right)$ for all $x \in \mathbb{R}_+$ if and only if

$$f_1(x) = f_2(x) = \alpha\left(x + x^2\right), \quad x \in \mathbb{R},$$

where α is a real constant.

Remark 1. Theorem 2.5 and Corollary 2.6 also hold if $y = \log_a(x)$ or $y = a^x$ for $a \in \mathbb{R}_+ \setminus \{1\}$.

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