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A NOTE ON THE FUZZY XOR

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Abstract. In this paper we will try to answer what conditions must be met by the fuzzy Xor to be used for cryptographic purposes. We will also show that defining the fuzzy Xor using other fuzzy connectives is not suitable for this purpose.

1. Introduction

The connective *exclusive or* plays an important role in computer programming especially in many encryption algorithms (see e.g. [9], [14], [15]), neural networks ([7], [13]), Quantum Computing ([10], [12]) or other fields ([4], [5], [16], [17]). The authors of [2] and [3] introduced an autonomous definition of the fuzzy Xor connective, which is independent of the other connectives. In the earlier literature we can find a fuzzy Xor operation as a composition of the fuzzy negation, and triangular conorms and norms (see [6], [8], [11], [13]).

In this paper we show what conditions must be met by the fuzzy Xor to be used for cryptographic purposes. We also provide new constructions of fuzzy Xor based on the composition of fuzzy implications and other fuzzy connectives.

2. Fuzzy connectives

In this article we use the notation $\overset{(property)}{=}$, which means that this equality holds due to the indicated property.

First, we recall definitions of some fuzzy connectives.

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Definition 2.1. A *t-norm* is a function $T\colon [0,1]^2 \to [0,1]$ satisfying
(T1) $T(x,y) = T(y,x)$, $x,y \in [0,1]$;
(T2) $T(x,T(y,z)) = T(T(x,y),z)$, $x,y,z \in [0,1]$;
(T3) $T(x,z) \le T(y,z)$, $x,y,z \in [0,1]$, $x \le y$;
(T4) $T(x,1) = x$, $x \in [0,1]$.

Definition 2.2. A *t-conorm* is a function $S\colon [0,1]^2 \to [0,1]$ satisfying
(S1) $S(x,y) = S(y,x)$, $x,y \in [0,1]$;
(S2) $S(x,S(y,z)) = S(S(x,y),z)$, $x,y,z \in [0,1]$;
(S3) $S(x,z) \le S(y,z)$, $x,y,z \in [0,1]$, $x \le y$;
(S4) $S(x,0) = x$, $x \in [0,1]$.

Definition 2.3. A non-increasing function $N\colon [0,1] \to [0,1]$ is called a *fuzzy negation* if $N(0) = 1$, $N(1) = 0$. A fuzzy negation N is called

- *strict* if it is strictly decreasing and continuous;
- *strong* if it is an involution, i.e., $N(N(x)) = x$, $x \in [0,1]$.

Given a fuzzy negation N, a value $e \in (0,1)$ such that $N(e) = e$ is said to be an *equilibrium point* of N. If the equilibrium point of N exists (e.g. when N is continuous), then it is unique.

Definition 2.4. A function $I\colon [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies the following properties:
(I1) $I(x_2,y) \le I(x_1,y)$, $x_1,x_2,y \in [0,1]$, $x_1 \le x_2$;
(I2) $I(x,y_1) \le I(x,y_2)$, $x,y_1,y_2 \in [0,1]$, $y_1 \le y_2$;
(I3) $I(0,0) = 1$;
(I4) $I(1,1) = 1$;
(I5) $I(1,0) = 0$.

The *natural negation* of a fuzzy implication I is defined by $N_I(x) := I(x,0)$, $x \in [0,1]$.

A fuzzy implication can have many properties (see [1]) but we will remind here only those that we will need, so for a fuzzy implication I and a fuzzy negation N we have the following properties:

(R-CP(N)) $$I\big(x, N(y)\big) = I\big(y, N(x)\big), \quad x,y \in [0,1];$$

(L-CP(N)) $$I\big(N(x), y\big) = I\big(N(y), x\big), \quad x,y \in [0,1];$$

(NP) $$I(1,x) = x, \quad x \in [0,1];$$

(IP) $$I(x,x) = 1, \quad x \in [0,1].$$

B. Bedregal, R.H.S. Reiser and G.P. Dimuro in their two works [2], [3] tried to introduce an intrinsic definition of the connective fuzzy exclusive or.

Definition 2.5. A function $E\colon [0,1]^2 \to [0,1]$ is called a *fuzzy Xor* if it satisfies the following properties:

(E0) $E(0,0) = E(1,1) = 0,\ E(1,0) = E(0,1) = 1$;

(E1) $E(x,y) = E(y,x),\ x,y \in [0,1]$;

(E2a) functions $E(0,\cdot), E(\cdot,0)$ are non-decreasing;

(E2b) functions $E(1,\cdot), E(\cdot,1)$ are non-increasing.

Remark 2.6. Let $E\colon [0,1]^2 \to [0,1]$ satisfy (E0) and (E2b). Then $N_E\colon [0,1] \to [0,1]$ given by

$$N_E(x) := E(x,1), \quad x \in [0,1]$$

is a fuzzy negation. When E is a fuzzy Xor, then it is called the natural fuzzy negation of the fuzzy Xor E.

In [2], [3] we can find also the following properties connected with a fuzzy Xor function E:

(E3) $E(x,x) \neq 1,\ x \in [0,1]$;

(E4) $E(0,x) = x,\ x \in [0,1]$;

(E5) $E(E(x,y),z) = E(x,E(y,z)),\ x,y,z \in [0,1]$;

(E6) $E(x,x) = 0,\ x \in [0,1]$;

(E7) $E(E(x,y),y) = x,\ x,y \in [0,1]$;

(E9) N_E is a strict fuzzy negation;

(E10) N_E is a strong fuzzy negation;

(E13) $E(N_E(x),x) = 1,\ x \in [0,1]$;

(E15) $N_E(E(x,y)) = E(x,N_E(y)),\ x,y \in [0,1]$.

In this work we will pay special attention to the property (E7), which is related to encryption. The authors of [3] showed that some of the above properties can be obtained from others (e.g. obvious implications (E6) $\Rightarrow$ (E3), (E10) $\Rightarrow$ (E9), (E5) $\Rightarrow$ (E15)) but they missed that some of them are incompatible.

Lemma 2.7. *Let $E\colon [0,1]^2 \to [0,1]$ satisfy* (E13) *and there exists an equilibrium point of N_E. Then* (E3) *does not hold.*

Proof. Let $e \in (0,1)$ be such that $N_E(e) = e$. We have

$$E(e,e) = E(N_E(e),e) = 1.$$ □

Theorem 2.8. *Let $E\colon [0,1]^2 \to [0,1]$ satisfy* (E7)*. Then, the following relations hold:*

(i) (E2b) $\Rightarrow$ $\big($ (E10)$\wedge\neg$(E15)$\wedge\neg$(E5)$\big)$;

(ii) (E1) $\Rightarrow$ (E13);

(iii) (E4) $\Leftrightarrow$ (E6)*;*
(iv) (E2a) $\Rightarrow$ $E(x,0)=x$ *for* $x\in[0,1]$*;*
(v) (E2b)$\wedge$(E13) $\Rightarrow$ $\big($ $\neg$(E3) $\wedge$ $\neg$(E4) $\wedge$ $\neg$(E6) $\big)$*;*
(vi) (E1)$\wedge$(E2b) $\Rightarrow$ $\big($ $\neg$(E2a) $\wedge$ $\neg$(E3) $\wedge$ $\neg$(E4) $\wedge$ $\neg$(E6) $\big)$*;*
(vii) (E1)$\wedge$(E2a) $\Rightarrow$ $\big($ $\neg$(E2b) $\wedge$ (E3) $\wedge$ (E4) $\wedge$ (E6) $\big)$*;*
(viii) $\big($(E2a)$\wedge$(E2b)$\wedge$(E7)$\big)$ $\Rightarrow$ (E0).

Proof. (i) We have

$$x \overset{(E7)}{=} E(E(x,1),1)=N_E^2(x),\ x\in[0,1].$$

Hence $N_E^2(0)=0$ and $N_E^2(1)=1$, so from (E2b) we get $N_E(0)=1$ and $N_E(1)=0$.

Suppose that (E15) holds. Since N_E is strong, there exists $y_0\in(0,1)$ such that $y_0=N_E(y_0)$. Then for $x_0:=E(1,y_0)$ we have

$$\begin{aligned}0=N_E(1) &\overset{(E7)}{=} N_E\Big(E\big(E(1,y_0),y_0\big)\Big)=N_E(E(x_0,y_0))\\ &\overset{(E15)}{=} E\big(x_0,N_E(y_0)\big)=E(x_0,y_0)=E\big(E(1,y_0),y_0\big)\overset{(E7)}{=}1,\end{aligned}$$

which gives us a contradiction. Hence (E15) and (E5) do not hold.

(ii) We observe that

$$E(N_E(x),x)=E(E(x,1),x)\overset{(E1)}{=}E(E(1,x),x)\overset{(E7)}{=}1,\quad x\in[0,1].$$

(iii) We have

$$E(x,x)\overset{(E4)}{=}E(E(0,x),x)\overset{(E7)}{=}0,\quad x\in[0,1]$$

and

$$E(0,x)\overset{(E6)}{=}E(E(x,x),x)\overset{(E7)}{=}x,\quad x\in[0,1].$$

(iv) Let $f(x):=E(x,0)$, $x\in[0,1]$. We notice that

$$f^2(x)=E(E(x,0),0)\overset{(E7)}{=}x,\quad x\in[0,1],$$

so since f is non-decreasing, $f(x)=x$, $x\in[0,1]$.

(v) From (i) and Lemma 2.7 we know that (E3) does not hold, so (E6) does not hold. Hence from (iii) (E4) does not hold.

(vi) From (ii) and (v) we obtain that (E3), (E4), (E6) do not hold.

Suppose that (E2a) holds. We have $E(0,x)\overset{(E1)}{=}E(x,0)\overset{(iv)}{=}x$, $x\in[0,1]$, so (E4) holds and we obtain the contradiction.

(vii) We have $E(0,x) \overset{(E1)}{=} E(x,0) \overset{(iv)}{=} x$, $x \in [0,1]$, so (E4) holds. From (iii) (E6) (and (E3)) holds. From (vi) (E2b) does not hold.

(viii) We have

$$0 \overset{(E7)}{=} E\big(E(0,0),0\big) \overset{(E2a)}{\geq} E(0,0),$$
$$1 \overset{(E7)}{=} E\big(E(1,0),0\big) \overset{(E2a)}{\leq} E(1,0),$$
$$1 \overset{(E7)}{=} E\big(E(1,1),1\big) \overset{(E2b)}{\leq} E(0,1),$$
$$0 \overset{(E7)}{=} E\big(E(0,1),1\big) \overset{(E2b)}{\geq} E(1,1).$$ □

Remark 2.9. Let

(E1-0) $E(x,0) = E(0,x)$, $x \in [0,1]$;

(E1-1) $E(x,1) = E(1,x)$, $x \in [0,1]$.

In the previous theorem instead of (E1) we can assume in (ii) that (E1-1) holds, in (vi) and (vii) that (E1-0), (E1-1) hold. Moreover, we have (E2a)$\wedge$ (E4) $\Rightarrow$ (E1-0).

Remark 2.10. There is no function that satisfies (E0), (E1), (E2a), (E2b) and (E7). Since conditions (E2a) and (E2b) are similar, then among functions satisfying (E7) it is better to look for these which do not satisfy (E1).

Remark 2.11. Let $E\colon [0,1]^2 \to [0,1]$ satisfy (E2a), (E2b) and (E7). Then we have the following possibilities:

- E satisfies (E3), (E4), (E6) and does not satisfy (E13).
- E satisfies (E13) and does not satisfy (E3), (E4), (E6).
- E does not satisfy (E3), (E4), (E6), (E13).

All of the above possibilities can occur, as illustrated by the following examples.

Example 2.12. Let $N_1, N_2\colon [0,1] \to [0,1]$ be fuzzy negations, N_1 be strong, $E\colon [0,1]^2 \to [0,1]$ be a function given by

$$E(x,y) = \begin{cases} N_2(y) + \big(1 - N_2(y)\big) N_1\Big(\frac{x - N_2(y)}{1 - N_2(y)}\Big), & x \geq N_2(y) \neq 1, \\ x, & x < N_2(y) \vee N_2(y) = 1. \end{cases}$$

We will show that E satisfies (E2a), (E2b), (E7) and does not satisfy (E3), (E4), (E6). Moreover (E13) holds iff $N_1 = N_2$. We have

$$E(x,0) = x, \quad x \in [0,1], \quad E(0,y) = \begin{cases} 1, & N_2(y) = 0, \\ 0, & N_2(y) \neq 0, \end{cases}$$

so (E2a) holds and (E4) does not hold,

$$\begin{aligned} N_E(x) = E(x,1) &= N_1(x), \quad x \in [0,1], \\ E(1,y) &= N_2(y), \quad y \in [0,1], \end{aligned}$$

so (E2b) holds. Finally, for $x, y \in [0,1]$ we have

- if $x \geq N_2(y) \neq 1$, then $E(x,y) \geq N_2(y)$ and

$$\begin{aligned} E(E(x,y),y) &= N_2(y) + \big(1 - N_2(y)\big) N_1\Big(\frac{E(x,y) - N_2(y)}{1 - N_2(y)}\Big) \\ &= N_2(y) + \big(1 - N_2(y)\big) N_1\left(\frac{N_2(y) + \big(1 - N_2(y)\big) N_1\Big(\frac{x - N_2(y)}{1 - N_2(y)}\Big) - N_2(y)}{1 - N_2(y)}\right) \\ &= N_2(y) + \big(1 - N_2(y)\big) N_1\Big(N_1\Big(\frac{x - N_2(y)}{1 - N_2(y)}\Big)\Big) \\ &= N_2(y) + \big(1 - N_2(y)\big) \frac{x - N_2(y)}{1 - N_2(y)} = x, \end{aligned}$$

- if $x < N_2(y) \vee N_2(y) = 1$, then $E(x,y) = x$ and

$$E(E(x,y),y) = E(x,y) = x,$$

so (E7) holds. From Theorem 2.8 (iii) (E6) and (E3) do not hold.

Now, we notice that

$$E\big(N_2(y), y\big) = 1,$$

so (E13) $\Leftrightarrow N_1 = N_2$.

EXAMPLE 2.13. Let $N_1, N_2 \colon [0,1] \to [0,1]$ be fuzzy negations, N_1 be strong, N_2 has not an equilibrium point. For $y \in (0,1)$ let $A_y := (0,1) \setminus \{y, N_2(y)\}$ and $\varphi_y \colon A_y \to A_y$ be a bijective involution. Let further $E \colon [0,1]^2 \to [0,1]$ be a function given by

$$E(x,y) = \begin{cases} \varphi_y(x), & y \in (0,1),\ x \in A_y, \\ y, & x = 0, \\ N_2(y), & x = 1, \\ 0, & x = y \in (0,1), \\ 1, & x = N_2(y) \in (0,1), \\ x, & y = 0,\ x \in (0,1), \\ N_1(x), & y = 1,\ x \in (0,1). \end{cases}$$

We will show that E satisfies (E2a), (E2b), (E3), (E4), (E6), (E7) and does not satisfy (E13).

First, we observe that

$$E\big(E(x,y),y\big) = \begin{cases} E\big(\varphi_y(x),y\big), & y \in (0,1),\ x \in A_y, \\ E(y,y), & x = 0, \\ E\big(N_2(y),y\big), & x = 1, \\ E(0,y), & x = y \in (0,1), \\ E(1,y), & x = N_2(y) \in (0,1), \\ E(x,y), & y = 0,\ x \in (0,1), \\ E\big(N_1(x),y\big), & y = 1,\ x \in (0,1). \end{cases}$$

$$= \begin{cases} \varphi_y\big(\varphi_y(x)\big), & y \in (0,1),\ x \in A_y, \\ 0, & x = 0, \\ 1, & x = 1, \\ y, & x = y \in (0,1), \\ N_2(y), & x = N_2(y) \in (0,1), \\ x, & y = 0,\ x \in (0,1), \\ N_1\big(N_1(x)\big), & y = 1,\ x \in (0,1). \end{cases} = x, \quad x,y \in [0,1],$$

so E satisfies (E7). We have also

$$\begin{aligned} &E(x,0) = x,\ x \in [0,1], \\ &E(0,y) = y,\ y \in [0,1], \\ &E(x,1) = N_1(x),\ x \in [0,1], \\ &E(1,y) = N_2(y),\ y \in [0,1], \end{aligned}$$

whence (E2a), (E4), (E2b) hold. From Theorem 2.8 we obtain that (E6), (E3) hold and (E13) does not hold.

3. Fuzzy Xor generated from the other fuzzy connectives

In [2] and [3] we can find some classes of functions:

$$\begin{aligned} &E_{S,T,N}(x,y) = S\Big(T\big(x,N(y)\big),T\big(N(x),y\big)\Big), \quad x,y \in [0,1]; \\ &E_{T,S,N}(x,y) = T\Big(S(x,y),S\big(N(x),N(y)\big)\Big), \quad x,y \in [0,1]; \\ &E_{S,T}(x,y) = S(x,y) - T(x,y), \quad x,y \in [0,1]; \end{aligned}$$

where $T\colon [0,1]^2 \to [0,1]$ is a t-norm, $S\colon [0,1]^2 \to [0,1]$ is a t-conorm, and $N\colon [0,1] \to [0,1]$ is a fuzzy negation. Each of them is a fuzzy Xor satisfying (E4). We would like to present others classes which may not have this property.

3.1. Fuzzy Xor generated from fuzzy implications and negation

In a classical logic we can obtain a Xor in the following way

$$\alpha \oplus \beta = ((\alpha \Rightarrow \beta) \Rightarrow \neg(\beta \Rightarrow \alpha)).$$

Hence, we can define the following class of functions.

Definition 3.1. Let $I\colon [0,1]^2 \to [0,1]$ be a fuzzy implication, $N\colon [0,1] \to [0,1]$ be a fuzzy negation. We define the function $E_{I,N}\colon [0,1]^2 \to [0,1]$ by the formula

$$E_{I,N}(x,y) = I\big(I(x,y), N\big(I(y,x)\big)\big), \quad x,y \in [0,1]. \tag{1}$$

We can generalize the above definition in the following way:

Definition 3.2. Let $I_1, I_2\colon [0,1]^2 \to [0,1]$ be fuzzy implications and $N\colon [0,1] \to [0,1]$ be a fuzzy negation. We define the function $E_{I_1,I_2,N}\colon [0,1]^2 \to [0,1]$ by the formula

$$E_{I_1,I_2,N}(x,y) = I_1\big(I_2(x,y), N\big(I_2(y,x)\big)\big), \quad x,y \in [0,1]. \tag{2}$$

Remark 3.3. Let $S\colon [0,1]^2 \to [0,1]$ be a t-conorm, $T\colon [0,1]^2 \to [0,1]$ be a t-norm, $N\colon [0,1] \to [0,1]$ be a fuzzy negation. Let further $\widetilde{N}\colon [0,1] \to [0,1]$ be any strong negation. We define

$$I_1(x,y) := S\big(\widetilde{N}(x), y\big), \quad x,y \in [0,1],$$
$$I_2(x,y) := \widetilde{N}\big(T\big(x, N(y)\big)\big), \quad x,y \in [0,1].$$

Then

$$\begin{aligned} E_{I_1,I_2,\widetilde{N}}(x,y) &= I_1\big(I_2(x,y), \widetilde{N}\big(I_2(y,x)\big)\big) \\ &= S\Big(\widetilde{N}\Big(\widetilde{N}\big(T\big(x,N(y)\big)\big)\Big), \widetilde{N}\Big(\widetilde{N}\big(T\big(y,N(x)\big)\big)\Big)\Big) \\ &= S\Big(T\big(x,N(y)\big), T\big(N(x),y\big)\Big) = E_{S,T,N}(x,y), \quad x,y \in [0,1]. \end{aligned}$$

Theorem 3.4. *Let $I_1, I_2\colon [0,1]^2 \to [0,1]$ be fuzzy implications, $N\colon [0,1] \to [0,1]$ be a fuzzy negation. Then for $E_{I_1,I_2,N}$ given by* (2) *we have:*

(i) (E0), (E2a), (E2b) *hold and*

$$E_{I_1,I_2,N}(x,0) = (N_{I_1} \circ N_{I_2})(x), \quad x \in [0,1], \tag{3}$$
$$E_{I_1,I_2,N}(0,x) = (g_{I_1} \circ N \circ N_{I_2})(x), \quad x \in [0,1], \tag{4}$$
$$E_{I_1,I_2,N}(x,1) = (g_{I_1} \circ N \circ g_{I_2})(x), \quad x \in [0,1], \tag{5}$$
$$E_{I_1,I_2,N}(1,x) = (N_{I_1} \circ g_{I_2})(x), \quad x \in [0,1], \tag{6}$$

where $g_{I_1}(x) := I_1(1,x)$, $g_{I_2}(x) := I_2(1,x)$ for $x \in [0,1]$;

(ii) *if* I_1 *satisfies* (R-CP(N)), *then* (E1) *holds,* (E7) *does not hold;*

(iii) (E4) *holds iff* $(g_{I_1} \circ N \circ N_{I_2})(x) = x$ *for* $x \in [0,1]$*;*

(iv) *if* I_1, I_2 *satisfy* (R-CP(N)), *then* $N_{E_{I_1,I_2,N}} \circ N = N_{I_1} \circ N_{I_2}$*;*

(v) *if* I_1, I_2 *satisfy* (R-CP(N)), N_{I_1}, N_{I_2}, N *are strict, then* (E9) *holds;*

(vi) *if* I_2 *satisfies* (IP), *then* (E6) *holds;*

(vii) *if* I_1 *satisfies* (R-CP(N)) *and* I_2 *satisfies* (IP), *then*

$$E_{I_1,I_2,N}(x,y) = N_{I_1}\Big(\min\Big(I_2(x,y), I_2(y,x)\Big)\Big), \quad x, y \in [0,1].$$

Proof. (i) We observe that for $x \in [0,1]$ we have

$$\begin{aligned}
E_{I_1,I_2,N}(x,0) &= I_1\Big(I_2(x,0), N\big(I_2(0,x)\big)\Big) = I_1\Big(N_{I_2}(x), N(1)\Big)\\
&= I_1(N_{I_2}(x), 0) = (N_{I_1} \circ N_{I_2})(x),\\
E_{I_1,I_2,N}(0,x) &= I_1\Big(I_2(0,x), N\big(I_2(x,0)\big)\Big) = I_1\Big(1, N(N_{I_2}(x))\Big)\\
&= (g_{I_1} \circ N \circ N_{I_2})(x),\\
E_{I_1,I_2,N}(x,1) &= I_1\Big(I_2(x,1), N\big(I_2(1,x)\big)\Big) = I_1\Big(1, N\big(I_2(1,x)\big)\Big)\\
&= (g_{I_1} \circ N \circ g_{I_2})(x),\\
E_{I_1,I_2,N}(1,x) &= I_1\Big(I_2(1,x), N\big(I_2(x,1)\big)\Big) = I_1\Big(I_2(1,x), N(1)\Big)\\
&= I_1(I_2(1,x), 0) = (N_{I_1} \circ g_{I_2})(x),
\end{aligned}$$

so we get (E0), (E2a), (E2b).

(ii) Assume that I_1 satisfies (R-CP(N)). Hence, using the above, we get

$$\begin{aligned}
E_{I_1,I_2,N}(y,x) &= I_1\Big(I_2(y,x), N\big(I_2(x,y)\big)\Big)\\
&\overset{\text{(R-CP(N))}}{=} I_1\Big(I_2(x,y), N\big(I_2(y,x)\big)\Big)\\
&= E_{I_1,I_2,N}(x,y),\ x, y \in [0,1].
\end{aligned}$$

Using Theorem 2.8 we get that (E7) does not hold.

(iii) It follows from i.

(iv) Assume that I_2 satisfies (R-CP(N)). Then we have

$$\begin{aligned}
N_{E_{I_1,I_2,N}}\big(N(x)\big) &\overset{ii}{=} N_{I_1}\Big(I_2\big(1, N(x)\big)\Big) \overset{\text{(R-CP(N))}}{=} N_{I_1}\Big(I_2\big(x, N(1)\big)\Big)\\
&= N_{I_1}\big(I_2(x,0)\big) = (N_{I_1} \circ N_{I_2})(x),\ x \in [0,1].
\end{aligned}$$

(v) It follows from iv.

(vi) Assume that I_2 satisfies (IP). Then

$$E_{I_1,I_2,N}(x,x) = I_1\big(I_2(x,x), N\big(I_2(x,x)\big)\big) \stackrel{\text{(IP)}}{=} I_1\big(1, N(1)\big) = I_1(1,0) = 0, \quad x \in [0,1],$$

so (E6) holds.

(vii) Let $x, y \in [0,1]$. From ii we have that $E_{I_1,I_2,N}$ is symmetric, so we can assume that $y \le x$ whence from (IP) for I_2 we get $I_2(y,x) = 1$ and

$$\begin{aligned} E_{I_1,I_2,N}(x,y) &= I_1\big(I_2(x,y), N\big(I_2(y,x)\big)\big) \\ &= I_1\big(I_2(x,y), N(1)\big) = I_1\big(I_2(x,y), 0\big) = N_{I_1}\big(I_2(x,y)\big) \\ &= N_{I_1}\big(\min\big(I_2(x,y), I_2(y,x)\big)\big). \end{aligned}$$

□

Lemma 3.5. *Let $I\colon [0,1]^2 \to [0,1]$ be a fuzzy implication, $N\colon [0,1] \to [0,1]$ be a fuzzy negation. Assume that $E_{I,N}$ is given by* (1). *Then* (E7) *does not hold.*

Proof. Suppose that (E7) holds. First, observe that N_I is strong. Let $x \in [0,1]$. Using Theorem 2.8 (iv) we have

$$x = E_{I,N}(x,0) \stackrel{(3)}{=} N_I\big(N_I(x)\big),$$

so N_I is strong.

Next, for $x \in [0,1]$ we notice that

$$N_{E_{I,N}}(x) = E_{I,N}(x,1) \stackrel{(5)}{=} (g_I \circ N \circ g_I)(x).$$

Since g_I is non-decreasing, $N_{E_{I,N}}$ is a strong fuzzy negation, then g_I is a bijection and $N = g_I^{-1} \circ N_{E_{I,N}} \circ g_I^{-1}$ is strict.

The function $g_I^{-1} \circ N_I \circ g_I \circ N \circ N_I$ is a strict fuzzy negation, so it has an equilibrium point $e \in (0,1)$. Hence

$$(g_I \circ N \circ N_I)(e) = (N_I \circ g_I)(e).$$

Using the previous theorem we get

$$\begin{aligned} 1 &\stackrel{\text{(E7)}}{=} E(E(1,e),e) \stackrel{(6)}{=} E\big((N_I \circ g_I)(e), e\big) \\ &= E\big((g_I \circ N \circ N_I)(e), e\big) \stackrel{(4)}{=} E(E(0,e),e) \stackrel{\text{(E7)}}{=} 0, \end{aligned}$$

which gives us a contradiction. □

Problem 3.6. Do exist fuzzy implications I_1,I_2 and a negation N such that $E_{I_1,I_2,N}$ satisfies (E7)?

3.2. Fuzzy Xor generated from t-norm, fuzzy implication and negation

In a classical logic we can obtain a Xor in the following way

$$\alpha \oplus \beta = ((\alpha \Rightarrow \neg\beta) \wedge (\neg\alpha \Rightarrow \beta)).$$

Hence, we can define the following class of functions.

Definition 3.7. Let $T\colon [0,1]^2 \to [0,1]$ be a t-norm, $I\colon [0,1]^2 \to [0,1]$ be a fuzzy implication, $N\colon [0,1] \to [0,1]$ be a fuzzy negation. We define the function $E_{T,I,N}\colon [0,1]^2 \to [0,1]$ by the formula

$$E_{T,I,N}(x,y) = T\Big(I\big(x,N(y)\big), I\big(N(x),y\big)\Big), \quad x,y \in [0,1]. \tag{7}$$

Remark 3.8. Let $S\colon [0,1]^2 \to [0,1]$ be a t-conorm, $T\colon [0,1]^2 \to [0,1]$ be a t-norm, $N\colon [0,1] \to [0,1]$ be a strong fuzzy negation. We define

$$I(x,y) := S\big(N(x),y\big), \quad x,y \in [0,1].$$

Then

$$\begin{aligned} E_{T,I,N}(x,y) &= T\Big(I\big(x,N(y)\big), I\big(N(x),y\big)\Big) \\ &= T\Big(S\big(N(x),N(y)\big), S\big(N^2(x),y\big)\Big) \\ &= T\Big(S\big(N(x),N(y)\big), S\big(x,y\big)\Big) = E_{T,S,N}(x,y),\ x,y \in [0,1]. \end{aligned}$$

Theorem 3.9. *Let $T\colon [0,1]^2 \to [0,1]$ be a t-norm, $I\colon [0,1]^2 \to [0,1]$ be a fuzzy implication, $N\colon [0,1] \to [0,1]$ be a fuzzy negation. Then for $E_{T,I,N}$ given by* (7) *we have:*

(i) (E0), (E2a), (E2b) *hold and*

$$E_{T,I,N}(x,0) = (N_I \circ N)(x),\ x \in [0,1], \tag{8}$$

$$E_{T,I,N}(0,x) = g_I(x),\ x \in [0,1], \tag{9}$$

$$E_{T,I,N}(x,1) = N_I(x),\ x \in [0,1], \tag{10}$$

$$E_{T,I,N}(1,x) = (g_I \circ N)(x),\ x \in [0,1], \tag{11}$$

where $g_I(x) := I(1,x)$ for $x \in [0,1]$;

(ii) *if I satisfies* (R-CP(N)) *and* (L-CP(N))*, then* (E1) *holds;*

(iii) *I satisfies* (NP) *iff* (E4) *holds;*

(iv) *N_I is strict (strong) iff* (E9) *(*(E10)*) holds;*

(v) *if I satisfies* (IP) *and N has an equilibrium point, then* (E3) *does not hold;*

(vi) (E7) *does not hold.*

Proof. (i) We observe that for $x \in [0,1]$ we have

$$\begin{aligned}
E_{T,I,N}(x,0) &= T\Big(I\big(x,N(0)\big), I\big(N(x),0\big)\Big) = T\Big(I\big(x,1\big),(N_I\circ N)(x)\Big)\\
&= T\big(1,(N_I\circ N)(x)\big) = (N_I\circ N)(x),\\
E_{T,I,N}(0,x) &= T\Big(I\big(0,N(x)\big), I\big(N(0),x\big)\Big) = T\Big(1, I(1,x)\big)\Big)\\
&= I(1,x) = g_I(x),\\
E_{T,I,N}(x,1) &= T\Big(I\big(x,N(1)\big), I\big(N(x),1\big)\Big)\\
&= T\big(I(x,0),1\big) = I(x,0) = N_I(x),\\
E_{T,I,N}(1,x) &= T\Big(I\big(1,N(x)\big), I\big(N(1),x\big)\Big) = T\Big((g_I\circ N)(x), I(0,x)\Big)\\
&= T\big((g_I\circ N)(x),1\big) = (g_I\circ N)(x),
\end{aligned}$$

so we get (E0), (E2a), (E2b).

(ii) Assume that I satisfies (R-CP(N)) and (L-CP(N)). Hence

$$\begin{aligned}
&E_{T,I,N}(y,x) = T\Big(I\big(y,N(x)\big), I\big(N(y),x\big)\Big)\\
&\overset{\text{(R-CP(N))}}{=} T\Big(I\big(x,N(y)\big), I\big(N(y),x\big)\Big)\\
&\overset{\text{(L-CP(N))}}{=} T\Big(I\big(x,N(y)\big), I\big(N(x),y\big)\Big) = E_{T,I,N}(x,y), \quad x,y\in[0,1].
\end{aligned}$$

(iii) It follows from (9).

(iv) It follows from (10).

(v) Assume that I satisfies (IP) and N has an equilibrium point $e \in (0,1)$. Then

$$\begin{aligned}
E_{T,I,N}(e,e) &= T\big(I\big(e,N(e)\big), I\big(N(e),e\big)\big) = T\big(I(e,e), I(e,e)\big)\\
&\overset{\text{(IP)}}{=} T(1,1) = 1, \quad x\in[0,1].
\end{aligned}$$

(vi) Suppose that (E7) holds. From Theorem 2.8 $N_{E_{T,I,N}}$ is strong, so from iv N_I is strong. From (8) we get

$$x \overset{\text{(E7)}}{=} E_{T,I,N}(E_{T,I,N}(x,0),0) \overset{(8)}{=} (N_I\circ N)^2(x), \quad x\in[0,1].$$

Since $N_I \circ N$ is non-decreasing $(N_I\circ N)(x) = x$ for $x\in[0,1]$, so $N = N_I$. Let $e\in(0,1)$ be an equilibrium point of N_I. Using i we have

$$\begin{aligned}
1 &\overset{\text{(E7)}}{=} E_{T,I,N}(E_{T,I,N}(1,e),e) \overset{(9)}{=} E_{T,I,N}\big((g_I\circ N)(e),e\big) = E_{T,I,N}\big(g_I(e),e\big),\\
0 &\overset{\text{(E7)}}{=} E_{T,I,N}(E_{T,I,N}(0,e),e) \overset{(11)}{=} E_{T,I,N}\big(g_I(e),e\big),
\end{aligned}$$

which gives us a contradiction. □

4. Examples

In [3, Table 2] we can find classification of fuzzy Xor connectives. Adding new classes of fuzzy Xor we can show that all these classes are different.

	$E_{I,N}$	$E_{T,I,N}$	$E_{S,T,N}$	$E_{T,S,N}$
$E_C(x,y)$	(I_{RS},N)	-	-	-
$E_1(x,y)$	-	(T,I_{LK},N_C)	(S_{LK},T_M,N_C)	(T,S_{LK},N_C)
$E_\perp(x,y)$	(I_1,N)	(T,I_0,N)	-	-
$E_2(x,y)$	-	-	(S_D,T,N_{D2})	(T,S_D,N_{D2})

where N is an arbitrary fuzzy negation, T is an arbitrary t-norm,

$$E_C(x,y)=\begin{cases}0, & x=y,\\ 1, & x\neq y,\end{cases}\qquad E_1(x,y)=\begin{cases}x+y, & x+y\leq 1,\\ 2-x-y, & x+y>1,\end{cases}$$

$$E_\perp(x,y)=\begin{cases}0, & |x-y|=1,\\ 1, & |x-y|\neq 1,\end{cases}\qquad E_2(x,y)=\begin{cases}0, & x=y=1,\\ \max(x,y), & x=0\vee y=0,\\ 1, & \text{otherwise},\end{cases}$$

$$I_{RS}(x,y)=\begin{cases}0, & x>y,\\ 1, & x\leq y,\end{cases}\qquad I_{LK}(x,y)=\begin{cases}1-x+y, & x>y,\\ 1, & x\leq y,\end{cases}$$

$$I_1(x,y)=\begin{cases}0, & x=1\wedge y=0,\\ 1, & x<1\vee y>0,\end{cases}\qquad I_0(x,y)=\begin{cases}0, & x>0\wedge y<1,\\ 1, & x=0\vee y=1,\end{cases}$$

$$S_{LK}(x,y)=\min(1,x+y),\quad x,y\in[0,1],$$

$$S_D(x,y)=\begin{cases}\max(x,y), & x=0\vee y=0,\\ 1, & x>0\wedge y>0,\end{cases}$$

$$T_M(x,y)=\min(x,y),\quad x,y\in[0,1],$$

$$N_C(x)=1-x,\quad x\in[0,1],\qquad N_{D2}(x)=\begin{cases}0, & x=1,\\ 1, & x<1.\end{cases}$$

It is easy to show the form of E if we know the generators, so we show only that the above fuzzy Xors do not belong to some classes. From [3, Table 2] we know that E_C and $E_\perp$ do not belong to classes $E_{S,T,N}$, $E_{T,S,N}$.

- Suppose that $E_C=E_{T,I,N}$. Using (8)–(11) we have

$$(N_I\circ N)(x)=g_I(x)=\begin{cases}0, & x=0,\\ 1, & x>0,\end{cases}\qquad N_I(x)=(g_I\circ N)(x)=\begin{cases}0, & x=1,\\ 1, & x<1.\end{cases}$$

Hence $I = I_1$. Let $x = y \in (0,1)$. Then $I(x, N(y)) = 1 = I(N(x), y)$ and

$$0 = E_C(x,y) = E_{T,I,N}(x,y) = T(1,1) = 1,$$

which give us a contradiction.

- Suppose that $E_1 = E_{I,N}$. Using (3)–(6) we have

$$N_I^2(x) = (g_I \circ N \circ N_I)(x) = x, \quad x \in [0,1],$$
$$(N_I \circ g_I)(x) = (g_I \circ N \circ g_I)(x) = 1 - x, \quad x \in [0,1].$$

Hence $N_I = g_I \circ N$ is strong negation and

$$x = N_I^2(x) = (g_I \circ N \circ g_I \circ N)(x) = 1 - N(x), \quad x \in [0,1],$$

so $N = N_C$.

Suppose that $I(x_0, y_0) = 1$ for some $x_0 > 0$, $y_0 < 1$. Then

$$g_I\Big(N_c\big(I(y,x)\big)\Big) = E_1(x,y) = \begin{cases} x+y, & x+y \le 1, \\ 2-x-y, & x+y > 1, \end{cases} \quad x < x_0,\ y > y_0.$$

The left hand side of the above identity is non-increasing in x and non-decreasing in y which is impossible, so $I(x_0, y_0) = 1$ only if $x_0 = 0$ or $y_0 = 1$.

Now, we observe that

$$1 = E_1(0.5, 0.5) = I\big(I(0.5, 0.5), 1 - I(0.5, 0.5)\big), \quad x \in [0,1],$$

so $I(0.5, 0.5) = 0$, whence

$$0 < N_I(0.5) = I(0.5, 0) \le I(0.5, 0.5) = 0,$$

which give us a contradiction.

- Suppose that $E_2 = E_{I,N}$. Using (3)–(6) we have

$$N_I^2(x) = (g_I \circ N \circ N_I)(x) = x, \quad x \in [0,1],$$
$$(N_I \circ g_I)(x) = (g_I \circ N \circ g_I)(x) = N_{D2}(x), \quad x \in [0,1].$$

Hence N_I is strong, g_I is surjective, so $N_{D2} = N_I \circ g_I$ is surjective, which give us a contradiction.

Suppose that $E_2 = E_{T,I,N}$. Using (8)–(11) we have

$$(N_I \circ N)(x) = g_I(x) = x, \quad x \in [0,1],$$
$$N_I(x) = (g_I \circ N)(x) = N_{D2}(x), \quad x \in [0,1].$$

Hence N_I is surjective, so $N_{D2} = N_I$ is surjective, which give us a contradiction.

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