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BOUNDEDNESS OF LASOTA–MYJAK ATTRACTORS FOR ITERATED FUNCTION SYSTEMS

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Abstract. We consider topological attractors of (possibly infinite) iterated function systems. It is shown that for some class of non-expansive mappings the boundedness of the set of all globally attractive fixed points is equivalent to the boundedness of the attractor. Moreover, such a system fullfils the so-called address theorem.

1. Introduction

A classical result by J. E. Hutchinson in [7] says, among others, what follows: Consider a finite family $S_1, \dots, S_N \colon X \to X$ of contractions of a complete metric space X and an associated operator F of the form $F(K) = S_1(K) \cup \dots \cup S_N(K)$ acting on the hyperspace $\mathcal{H}(X)$ of all nonempty compact subsets of X endowed with the Hausdorff–Pompeiu metric. Then F is contractive and has the unique fixed point $A_* \in \mathcal{H}(X)$. Moreover, for every $K \in \mathcal{H}(X)$ we have $F^n(K) \to A_*$ if $n \to \infty$ (here and in what follows the symbol F^n stands for the composition of n copies of the mapping F). The set A_* is called the *attractor* of an *iterated function system* $\{S_1, \dots, S_N\}$.

During the last four decades the theory of compact attractors of iterated function systems was deeply studied (see, for example, [1] and [11] for a survey and references to the literature of the subject). It has also evolved in different directions. One of them comes from the problem of what may happen when one considers infinite familes of mappings and spaces where compact sets are

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too 'thin'. The most beautiful and complete theory was presented in [10] by A. Lasota and J. Myjak. They introduced the notion of an attractor which is not necessarily compact and it attracts bounded sets (instead of compacta) with respect to the topological convergence (instead of the Hausdorff–Pompeiu metric). Notice that such an attractor, if it exists, is the smallest invariant closed set with respect to the Barnsley–Hutchinson multifunction, which is a lower semicontinuous multifunction associated to the system (for deatils see Section 3 below).

It is remarkable that one can easily construct systems having such attractor which is unbounded, so non-compact. Indeed, consider the following example: Fix a number $a \in (0,1)$, a nonempty set $\Sigma \subset \mathbb{R}$ and a family of contractive affine transformations on $\mathbb{R}$ of the form $S_\sigma(x) = ax + \sigma,\ \sigma \in \Sigma$. It is shown below (see Theorem 4.1) that the set of attractive fixed points of members of the system is always contained in its semiattractor, so, in this particular case, in the attractor. Therefore it is easy to see that if Σ is unbounded, so is the set of fixed points of mappings S_σ and, consequently, the attractor of the considered system is also unbounded.

A natural question arrises as to whether it is possible to obtain unbounded attractor for a family of mappings having bounded set of fixed points. In this paper we give the negative answer for a vast class of iterated function systems consisting of so-called ϕ-contractions. More precisely, the boundedness of the set of fixed points of all members of the system is equivalent to the boundedness of its attractor (see Theorem 4.8). It is also proved that, under considered assumptions, so-called address property holds. This means that each trajectory of the system is convergent to the unique point in the attractor and the limit does not depend on a starting point (or, equivalently, a bounded set). Such a system (or its attractor) is called *point-fibred* (see [8], and also [4]).

To prove the main result we adopt some ideas from [6] as well as some facts proved by the present author in [3].

2. Preliminaries

First we introduce a general definition of a multifunction. Let $\mathcal{X}, \mathcal{Y}$ be nonempty sets. By a *multifunction* $F\colon \mathcal{X} \rightsquigarrow \mathcal{Y}$ we mean a subset of the product $\mathcal{X} \times \mathcal{Y}$ (a relation) such that for every $x \in \mathcal{X}$ the set $F(x) = \{y \in \mathcal{X} : (x,y) \in F\}$ is nonempty.

Given multifunction $F\colon \mathcal{X} \rightsquigarrow \mathcal{Y}$ and subset $A \subset \mathcal{X}$ we define the set

$$F(A) := \bigcup_{x \in A} F(x).$$

If in addition $\mathcal{Z}$ is a nonempty set and $F\colon \mathcal{X} \rightsquigarrow \mathcal{Y}$, $G\colon \mathcal{Y} \rightsquigarrow \mathcal{Z}$ are multifunctions, we define the *composition* $G \circ F$ of F and G as a multifunction $G \circ F\colon \mathcal{X} \rightsquigarrow \mathcal{Z}$ given by $G \circ F(x) = G(F(x))$.

In the present paper we deal with lower semicontinuous multifunctions. Assume that $\mathcal{X}, \mathcal{Y}$ are topological spaces. A multifunction $F\colon \mathcal{X} \rightsquigarrow \mathcal{Y}$ is said to be *lower semicontinuous* (we will write *l.s.c.* for short) if $F(\mathrm{cl}B) \subset \mathrm{cl}F(B)$ for every $B \subset \mathcal{X}$, where $\mathrm{cl}A$ stands for a closure of a set A in the proper topological space. Notice that there are many equivalent definitions of the lower semicontinuity (see, for example, [10, Proposition 2.1]).

Now and in what follows let (X, ϱ) be a metric space. By $B^o(x, \varepsilon)$ (resp. $B(x, \varepsilon)$) we denote the open (resp. closed) ball with center x and radius ε. If $A \subset X$ is a nonempty set, then we put

$$B^o(A, \varepsilon) := \{y \in X : \varrho(y, A) < \varepsilon\} = \bigcup_{x \in A} B^o(x, \varepsilon).$$

Similarly we denote

$$B(A, \varepsilon) := \{y \in X : \varrho(y, A) \leq \varepsilon\} = \bigcup_{x \in A} B(x, \varepsilon).$$

In what follows, all sequences are indexed by elements of the set $\mathbb{N}$ of all positive integers. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X. We define the *lower limit* $\mathrm{Li}\ A_n$ and the *upper limit* $\mathrm{Ls}\ A_n$ as follows: $x \in \mathrm{Li}\ A_n$ if for every $\varepsilon > 0$ there is a positive integer n_0 such that for every $n \geq n_0$

$$A_n \cap B^o(x, \varepsilon) \neq \emptyset, \tag{2.1}$$

and $x \in \mathrm{Ls}\ A_n$ if for every $\varepsilon > 0$ condition (2.1) is satisfied for infinitely many $n \in \mathbb{N}$.

The following characterization of lower and upper topological limits is valid: $x \in \mathrm{Li}\ A_n$ if and only if x is the limit of some sequence (x_n) of points $x_n \in A_n, n \in \mathbb{N}$, and $x \in \mathrm{Ls}\ A_n$ if and only if x is a cluster point of some sequence (x_n) of points $x_n \in A_n, n \in \mathbb{N}$.

Obviously, lower and upper topological limits are closed sets. Moreover, $\mathrm{Li}\ A_n \subset \mathrm{Ls}\ A_n$. Moreover, if a sequence $(A_n)_{n\in\mathbb{N}}$ contains the empty set, then $\mathrm{Li}\ A_n = \mathrm{Ls}\ A_n = \emptyset$.

If $\mathrm{Li}\ A_n = \mathrm{Ls}\ A_n$ we say that the sequence $(A_n)_{n\in\mathbb{N}}$ is *topologically convergent* and we denote this common limit as $\mathrm{Lt}\ A_n$. It is called the *topological limit* of the sequence $(A_n)_{n\in\mathbb{N}}$. Observe also that if $A_n = A$ for every $n \in \mathbb{N}$, then $\mathrm{Li}\ A_n = \mathrm{Ls}\ A_n = \mathrm{cl}\ A$, moreover $\mathrm{Li}\ A_n = \mathrm{Li}\ \mathrm{cl}A_n$ (the same is true for the upper limit).

If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of sets, i.e. $A_{n+1} \subset A_n$ for every $n \in \mathbb{N}$, then it is topologically convergent and

$$\mathrm{Lt}\ A_n = \bigcap_{n\in\mathbb{N}} \mathrm{cl} A_n.$$

On the other hand, if $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of sets, i.e. $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$, then it is topologically convergent and

$$\mathrm{Lt}\ A_n = \mathrm{cl} \bigcup_{n\in\mathbb{N}} A_n.$$

Other properties of topological limits can be found in [9].

3. Lasota–Myjak attractors and semiattractors of IFSs

Let Σ be a nonempty set (of indexes). Consider a family $\mathcal{S} = \{S_\sigma \colon X \to X : \sigma \in \Sigma\}$ of continuous selfmappings of X. Such a family is called an *iterated function system* (IFS for short). With a given IFS $\mathcal{S}$ we associate its *Barnsley–Hutchinson multifunction* $F \colon X \rightsquigarrow X$ given by

$$F(x) := \{S_\sigma(x) : \sigma \in \Sigma\} \quad \text{for} \ \ x \in X.$$

One can prove that since transformations S_σ are continuous, the Barnsley–Hutchinson multifunction F associated with $\mathcal{S}$ is l.s.c..

In particular, if Σ is finite, that is for some $N \in \mathbb{N}$ we have $\Sigma = \{1, \ldots, N\}$, an IFS $\mathcal{S}$ is called *classical*. In this case the Barnsley–Hutchinson multifunction F associated with $\mathcal{S}$ has closed values

$$F(x) = \{S_1(x), \ldots, S_N(x)\} \quad \text{for} \ \ x \in X.$$

Notice that for every $n \in \mathbb{N}$ and $x \in X$ we have

$$F^n(x) := \{S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(x) : \sigma_1, \ldots, \sigma_n \in \Sigma\}.$$

In [10] A. Lasota and J. Myjak proposed the following generalization of classical definition of attractors for IFSs. The generalization goes in two ways. Let $\mathcal{S} = \{S_\sigma \colon X \to X : \sigma \in \Sigma\}$ be an IFS with its Barnsley–Hutchinson multifunction $F \colon X \rightsquigarrow X$. If the following set

$$C = \bigcap_{x\in X} \mathrm{Li}\ F^n(x)$$

is nonempty, then it is called the *semiattractor* of $\mathcal{S}$.

The semiattractor is unique whenever it exists and it is a closed set. The following properties of semiattractors were shown in [10, Proposition 3.1, Theorem 3.2] (see also [2, Proposition 5.6, Theorem 5.7], [5, Theorem 3.4, Theorem 3.5]).

Proposition 3.1. *If $\mathcal{S}$ admits the semiattractor C then the following conditions hold:*

(i) *if a nonempty closed set A is such that $F(A) \subset A$, then $C \subset A$;*
(ii) $\mathrm{cl} F(C) = C$*;*
(iii) $\mathrm{Lt}\ F^n(A) = C$ *for every non-empty $A \subset C$; in particular,* $\mathrm{Lt}\ F^n(x) = C$ *for every $x \in C$.*

According to [10] a nonempty set $A_* \subset X$ is called an *attractor* of $\mathcal{S}$ if for every nonempty bounded subset D of X we have

$$A_* = \mathrm{Lt}\ F^n(D)$$

independent of the choice of D. To contrast this notion to classical compact attractors we will use the name *Lasota–Myjak attractor* or simply *L-M attractor*. The L-M attractor is obviously the semiattractor of the system.

4. Bounded L-M attractors

A fixed point $x_* \in X$ of the mapping $S\colon X \to X$ is called *globally attractive* if $\lim_{n\to\infty} S^n(x) = x_*$ for every $x \in X$.

For a given IFS $\mathcal{S}$ we denote the set of all globally attractive fixed points of transformations $S_\sigma, \sigma \in \Sigma$, by $\mathrm{Attr}\,\mathcal{S}$. Obviously $\mathrm{Attr}\,\mathcal{S} \subset \mathrm{Fix}\,\mathcal{S} = \{x \in X : x = S_\sigma(x)$ for some $\sigma \in \Sigma\}$.

Let us recall the following result (see [2, Corollary 5.11]).

Proposition 4.1. *Let $\mathcal{S} = \{S_\sigma\colon X \to X : \sigma \in \Sigma\}$ be an IFS. If* $\mathrm{Attr}\,\mathcal{S} \neq \emptyset$*, then $\mathcal{S}$ has the semiattractor C. Moreover*

$$C = \mathrm{Lt}\ F^n(x_*) = \mathrm{cl} \bigcup_{n=1}^{\infty} F^n(x_*)$$

for every $x_* \in \mathrm{Attr}\,\mathcal{S}$. In particular,

$$\mathrm{Attr}\,\mathcal{S} \subset C.$$

In what follows let $\mathcal{S} = \{S_\sigma\colon X \to X : \sigma \in \Sigma\}$ be an IFS consisting of Lipschitz mappings on a complete metric space (X, ϱ). In particular, for every $\sigma \in \Sigma$ there exists a number $L_\sigma \geq 0$ (called a *Lipschitz constant* of S_σ) such that

$$\varrho(S_\sigma(x), S_\sigma(y)) \leq L_\sigma \varrho(x, y) \quad \text{for } x, y \in X.$$

Let us denote

$$L := \sup \{L_\sigma : \sigma \in \Sigma\}.$$

The following result was proved in [10] (see Corollary 4.1 therein).

Proposition 4.2. *If (X, ϱ) is complete and $L < 1$, then $\mathcal{S}$ admits the L-M attractor.*

In [10, Remark 4.2] the example was presented that the condition $L < 1$ cannot be replaced by a weaker $L_\sigma < 1$ for every $\sigma \in \Sigma$. On the other hand, there is a vast class of IFSs consisting of non-expansive mappings (i.e. mappings with Lipschitz constants not greater than 1) and admitting the L-M attractor. In particular, such a property is entitled to IFSs consisting of so-called ϕ-contractions.

Namely, let $\phi\colon [0,\infty) \to [0,\infty)$ be an upper semicontinuous and non-decreasing function satisfying $\phi(t) < t$ for $t > 0$. Observe that $\phi^n(x) \to 0$ as $n \to \infty$ for every $x \in X$.

We say that a transformation $S\colon X \to X$ is *ϕ-contraction* if

$$\varrho(S(x), S(y)) \le \phi(\varrho(x, y)) \quad \text{for } x, y \in X.$$

It is known that each ϕ-contraction of a complete metric space into itself has the unique fixed point and it is globally attractive.

In [3, Corollary 6.4] it is proved that.

Proposition 4.3. *An IFS $\mathcal{S} = \{S_\sigma\colon X \to X : \sigma \in \Sigma\}$ consisting of ϕ-contractions of a complete metric space X with ϕ independent of $\sigma \in \Sigma$ admits the L-M attractor.*

Moreover, from this and [3, Theorem 6.3] the following result follows:

Proposition 4.4. *Let $\mathcal{S} = \{S_\sigma\colon X \to X : \sigma \in \Sigma\}$ be an IFS consisting of ϕ-contractions of complete metric space X with ϕ independent of $\sigma \in \Sigma$. Let A_* be its L-M attractor. Assume that $B \subset X$ is a nonempty bounded set such that*

$$S_\sigma(B) \subset B \quad \text{for } \sigma \in \Sigma.$$

Then:

(i) *for every sequence $\omega = (\sigma_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ there exists the unique point $x_\omega \in X$ such that*

$$\bigcap_{n\in\mathbb{N}} \operatorname{cl}\big(S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(B)\big) = \{x_\omega\};$$

(ii) *for every sequence* $\omega = (\sigma_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ *the limit*

$$\lim_{n\to\infty} S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(x)$$

exists and does not depend on $x \in X$ *and it is equal to* x_ω*;*

(iii) *the L-M attractor* A_* *is bounded.*

COROLLARY 4.5. *Assume that* $\mathcal{S} = \{S_\sigma \colon X \to X : \sigma \in \Sigma\}$ *is an IFS consisting of* ϕ*-contractions of complete metric space* X *with* ϕ *independent of* $\sigma \in \Sigma$*. If its L-M attractor* A_* *is bounded, then conditions* (i) *and* (ii) *of Proposition* 4.4 *hold.*

PROOF. It is enough to put A_* instead of B in Proposition 4.4. □

PROPOSITION 4.6. *Let* $\mathcal{S} = \{S_\sigma \colon X \to X : \sigma \in \Sigma\}$ *be an IFS with the L-M attracor* A_**. If for every sequence* $\omega = (\sigma_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ *the limit*

$$x_\omega = \lim_{n\to\infty} S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(x)$$

exists and does not depend on $x \in X$*, then*

$$A_* = \mathrm{cl}\{x_\omega : \omega \in \Sigma^{\mathbb{N}}\}. \tag{4.1}$$

PROOF. Denote $\mathcal{L} := \{x_\omega : \omega \in \Sigma^{\mathbb{N}}\}$. Evidently $\mathcal{L} \subset A_*$, and since L-M attractor is a closed set, so also $\mathrm{cl}\mathcal{L} \subset A_*$.

To prove the opposite inclusion we show first that $\mathcal{L}$ is positively invariant with respect to the Barnsley-Hutchinson multifunction F associated with an IFS $\mathcal{S}$, i.e.

$$F(\mathcal{L}) \subset \mathcal{L}. \tag{4.2}$$

To this aim fix $x \in F(\mathcal{L})$. Therefore there exists $\omega = (\sigma_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ such that $x = S_\sigma(x_\omega)$ for some $\sigma \in \Sigma$, where $x_\omega = \lim_{n\to\infty} S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(y)$ is independent of $y \in X$. According to the continuity of a mapping $S_\sigma \colon X \to X$ we get

$$x = S_\sigma\big(\lim_{n\to\infty} S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(y)\big) = \lim_{n\to\infty} S_\sigma \circ S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(y).$$

The last limit does not depend on y, hence $x = x_{\omega'}$ where $\omega' = (\sigma, \sigma_1, \sigma_2, \dots)$. This means that $x \in \mathcal{L}$. But $x \in F(\mathcal{L})$ was arbitrary, so we infer that the desired inclusion (4.2) holds.

Since F is a l.s.c. multifunction, from the inclusion (4.2) we get that

$$F(\mathrm{cl}\mathcal{L}) \subset \mathrm{cl}F(\mathcal{L}) \subset \mathrm{cl}\mathcal{L}.$$

From this and (i) in Proposition 3.1 we obtain that $A_* \subset \mathrm{cl}\mathcal{L}$. This completes the proof of the equality (4.1). □

Remark 4.7. The proposition above improves the last assertion in [3, Theorem 6.3]. Notice that the argument used in the proof therein was false.

Now we are in a position to formulate the main result of the paper.

Theorem 4.8. *Let* (X,ϱ) *be a complete metric space and* $\mathcal{S}=\{S_\sigma\colon X\to X : \sigma\in\Sigma\}$ *be an IFS consisting of* ϕ*-contractions of* X *with* $\phi\colon[0,\infty)\to[0,\infty)$ *independent from* $\sigma\in\Sigma$*, satisfying additionally*

$$\lim_{t\to\infty}(t-\phi(t))=\infty. \tag{4.3}$$

Then the following statements are equivalent:
- (i) *there exists* $x_0\in X$ *such that the set* $\{S_\sigma(x_0):\sigma\in\Sigma\}$ *is bounded;*
- (ii) *the set* $\{S_\sigma(x):\sigma\in\Sigma\}$ *is bounded for every* $x\in X$*;*
- (iii) *the set* $\operatorname{Fix}\mathcal{S}$ *is bounded;*
- (iv) *the L-M attractor of* $\mathcal{S}$ *is bounded.*

Before we will prove the theorem let us show some auxiliary results.

Lemma 4.9. *Assume that* (X,ϱ) *is a metric space and an IFS* $\mathcal{S}$ *consists of Lipschitz mappings* $S_\sigma\colon X\to X$, $\sigma\in\Sigma$, *with* $L=\sup\{L_\sigma:\sigma\in\Sigma\}<\infty$. *Consider* (i), (ii) *and* (iii) *of Theorem* 4.8. *Then* (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

Proof. Assume that (i) holds for some $x_0\in X$ and let $x\in X$. Then

$$\varrho(S_\sigma(x),S_\sigma(x_0))\le L\varrho(x,x_0)$$

for all $\sigma\in\Sigma$. Put $A:=\{S_\sigma(x_0):\sigma\in\Sigma\}$ and $\varepsilon:=L\varrho(x,x_0)$. Hence, since A is bounded, so is $B^o(A,\varepsilon)$. But $\{S_\sigma(x):\sigma\in\Sigma\}\subset B(A,\varepsilon)$, and this means that (ii) holds.

The implication (ii) $\Rightarrow$ (i) is obvious.

Assume now that (iii) is satisfied. If $x\in X$, $\sigma\in\Sigma$ and x_σ is a fixed point of $S_\sigma\colon X\to X$, then

$$\varrho(S_\sigma(x),x_\sigma)=\varrho(S_\sigma(x),S_\sigma(x_\sigma))\le L\varrho(x,x_\sigma).$$

By (iii), we have that $\varepsilon:=L\sup\{\varrho(x,x_\sigma):\sigma\in\Sigma\}$ is finite, so $\{S_\sigma(x):\sigma\in\Sigma\}\subset B(\operatorname{Fix}\mathcal{S},\varepsilon)$. This means that (ii) holds. □

Lemma 4.10. *Under assumptions of Theorem* 4.8, *if* (i) *holds, then*

$$\sup\{\varrho(x,S_{\sigma_1}\circ\cdots\circ S_{\sigma_n}(x):n\in\mathbb{N},\ \sigma_1,\dots,\sigma_n\in\Sigma\}<\infty$$

for every $x\in X$.

Proof. Fix $x\in X$. Given $n\in\mathbb{N}$ define

$$a_n:=\sup\{\varrho(x,S_{\sigma_1}\circ\cdots\circ S_{\sigma_n}(x)):\sigma_1,\dots,\sigma_n\in\Sigma\}.$$

In particular, $a_1 = \sup\{\varrho(x, S_\sigma(x)) : \sigma \in \Sigma\}$. Since S_σ, $\sigma \in \Sigma$, are non-expansive mappings, Lemma 4.9 implies that a_1 is finite.

By property (4.3), there is $M > 0$ such that $M - \phi(M) \geq a_1$. Using induction we show that

$$a_n \leq M \quad \text{for } n \in \mathbb{N}.$$

Clearly $a_1 \leq M$. Let $k \in \mathbb{N}$ be such that $a_k \leq M$. Then for any $\sigma_1, \dots, \sigma_{k+1} \in \Sigma$ we have

$$\begin{aligned}\varrho(x, S_{\sigma_1} \circ \cdots \circ S_{\sigma_{k+1}}(x)) &\leq \varrho(x, S_{\sigma_1}(x)) + \phi\big(\varrho(x, S_{\sigma_2} \circ \cdots \circ S_{\sigma_{k+1}}(x))\big) \\ &\leq a_1 + \phi(a_k) \leq a_1 + \phi(M) \leq M.\end{aligned}$$

Since $\sigma_1, \dots, \sigma_{k+1} \in \Sigma$ were arbitrary, we infer that $a_{k+1} \leq M$. This completes the inductive proof that $a_n \leq M$ for every $n \in \mathbb{N}$. □

Lemma 4.11. *Under assumptions of Theorem 4.8, if* (i) *holds, then for every $x \in X$ and $\omega = (\sigma_n)_{n\in\mathbb{N}}$ there is a unique point $x_\omega \in X$ such that*

$$x_\omega = \lim_{n\to\infty} S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(x)$$

and this limit does not depend on x.

Proof. Fix $x \in X$ and $\omega = (\sigma_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$. Define the constant

$$M := \sup\{\varrho(x, S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(x)) : \sigma_1, \dots, \sigma_n \in \Sigma,\ n \in \mathbb{N}\}.$$

By Lemma 4.10, it is finite.

Hence, if $j, k \in \mathbb{N}$, we have

$$\varrho\big(S_{\sigma_1}\circ\cdots\circ S_{\sigma_j}(x), S_{\sigma_1}\circ\cdots\circ S_{\sigma_{j+k}}(x)\big) \leq \phi^j\left(\varrho(x, S_{\sigma_{j+1}}\circ\cdots\circ S_{\sigma_{j+k}}(x)\right) \leq \phi^j(M),$$

since $S_{\sigma_1}, \dots, S_{\sigma_j}$ are ϕ-contractions and ϕ is non-decreasing. Since $\phi^j(M) \to 0$ as $j \to \infty$, given $\varepsilon > 0$, there is $l \in \mathbb{N}$ such that $\phi^j(M) < \varepsilon$ for every $j \geq l$. Consequently

$$\varrho\big(S_{\sigma_1} \circ \cdots \circ S_{\sigma_j}(x), S_{\sigma_1} \circ \cdots \circ S_{\sigma_{j+k}}(x)\big) < \varepsilon \quad \text{for } j \geq l,\ k \in \mathbb{N}.$$

This means that $(S_{\sigma_1}\circ\cdots\circ S_{\sigma_n}(x))_{n\in\mathbb{N}}$ is a Cauchy sequenc, so it is convergent to some element in X. Moreover, if $x, x' \in X$, then

$$\varrho\big(S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(x), S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(x')\big) \leq \phi^n(\varrho(x, x')) \to 0 \quad \text{as } n \to \infty.$$

But this means that the limit of the considered sequence does not depend on x and this completes the proof. □

Proof of Theorem 4.8. We get (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii) by Lemma 4.9.

(iv) $\Rightarrow$ (iii) In the considered case we have that $\operatorname{Attr}\mathcal{S} = \operatorname{Fix}\mathcal{S} \neq \emptyset$. The L-M attractor is the semiattractor of the system, hence, by Proposition 4.1, we obtain that $\operatorname{Fix}\mathcal{S} \subset A_*$. Finally, since A_* is bounded, we infer that $\operatorname{Fix}\mathcal{S}$ is so.

(i) $\Rightarrow$ (iv) Assume (i). By Lemma 4.11 and Proposition 4.6 it is enough to show that the set $\mathcal{L} = \{x_\omega : \omega \in \Sigma^{\mathbb{N}}\}$ is bounded. Fix $x \in X$. Since (i) $\Leftrightarrow$ (ii), there exists $M > 0$ such that for any $\omega = (\sigma_n)_{n\in\mathbb{N}}$ we have

$$\varrho\big(x, S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(x)\big) \leq M \quad \text{for} \quad n \in \mathbb{N}.$$

Now, if $n \to \infty$, we also get $\varrho(x, x_\omega) \leq M$, where $x_\omega = \lim_{n\to\infty} S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(x)$ does not depend on x. But, since M is independent of $\omega \in \Sigma^{\mathbb{N}}$, this implies that $\mathcal{L} \subset B(x, M)$.

Finally all conditions (i), (ii), (iii) and (iv) are equivalent. □

At the very end we prove the following corollary (cf. [6, Corollary 1]).

Corollary 4.12. *Under the assumptions of Theorem 4.8 on IFS $\mathcal{S}$ the following statements are equivalent:*

(i) *L-M attractor of $\mathcal{S}$ is bounded;*

(ii) *for every nonempty and bounded set $D \subset X$ its image $F(D)$ under the Barnsley–Hutchinson multifuncion F is also bounded.*

Proof. (i) $\Rightarrow$ (ii) Implication (iv) $\Rightarrow$ (iii) of Theorem 4.8 says that if L-M attractor of $\mathcal{S}$ is bounded, then the set $\operatorname{Fix}\mathcal{S}$ is so. Let $D \subset X$ be nonempty and bounded and $x \in D$. Therefore $\operatorname{Fix}\mathcal{S} \cup D$ is bounded and $r := \operatorname{diam}(\operatorname{Fix}\mathcal{S} \cup D) < \infty$. If $\sigma \in \Sigma$ and x_σ is the fixed point of S_σ, then

$$\varrho\big(S_\sigma(x), x_\sigma\big) = \varrho\big(S_\sigma(x), S_\sigma(x_\sigma)\big) \leq \phi\big(\varrho(x, x_\sigma)\big) \leq \phi(r).$$

This means that $S_\sigma(x) \in B(\operatorname{Fix}\mathcal{S}, \phi(r))$. But $\sigma \in \Sigma$ was arbitrary, hence $F(x) = \{S_\sigma(x) : \sigma \in \Sigma\} \subset B(\operatorname{Fix}\mathcal{S}, \phi(r))$. And since $x \in D$ was arbitrary, we obtain that $F(D) = \{S_\sigma(x) : \sigma \in \Sigma\} \subset B(\operatorname{Fix}\mathcal{S}, \phi(r))$. Consequently, $F(D)$ is bounded.

(ii) $\Rightarrow$ (i) It is clear. Indeed, if $x \in X$ is arbitrary, then putting $D = \{x\}$ we have that the set $F(D) = F(x) = \{S_\sigma(x) : \sigma \in \Sigma\}$ is bounded. Now it is enough to apply the implication (ii) $\Rightarrow$ (iv) of Theorem 4.8. □

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