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GENERALIZED COMMUTATIVE MERSENNE AND MERSENNE–LUCAS QUATERNION POLYNOMIALS

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Abstract. Generalized commutative quaternions generalize elliptic, parabolic and hyperbolic quaternions, bicomplex numbers, complex hyperbolic numbers and hyperbolic complex numbers. In this paper, we use the Mersenne numbers and polynomials in the theory of these quaternions. We introduce and study generalized commutative Mersenne quaternion polynomials and generalized commutative Mersenne–Lucas quaternion polynomials.

1. Introduction

Let p, q, n be integers. In [6], Horadam introduced a sequence $\{W_n(W_0, W_1; p, q)\}$ defined by the second-order linear recurrence relation

$$W_n = pW_{n-1} - qW_{n-2} \quad \text{for } n \geq 2 \tag{1}$$

with fixed real numbers W_0, W_1. For special values of W_0, W_1, p, q the equation (1) defines the well-known sequences of numbers, for example, the Fibonacci sequence $F_n = W_n(0,1;1,-1)$, the Pell sequence $P_n = W_n(0,1;2,-1)$ or the Jacobsthal sequence $J_n = W_n(0,1;1,-2)$. Other examples of the Horadam sequence are the sequence of Mersenne numbers and the sequence of Lucas-Mersenne numbers.

The Mersenne numbers M_n are defined by the recurrence

$$M_n = 3M_{n-1} - 2M_{n-2} \quad \text{for } n \geq 2$$

with $M_0 = 0,\ M_1 = 1$ or

$$M_n = 2M_{n-1} + 1 \quad \text{for } n \geq 1$$

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with initial condition $M_0 = 0$. The sequence of Mersenne–Lucas numbers $\{m_n\}$ is defined by the same recurrence

$$m_n = 3m_{n-1} - 2m_{n-2} \quad \text{for } n \geq 2$$

with $m_0 = 2,\ m_1 = 3$.

The Binet formula of the Mersenne numbers and Mersenne–Lucas numbers has the form

$$M_n = 2^n - 1,$$
$$m_n = 2^n + 1,$$

respectively. Some interesting properties of the Mersenne numbers can be found in [3, 15].

Sequences defined by the second-order linear homogeneous recurrence equation of the form

$$h_n(x) = p(x)h_{n-1}(x) + q(x)h_{n-2}(x),$$

for $n \geq 3$ with $h_1(x) = a$ and $h_2(x) = bx$ are named as Horadam polynomials, see [7, 8]. One of them is the sequence of Mersenne polynomials $\{M_n(x)\}$, defined as follows

$$M_n(x) = 3xM_{n-1}(x) - 2M_{n-2}(x) \quad \text{for } n \geq 2 \tag{2}$$

with $M_0(x) = 0$, $M_1(x) = 1$. Hence, we get

$$M_2(x) = 3x,$$
$$M_3(x) = 9x^2 - 2,$$
$$M_4(x) = 27x^3 - 12x,$$
$$M_5(x) = 81x^4 - 54x^2 + 4.$$

The Mersenne–Lucas polynomials $m_n(x)$ are defined by the same recurrence relation

$$m_n(x) = 3xm_{n-1}(x) - 2m_{n-2}(x) \quad \text{for } n \geq 2$$

with $m_0(x) = 2$, $m_1(x) = 3$. Hence, we obtain

$$m_2(x) = 9x - 4,$$
$$m_3(x) = 27x^2 - 12x - 6,$$
$$m_4(x) = 81x^3 - 36x^2 - 36x + 8,$$
$$m_5(x) = 243x^4 - 108x^3 - 162x^2 + 48x + 12.$$

Binet formula for the Mersenne polynomials has the form

$$M_n(x) = \frac{\lambda_1^n(x) - \lambda_2^n(x)}{\lambda_1(x) - \lambda_2(x)}, \tag{3}$$

where

$$\lambda_1(x) = \frac{1}{2}\big(3x + \sqrt{9x^2-8}\big), \quad \lambda_2(x) = \frac{1}{2}\big(3x - \sqrt{9x^2-8}\big), \quad 9x^2-8>0 \tag{4}$$

are the roots of the characteristic equation

$$\lambda^2 - 3x\lambda + 2 = 0.$$

Binet formula for the Mersenne–Lucas polynomials has the form

$$m_n(x) = A\lambda_1^n(x) + B\lambda_2^n(x),$$

where

$$A = 1 + \frac{3-3x}{\sqrt{9x^2-8}}, \quad B = 1 + \frac{3x-3}{\sqrt{9x^2-8}}. \tag{5}$$

2. The generalized commutative Mersenne and Mersenne–Lucas quaternions

In 1843, Hamilton ([4]) introduced the set $\mathbb{H}$ of quaternions q of the form

$$q = x_0 + x_1 i + x_2 j + x_3 k,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

In [5], Horadam introduced the concept of Fibonacci and Lucas quaternions. Moreover, Iyer in [11] gave relations between the Fibonacci and Lucas quaternions. Iakin in [9, 10] introduced the concept of a higher-order quaternion, and established some identities for these quaternions.

Non-commutative quaternions and commutative quaternions were generalized and studied recently by Jafari and Yayli, see [12]. Generalized commutative quaternions were introduced by Szynal-Liana and Włoch in [17], where the authors studied generalized commutative quaternions in the special subfamily of quaternions of Fibonacci-type. Some properties of other generalized commutative quaternions can be found in [2, 13].

Let $\mathbb{H}^c_{\alpha\beta}$ be the set of generalized commutative quaternions $\mathbf{x}$ of the form

$$\mathbf{x} = x_0 + x_1e_1 + x_2e_2 + x_3e_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$, quaternionic units e_1, e_2, e_3 satisfy the equalities

$$e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta,$$
$$e_1e_2 = e_2e_1 = e_3, \quad e_2e_3 = e_3e_2 = \beta e_1 \quad \text{and} \quad e_3e_1 = e_1e_3 = \alpha e_2,$$

and $\alpha, \beta \in \mathbb{R}$.

Generalized commutative quaternions generalize elliptic quaternions ($\alpha < 0$, $\beta = 1$), parabolic quaternions ($\alpha = 0$, $\beta = 1$), hyperbolic quaternions ($\alpha > 0$, $\beta = 1$), bicomplex numbers ($\alpha = -1$, $\beta = -1$), complex hyperbolic numbers ($\alpha = -1$, $\beta = 1$) and hyperbolic complex numbers ($\alpha = 1$, $\beta = -1$).

Let $n \geq 0$ be an integer. The n-th generalized commutative Mersenne quaternion $gc\mathcal{M}_n$ and the n-th generalized commutative Mersenne–Lucas quaternion $gc\mathcal{ML}_n$ are defined as follows

$$gc\mathcal{M}_n = M_n + M_{n+1}e_1 + M_{n+2}e_2 + M_{n+3}e_3,$$
$$gc\mathcal{ML}_n = m_n + m_{n+1}e_1 + m_{n+2}e_2 + m_{n+3}e_3.$$

These quaternions are special types of generalized commutative Horadam quaternions defined in [17].

Let $n \geq 0$ be an integer and x be a real variable. The n-th generalized commutative Mersenne quaternion polynomial $gc\mathcal{M}_n(x)$ and the n-th generalized commutative Mersenne–Lucas quaternion polynomial $gc\mathcal{ML}_n(x)$ are defined as follows

$$\text{(6)} \qquad gc\mathcal{M}_n(x) = M_n(x) + M_{n+1}(x)e_1 + M_{n+2}(x)e_2 + M_{n+3}(x)e_3,$$
$$gc\mathcal{ML}_n(x) = m_n(x) + m_{n+1}(x)e_1 + m_{n+2}(x)e_2 + m_{n+3}(x)e_3.$$

For $x = 1$, we have $gc\mathcal{M}_n(1) = gc\mathcal{M}_n$ and $gc\mathcal{ML}_n(1) = gc\mathcal{ML}_n$.

Generalized commutative quaternion polynomials of the Fibonacci-type are introduced in [18]. In [19], the authors considered generalized Pauli Fibonacci polynomial quaternions.

3. Main results

In this section, we give some identities for the generalized commutative Mersenne quaternion polynomials and the generalized commutative Mersenne–Lucas quaternion polynomials. We start with recurrence relations and Binet-type formulas for these quaternion polynomials.

THEOREM 1. *Let $n \geq 2$ be an integer and x be a real variable. Then*
(i) $gc\mathcal{M}_n(x) = 3xgc\mathcal{M}_{n-1}(x) - 2gc\mathcal{M}_{n-2}(x)$,
(ii) $gc\mathcal{ML}_n(x) = 3xgc\mathcal{ML}_{n-1}(x) - 2gc\mathcal{ML}_{n-2}(x)$,
where

$$\begin{aligned} gc\mathcal{M}_0(x) &= e_1 + 3xe_2 + (9x^2 - 2)e_3, \\ gc\mathcal{M}_1(x) &= 1 + 3xe_1 + (9x^2 - 2)e_2 + (27x^3 - 12x)e_3, \\ gc\mathcal{ML}_0(x) &= 2 + 3e_1 + (9x - 4)e_2 + (27x^2 - 12x - 6)e_3, \\ gc\mathcal{ML}_1(x) &= 3 + (9x - 4)e_1 + (27x^2 - 12x - 6)e_2 \\ &\quad + (81x^3 - 36x^2 - 36x + 8)e_3. \end{aligned}$$

PROOF. For $n = 2$ we get

$$\begin{aligned} gc\mathcal{M}_2(x) &= 3xgc\mathcal{M}_1(x) - 2gc\mathcal{M}_0(x) \\ &= 3x + 9x^2e_1 + (27x^3 - 6x)e_2 + (81x^4 - 36x^2)e_3 \\ &\quad - 2e_1 - 6xe_2 - (18x^2 - 4)e_3 \\ &= 3x + (9x^2 - 2)e_1 + (27x^3 - 12x)e_2 + (81x^4 - 54x^2 + 4)e_3. \end{aligned}$$

Let $n \geq 3$. By formulas (6) and (2) we get

$$\begin{aligned} gc\mathcal{M}_n(x) &= M_n(x) + M_{n+1}(x)e_1 + M_{n+2}(x)e_2 + M_{n+3}(x)e_3 \\ &= 3xM_{n-1}(x) - 2M_{n-2}(x) + (3xM_n(x) - 2M_{n-1}(x))e_1 \\ &\quad + (3xM_{n+1}(x) - 2M_n(x))e_2 + (3xM_{n+2}(x) - 2M_{n+1}(x))e_3 \\ &= 3x(M_{n-1}(x) + M_n(x)e_1 + M_{n+1}(x)e_2 + M_{n+2}(x)e_3) \\ &\quad - 2(M_{n-2}(x) + M_{n-1}(x)e_1 + M_n(x)e_2 + M_{n+1}(x)e_3) \\ &= 3xgc\mathcal{M}_{n-1}(x) - 2gc\mathcal{M}_{n-2}(x), \end{aligned}$$

which ends the proof of (i).
The second part can be proved similarly. □

COROLLARY 1. *Let $n \geq 2$ be an integer. Then*
(i) $gc\mathcal{M}_n = 3gc\mathcal{M}_{n-1} - 2gc\mathcal{M}_{n-2}$,
(ii) $gc\mathcal{ML}_n = 3gc\mathcal{ML}_{n-1} - 2gc\mathcal{ML}_{n-2}$,
where

$$\begin{aligned} gc\mathcal{M}_0 &= e_1 + 3e_2 + 7e_3, \\ gc\mathcal{M}_1 &= 1 + 3e_1 + 7e_2 + 15e_3, \\ gc\mathcal{ML}_0 &= 2 + 3e_1 + 5e_2 + 9e_3, \\ gc\mathcal{ML}_1 &= 3 + 5e_1 + 9e_2 + 17e_3. \end{aligned}$$

Theorem 2 (Binet-type formula for generalized commutative Mersenne quaternion polynomials). *Let $n \geq 0$ be an integer, x be a real variable and $9x^2 - 8 > 0$. Then*

$$gc\mathcal{M}_n(x) = \frac{\lambda_1^n(x)\widehat{\lambda_1(x)} - \lambda_2^n(x)\widehat{\lambda_2(x)}}{\lambda_1(x) - \lambda_2(x)}, \tag{7}$$

where $\lambda_1(x)$, $\lambda_2(x)$ are given by (4) *and*

$$\begin{aligned} \widehat{\lambda_1(x)} &= 1 + \lambda_1(x)e_1 + \lambda_1^2(x)e_2 + \lambda_1^3(x)e_3, \\ \widehat{\lambda_2(x)} &= 1 + \lambda_2(x)e_1 + \lambda_2^2(x)e_2 + \lambda_2^3(x)e_3. \end{aligned} \tag{8}$$

Proof. By (6) and (3) we get

$$\begin{aligned} gc\mathcal{M}_n(x) &= M_n(x) + M_{n+1}(x)e_1 + M_{n+2}(x)e_2 + M_{n+3}(x)e_3 \\ &= \frac{\lambda_1^n(x) - \lambda_2^n(x)}{\lambda_1(x) - \lambda_2(x)} + \frac{\lambda_1^{n+1}(x) - \lambda_2^{n+1}(x)}{\lambda_1(x) - \lambda_2(x)}e_1 \\ &\quad + \frac{\lambda_1^{n+2}(x) - \lambda_2^{n+2}(x)}{\lambda_1(x) - \lambda_2(x)}e_2 + \frac{\lambda_1^{n+3}(x) - \lambda_2^{n+3}(x)}{\lambda_1(x) - \lambda_2(x)}e_3 \\ &= \frac{\lambda_1^n(x)(1 + \lambda_1(x)e_1 + \lambda_1^2(x)e_2 + \lambda_1^3(x)e_3)}{\lambda_1(x) - \lambda_2(x)} \\ &\quad - \frac{\lambda_2^n(x)(1 + \lambda_2(x)e_1 + \lambda_2^2(x)e_2 + \lambda_2^3(x)e_3)}{\lambda_1(x) - \lambda_2(x)}. \end{aligned}$$

Hence, we get the result. □

In the same way, we can prove the following theorem.

Theorem 3 (Binet-type formula for generalized commutative Mersenne–Lucas quaternion polynomials). *Let $n \geq 0$ be an integer, x be a real variable and $9x^2 - 8 > 0$. Then*

$$gc\mathcal{ML}_n(x) = A\lambda_1^n(x)\widehat{\lambda_1(x)} + B\lambda_2^n(x)\widehat{\lambda_2(x)},$$

where A, B, $\lambda_1(x)$, $\lambda_2(x)$, $\widehat{\lambda_1(x)}$, $\widehat{\lambda_2(x)}$ are given by (5), (4), (8), *respectively.*

Corollary 2. *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} gc\mathcal{M}_n &= 2^n(1 + 2e_1 + 4e_2 + 8e_3) - (1 + e_1 + e_2 + e_3), \\ gc\mathcal{ML}_n &= 2^n(1 + 2e_1 + 4e_2 + 8e_3) + (1 + e_1 + e_2 + e_3). \end{aligned}$$

By simple calculations we have

$$\begin{aligned} &\lambda_1(x) - \lambda_2(x) = \sqrt{9x^2 - 8}, \quad \lambda_1(x) + \lambda_2(x) = 3x, \quad \lambda_1(x) \cdot \lambda_2(x) = 2, \\ &\lambda_1^2(x) + \lambda_2^2(x) = 9x^2 - 4, \quad \lambda_1^3(x) + \lambda_2^3(x) = 27x^3 - 18x, \end{aligned}$$

$$\begin{aligned}
\widehat{\lambda_1(x)}\widehat{\lambda_2(x)} &= \widehat{\lambda_2(x)}\widehat{\lambda_1(x)} \\
&= 1 + \lambda_2(x)e_1 + \lambda_2^2(x)e_2 + \lambda_2^3(x)e_3 \\
&\quad + \lambda_1(x)e_1 + \lambda_1(x)\lambda_2(x)\alpha + \lambda_1(x)\lambda_2^2(x)e_3 + \lambda_1(x)\lambda_2^3(x)\alpha e_2 \\
&\quad + \lambda_1^2(x)e_2 + \lambda_1^2(x)\lambda_2(x)e_3 + \lambda_1^2(x)\lambda_2^2(x)\beta + \lambda_1^2(x)\lambda_2^3(x)\beta e_1 \\
&\quad + \lambda_1^3(x)e_3 + \lambda_1^3(x)\lambda_2(x)\alpha e_2 + \lambda_1^3(x)\lambda_2^2(x)\beta e_1 + \lambda_1^3(x)\lambda_2^3(x)\alpha\beta \\
&= 1 + \lambda_1(x)\lambda_2(x)\alpha + \lambda_1^2(x)\lambda_2^2(x)\beta + \lambda_1^3(x)\lambda_2^3(x)\alpha\beta \\
&\quad + \lambda_2(x)e_1 + \lambda_1(x)e_1 + \lambda_1^2(x)\lambda_2^3(x)\beta e_1 + \lambda_1^3(x)\lambda_2^2(x)\beta e_1 \\
&\quad + \lambda_2^2(x)e_2 + \lambda_1^2(x)e_2 + \lambda_1(x)\lambda_2^3(x)\alpha e_2 + \lambda_1^3(x)\lambda_2(x)\alpha e_2 \\
&\quad + \lambda_2^3(x)e_3 + \lambda_1^3(x)e_3 + \lambda_1(x)\lambda_2^2(x)e_3 + \lambda_1^2(x)\lambda_2(x)e_3.
\end{aligned}$$

Hence, we get

$$\widehat{\lambda_1(x)}\widehat{\lambda_2(x)} = 1 + 2\alpha + 4\beta + 8\alpha\beta + (3x + 12x\beta)e_1 + (9x^2 - 4)(1 + 2\alpha)e_2 + (27x^3 - 12x)e_3. \tag{9}$$

The next theorems present general bilinear index-reduction formulas for generalized commutative Mersenne quaternion polynomials and generalized commutative Mersenne–Lucas quaternion polynomials.

THEOREM 4. *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Assume that x is a real variable and $9x^2 - 8 > 0$. Then*

$$\begin{aligned}
&gc\mathcal{M}_a(x) \cdot gc\mathcal{M}_b(x) - gc\mathcal{M}_c(x) \cdot gc\mathcal{M}_d(x) \\
&\quad = \frac{\left[\lambda_1^c(x)\lambda_2^d(x) + \lambda_2^c(x)\lambda_1^d(x) - \lambda_1^a(x)\lambda_2^b(x) - \lambda_2^a(x)\lambda_1^b(x)\right]\widehat{\lambda_1(x)}\widehat{\lambda_2(x)}}{9x^2 - 8},
\end{aligned}$$

where $\lambda_1(x)$, $\lambda_2(x)$ are given by (4) *and $\widehat{\lambda_1(x)}\widehat{\lambda_2(x)}$ is given by* (9).

PROOF. By formula (7) we have

$$\begin{aligned}
&gc\mathcal{M}_a(x) \cdot gc\mathcal{M}_b(x) - gc\mathcal{M}_c(x) \cdot gc\mathcal{M}_d(x) \\
&= \frac{-\lambda_1^a(x)\lambda_2^b(x)\widehat{\lambda_1(x)}\widehat{\lambda_2(x)} - \lambda_2^a(x)\lambda_1^b(x)\widehat{\lambda_2(x)}\widehat{\lambda_1(x)}}{9x^2 - 8} \\
&\quad + \frac{\lambda_1^c(x)\lambda_2^d(x)\widehat{\lambda_1(x)}\widehat{\lambda_2(x)} + \lambda_2^c(x)\lambda_1^d(x)\widehat{\lambda_2(x)}\widehat{\lambda_1(x)}}{9x^2 - 8} \\
&= \frac{\left[\lambda_1^c(x)\lambda_2^d(x) + \lambda_2^c(x)\lambda_1^d(x) - \lambda_1^a(x)\lambda_2^b(x) - \lambda_2^a(x)\lambda_1^b(x)\right]\widehat{\lambda_1(x)}\widehat{\lambda_2(x)}}{9x^2 - 8},
\end{aligned}$$

which ends the proof. □

Theorem 5. *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b = c+d$. Assume that x is a real variable and $9x^2-8>0$. Then*

$$gc\mathcal{ML}_a(x)\cdot gc\mathcal{ML}_b(x) - gc\mathcal{ML}_c(x)\cdot gc\mathcal{ML}_d(x)$$
$$= \frac{18x-17}{9x^2-8}\left(\lambda_1^a(x)\lambda_2^b(x)+\lambda_2^a(x)\lambda_1^b(x)-\lambda_1^c(x)\lambda_2^d(x)-\lambda_2^c(x)\lambda_1^d(x)\right)\widehat{\lambda_1(x)}\widehat{\lambda_2(x)},$$

where $\lambda_1(x)$, $\lambda_2(x)$ are given by (4) and $\widehat{\lambda_1(x)}\widehat{\lambda_2(x)}$ is given by (9).

Corollary 3. *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b=c+d$. Then*

$$gc\mathcal{M}_a\cdot gc\mathcal{M}_b - gc\mathcal{M}_c\cdot gc\mathcal{M}_d = \left(2^c+2^d-2^a-2^b\right)\widehat{\mathbf{1}}\cdot\widehat{\mathbf{2}},$$
$$gc\mathcal{ML}_a\cdot gc\mathcal{ML}_b - gc\mathcal{ML}_c\cdot gc\mathcal{ML}_d = \left(2^a+2^b-2^c-2^d\right)\widehat{\mathbf{1}}\cdot\widehat{\mathbf{2}},$$

where

$$\widehat{\mathbf{1}} = 1+e_1+e_2+e_3, \quad \widehat{\mathbf{2}} = 1+2e_1+4e_2+8e_3$$

and

$$\widehat{\mathbf{1}}\cdot\widehat{\mathbf{2}} = 1+2\alpha+4\beta+8\alpha\beta+(3+12\beta)e_1+(5+10\alpha)e_2+15e_3. \tag{10}$$

It is easily seen that for special values of a, b, c, d, by Theorem 4 and Theorem 5 we get new identities for generalized commutative Mersenne quaternion polynomials and generalized commutative Mersenne–Lucas quaternion polynomials. Assume that $\widehat{\lambda_1(x)}\widehat{\lambda_2(x)}$ is given by (9) and $9x^2-8>0$.

- Catalan-type identities for $a=n+r$, $b=n-r$, $c=d=n$, $r\geq 0$ and $n \geq r$

$$gc\mathcal{M}_{n+r}(x)\cdot gc\mathcal{M}_{n-r}(x) - \left(gc\mathcal{M}_n(x)\right)^2$$
$$= \frac{2^n}{9x^2-8}\left[2-\left(\frac{\lambda_1(x)}{\lambda_2(x)}\right)^r-\left(\frac{\lambda_2(x)}{\lambda_1(x)}\right)^r\right]\widehat{\lambda_1(x)}\widehat{\lambda_2(x)},$$
$$gc\mathcal{ML}_{n+r}(x)\cdot gc\mathcal{ML}_{n-r}(x) - \left(gc\mathcal{ML}_n(x)\right)^2$$
$$= \frac{2^n(18x-17)}{9x^2-8}\left[\left(\frac{\lambda_1(x)}{\lambda_2(x)}\right)^r+\left(\frac{\lambda_2(x)}{\lambda_1(x)}\right)^r-2\right]\widehat{\lambda_1(x)}\widehat{\lambda_2(x)},$$

- Cassini-type identities for $a=n+1$, $b=n-1$, $c=d=n$ and $n\geq 1$

$$gc\mathcal{M}_{n+1}(x)\cdot gc\mathcal{M}_{n-1}(x) - \left(gc\mathcal{M}_n(x)\right)^2$$
$$= \frac{2^n}{9x^2-8}\left[2-\frac{\lambda_1(x)}{\lambda_2(x)}-\frac{\lambda_2(x)}{\lambda_1(x)}\right]\widehat{\lambda_1(x)}\widehat{\lambda_2(x)},$$
$$gc\mathcal{ML}_{n+1}(x)\cdot gc\mathcal{ML}_{n-1}(x) - \left(gc\mathcal{ML}_n(x)\right)^2$$
$$= \frac{2^n(18x-17)}{9x^2-8}\left[\frac{\lambda_1(x)}{\lambda_2(x)}+\frac{\lambda_2(x)}{\lambda_1(x)}-2\right]\widehat{\lambda_1(x)}\widehat{\lambda_2(x)},$$

- d'Ocagne-type identities for $a = n$, $b = m + 1$, $c = n + 1$ and $d = m$ $n \geq m$

$$gc\mathcal{M}_n(x) \cdot gc\mathcal{M}_{m+1}(x) - gc\mathcal{M}_{n+1}(x) \cdot gc\mathcal{M}_m(x) = \frac{[\lambda_1^n(x)\lambda_2^m(x) - \lambda_2^n(x)\lambda_1^m(x)]\, \widehat{\lambda_1(x)}\widehat{\lambda_2(x)}}{\sqrt{9x^2-8}},$$

$$gc\mathcal{ML}_n(x) \cdot gc\mathcal{ML}_{m+1}(x) - gc\mathcal{ML}_{n+1}(x) \cdot gc\mathcal{ML}_m(x) = \frac{18x-17}{\sqrt{9x^2-8}} \left(\lambda_2^n(x)\lambda_1^m(x) - \lambda_1^n(x)\lambda_2^m(x)\right) \widehat{\lambda_1(x)}\widehat{\lambda_2(x)},$$

- Vajda-type identities for $a = m + p$, $b = n - p$, $c = m$, $d = n$ and $m \geq 0$, $p \geq 0$, $n \geq p$

$$gc\mathcal{M}_{m+p}(x) \cdot gc\mathcal{M}_{n-p}(x) - gc\mathcal{M}_m(x) \cdot gc\mathcal{M}_n(x) = \frac{\left[\lambda_1^m(x)\lambda_2^n(x)\left(1 - \left(\frac{\lambda_1(x)}{\lambda_2(x)}\right)^p\right) + \lambda_2^m(x)\lambda_1^n(x)\left(1 - \left(\frac{\lambda_2(x)}{\lambda_1(x)}\right)^p\right)\right]}{9x^2-8} \widehat{\lambda_1(x)}\widehat{\lambda_2(x)},$$

$$gc\mathcal{ML}_{m+p}(x) \cdot gc\mathcal{ML}_{n-p}(x) - gc\mathcal{ML}_m(x) \cdot gc\mathcal{ML}_n(x) = \frac{(18x-17)\left[\lambda_1^m(x)\lambda_2^n(x)\left(\left(\frac{\lambda_1(x)}{\lambda_2(x)}\right)^p - 1\right) + \lambda_2^m(x)\lambda_1^n(x)\left(\left(\frac{\lambda_2(x)}{\lambda_1(x)}\right)^p - 1\right)\right] \widehat{\lambda_1(x)}\widehat{\lambda_2(x)}}{9x^2-8}.$$

Now, we give such identities for generalized commutative Mersenne quaternions and generalized commutative Mersenne–Lucas quaternions. Assume that $\widehat{\mathbf{1}} \cdot \widehat{\mathbf{2}}$ is given by (10).

- Catalan-type identities for $r \geq 0$ and $n \geq r$

$$gc\mathcal{M}_{n+r} \cdot gc\mathcal{M}_{n-r} - (gc\mathcal{M}_n)^2 = \left(2^{n+1} - 2^{n+r} - 2^{n-r}\right) \widehat{\mathbf{1}} \cdot \widehat{\mathbf{2}},$$

$$gc\mathcal{ML}_{n+r} \cdot gc\mathcal{ML}_{n-r} - (gc\mathcal{ML}_n)^2 = \left(2^{n+r} + 2^{n-r} - 2^{n+1}\right) \widehat{\mathbf{1}} \cdot \widehat{\mathbf{2}},$$

- Cassini-type identities for $n \geq 1$

$$gc\mathcal{M}_{n+1} \cdot gc\mathcal{M}_{n-1} - (gc\mathcal{M}_n)^2 = -2^{n-1} \cdot \widehat{\mathbf{1}} \cdot \widehat{\mathbf{2}},$$

$$gc\mathcal{ML}_{n+1} \cdot gc\mathcal{ML}_{n-1} - (gc\mathcal{ML}_n)^2 = 2^{n-1} \cdot \widehat{\mathbf{1}} \cdot \widehat{\mathbf{2}},$$

- d'Ocagne-type identities for $n \geq m$

$$gc\mathcal{M}_n \cdot gc\mathcal{M}_{m+1} - gc\mathcal{M}_{n+1} \cdot gc\mathcal{M}_m = (2^n - 2^m)\, \widehat{\mathbf{1}} \cdot \widehat{\mathbf{2}},$$

$$gc\mathcal{ML}_n \cdot gc\mathcal{ML}_{m+1} - gc\mathcal{ML}_{n+1} \cdot gc\mathcal{ML}_m = (2^m - 2^n)\, \widehat{\mathbf{1}} \cdot \widehat{\mathbf{2}},$$

- Vajda-type identities for $m \geq 0$, $p \geq 0$, $n \geq p$

$$gc\mathcal{M}_{m+p} \cdot gc\mathcal{M}_{n-p} - gc\mathcal{M}_m \cdot gc\mathcal{M}_n = \left(2^m\left(1 - 2^p\right) + 2^n\left(1 - 2^{-p}\right)\right) \widehat{\mathbf{1}} \cdot \widehat{\mathbf{2}},$$

$$gc\mathcal{ML}_{m+p} \cdot gc\mathcal{ML}_{n-p} - gc\mathcal{ML}_m \cdot gc\mathcal{ML}_n = \left(2^m\left(2^p - 1\right) + 2^n\left(2^{-p} - 1\right)\right) \widehat{\mathbf{1}} \cdot \widehat{\mathbf{2}}.$$

4. Matrix generators and generating functions

Now, we give the matrix representations of $gc\mathcal{M}_n(x)$. By Theorem 1 we get the following result.

THEOREM 6. *Let $n \geq 1$ be an integer and x be a real variable. Then*

$$\begin{bmatrix} gc\mathcal{M}_{n+1}(x) \\ gc\mathcal{M}_n(x) \end{bmatrix} = \begin{bmatrix} 3x & -2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} gc\mathcal{M}_n(x) \\ gc\mathcal{M}_{n-1}(x) \end{bmatrix}.$$

THEOREM 7. *Let $n \geq 0$ be an integer and x be a real variable. Then*

$$\begin{bmatrix} gc\mathcal{M}_{n+2}(x) & gc\mathcal{M}_{n+1}(x) \\ gc\mathcal{M}_{n+1}(x) & gc\mathcal{M}_n(x) \end{bmatrix} = \begin{bmatrix} gc\mathcal{M}_2(x) & gc\mathcal{M}_1(x) \\ gc\mathcal{M}_1(x) & gc\mathcal{M}_0(x) \end{bmatrix} \cdot \begin{bmatrix} 3x & 1 \\ -2 & 0 \end{bmatrix}^n. \tag{11}$$

PROOF. We use induction on n. If $n = 0$ then the result is obvious. Assuming the formula (11) holds for $n \geq 0$, we shall prove it for $n + 1$.

Using induction's hypothesis and Theorem 1, we have

$$\begin{aligned}
&\begin{bmatrix} gc\mathcal{M}_2(x) & gc\mathcal{M}_1(x) \\ gc\mathcal{M}_1(x) & gc\mathcal{M}_0(x) \end{bmatrix} \cdot \begin{bmatrix} 3x & 1 \\ -2 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 3x & 1 \\ -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} gc\mathcal{M}_{n+2}(x) & gc\mathcal{M}_{n+1}(x) \\ gc\mathcal{M}_{n+1}(x) & gc\mathcal{M}_n(x) \end{bmatrix} \cdot \begin{bmatrix} 3x & 1 \\ -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3xgc\mathcal{M}_{n+2}(x) - 2gc\mathcal{M}_{n+1}(x) & gc\mathcal{M}_{n+2}(x) \\ 3xgc\mathcal{M}_{n+1}(x) - 2gc\mathcal{M}_n(x) & gc\mathcal{M}_{n+1}(x) \end{bmatrix} \\
&= \begin{bmatrix} gc\mathcal{M}_{n+3}(x) & gc\mathcal{M}_{n+2}(x) \\ gc\mathcal{M}_{n+2}(x) & gc\mathcal{M}_{n+1}(x) \end{bmatrix},
\end{aligned}$$

which ends the proof. □

COROLLARY 4. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} gc\mathcal{M}_{n+2} & gc\mathcal{M}_{n+1} \\ gc\mathcal{M}_{n+1} & gc\mathcal{M}_n \end{bmatrix} = \begin{bmatrix} gc\mathcal{M}_2 & gc\mathcal{M}_1 \\ gc\mathcal{M}_1 & gc\mathcal{M}_0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}^n.$$

In the same way, we can prove the following results.

THEOREM 8. *Let $n \geq 0$ be an integer and x be a real variable. Then*

$$\begin{bmatrix} gc\mathcal{ML}_{n+2}(x) & gc\mathcal{ML}_{n+1}(x) \\ gc\mathcal{ML}_{n+1}(x) & gc\mathcal{ML}_n(x) \end{bmatrix} = \begin{bmatrix} gc\mathcal{ML}_2(x) & gc\mathcal{ML}_1(x) \\ gc\mathcal{ML}_1(x) & gc\mathcal{ML}_0(x) \end{bmatrix} \cdot \begin{bmatrix} 3x & 1 \\ -2 & 0 \end{bmatrix}^n.$$

COROLLARY 5. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} gc\mathcal{ML}_{n+2} & gc\mathcal{ML}_{n+1} \\ gc\mathcal{ML}_{n+1} & gc\mathcal{ML}_n \end{bmatrix} = \begin{bmatrix} gc\mathcal{ML}_2 & gc\mathcal{ML}_1 \\ gc\mathcal{ML}_1 & gc\mathcal{ML}_0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}^n.$$

THEOREM 9. *The generating function of the generalized commutative Mersenne quaternion polynomials has the following form*

$$f(t) = \frac{e_1 + 3xe_2 + (9x^2 - 2)e_3 + (1 - 2e_2 - 6xe_3)t}{1 - 3xt + 2t^2}.$$

PROOF. Let

$$f(t) = gc\mathcal{M}_0(x) + tgc\mathcal{M}_1(x) + t^2 gc\mathcal{M}_2(x) + \ldots + t^n gc\mathcal{M}_n(x) + \ldots$$

be the generating function of the generalized commutative Mersenne quaternion polynomials. Then

$$\begin{aligned} 3xtf(t) &= 3txgc\mathcal{M}_0(x) + 3t^2 xgc\mathcal{M}_1(x) + 3t^3 xgc\mathcal{M}_2(x) \\ &\quad + \ldots + 3t^n xgc\mathcal{M}_n(x) + \ldots \\ 2t^2 f(t) &= 2t^2 gc\mathcal{M}_0(x) + 2t^3 gc\mathcal{M}_1(x) + 2t^4 gc\mathcal{M}_2(x) + \ldots \\ &\quad + 2t^n gc\mathcal{M}_{n-2}(x) + \ldots . \end{aligned}$$

Hence, by the recurrence $gc\mathcal{M}_n(x) = 3xgc\mathcal{M}_{n-1}(x) - 2gc\mathcal{M}_{n-2}(x)$, we get

$$\begin{aligned} f(t) - 3xtf(t) + 2t^2 f(t) &= gc\mathcal{M}_0(x) + (gc\mathcal{M}_1(x) - 3xgc\mathcal{M}_0(x))t \\ &\quad + (2gc\mathcal{M}_0(x) + gc\mathcal{M}_2(x) - 3xgc\mathcal{M}_1(x))t^2 + \ldots \\ &= gc\mathcal{M}_0(x) + (gc\mathcal{M}_1(x) - 3xgc\mathcal{M}_0(x))t. \end{aligned}$$

Thus

$$f(t) = \frac{gc\mathcal{M}_0(x) + (gc\mathcal{M}_1(x) - 3xgc\mathcal{M}_0(x))t}{1 - 3xt + 2t^2}.$$

After simple calculations we obtain

$$f(t) = \frac{e_1 + 3xe_2 + (9x^2 - 2)e_3 + (1 - 2e_2 - 6xe_3)t}{1 - 3xt + 2t^2}.$$

□

THEOREM 10. *The generating function of the generalized commutative Mersenne–Lucas quaternion polynomials has the following form*

$$g(t) = \frac{gc\mathcal{ML}_0(x) + (gc\mathcal{ML}_1(x) - 3xgc\mathcal{ML}_0(x))t}{1 - 3xt + 2t^2},$$

where

$$\begin{aligned} gc\mathcal{ML}_0(x) &= 2 + 3e_1 + (9x - 4)e_2 + (27x^2 - 12x - 6)e_3, \\ gc\mathcal{ML}_1(x) - 3xgc\mathcal{ML}_0(x) &= 3 + (3x - 4)e_1 - 6e_2 + (-18x + 8)e_3. \end{aligned}$$

COROLLARY 6. *The generating function of the generalized commutative Mersenne quaternions has the following form*

$$f_M(t) = \frac{e_1 + 3e_2 + 7e_3 + (1 - 2e_2 - 6e_3)t}{1 - 3t + 2t^2}.$$

COROLLARY 7. *The generating function of the generalized commutative Mersenne–Lucas quaternions has the following form*

$$g_L(t) = \frac{2 + 3e_1 + 5e_2 + 9e_3 + (3 - e_1 - 6e_2 - 10e_3)t}{1 - 3t + 2t^2}.$$

Concluding remarks

For any positive integer n, the n-th bivariate Horadam polynomial $h_n(x, y)$ was defined in [16] as $h_n(x, y) = pxh_{n-1}(x, y) + qyh_{n-2}(x, y)$ for $n \geq 3$ with the initial values $h_1(x, y) = a$ and $h_2(x, y) = bx$. It is easy to see that $h_n(x, 1) = h_n(x)$. Bivariate Mersenne polynomials $M_n(x, y)$ and bivariate Mersenne Lucas polynomials $m_n(x, y)$ were defined in [1] and [14], respectively, as follows

$$M_n(x, y) = 3yM_{n-1}(x, y) - 2xM_{n-2}(x, y) \quad \text{for } n \geq 2$$

with $M_0(x, y) = 0$, $M_1(x, y) = 1$ and

$$m_n(x, y) = 3ym_{n-1}(x, y) - 2xm_{n-2}(x, y) \quad \text{for } n \geq 2$$

with $m_0(x, y) = 2$, $m_1(x, y) = 3y$.

It is worth noting that, unlike before, $M_n(1, x) = M_n(x)$ and $m_n(1, x) = m_n(x)$. Using the above definitions, we can define, for any variables x, y and any nonnegative integer n, the n-th bivariate generalized commutative Mersenne quaternion polynomial $gc\mathcal{M}_n(x, y)$ and the n-th bivariate generalized commutative Mersenne–Lucas quaternion polynomial $gc\mathcal{ML}_n(x, y)$ as follows

$$gc\mathcal{M}_n(x, y) = M_n(x, y) + M_{n+1}(x, y)e_1 + M_{n+2}(x, y)e_2 + M_{n+3}(x, y)e_3,$$
$$gc\mathcal{ML}_n(x, y) = m_n(x, y) + m_{n+1}(x, y)e_1 + m_{n+2}(x, y)e_2 + m_{n+3}(x, y)e_3.$$

Further work may involve research on these polynomials.

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