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CLASSIFICATION OF LIPSCHITZ DERIVATIVES IN TERMS OF SEMICONTINUITY AND THE BAIRE LIMIT FUNCTIONS

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Abstract. We introduce the generalized notion of semicontinuity of a function defined on a topological space and derive the useful classification of the so-called Lipschitz derivatives of functions defined on a metric space. Secondly, we investigate some connections of the Lipschitz derivatives defined on normed spaces to the Fréchet derivative and relations between little, big and local Lipschitz derivatives (denoted by $\operatorname{lip} f$, $\operatorname{Lip} f$ and $\mathbb{L}\mathrm{ip}\, f$ respectively) in terms of Baire limit functions. In particular, we prove that $\operatorname{lip} f$ is $\mathcal{F}_\sigma$-lower, $\operatorname{Lip} f$ is $\mathcal{F}_\sigma$-upper, $\mathbb{L}\mathrm{ip}\, f$ is upper semicontinuous. Moreover, for a function f defined on an open or convex subset of a normed space, the upper Baire limit function of functions $\operatorname{lip} f$ and $\operatorname{Lip} f$ are equal to $\mathbb{L}\mathrm{ip}\, f$.

1. Introduction

The Lipschitz derivatives are useful tools for investigation of different notions of differentiability. For example, the big Lipschitz derivative $\operatorname{Lip} f$ of a given function f often occurs in theorems of Rademacher–Stepanov type (see for example [11, 14, 9, 8, 5]). Statements of this type usually involves the set $L(f) = \{x \in \mathbb{R} \colon \operatorname{Lip} f(x) < \infty\}$. The local Lipschitz derivative $\mathbb{L}\mathrm{ip}\, f$ together with $\operatorname{Lip} f$ characterize the local and pointwise Lipschitzness of functions defined on a metric space [7]. The little Lipschitz derivative was introduced by Cheeger in [4] and together with $\operatorname{Lip} f$ play important role in the research of the first order differential calculus in metric spaces. The crucial

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fact is the purely metric character of the definitions of Lipschitz derivatives. The above considerations lead to the natural question of characterizing the sets $\ell(f)$, $L(f)$ and $\mathbb{L}(f)$ for functions defined on metric spaces. In recent years, this problem has been investigated in many articles, such as [12, 3, 6]. In particular, in [3] it was shown that $\ell(f)$ is a $G_{\delta\sigma}$-set and $L(f)$ is an F_σ-set for a function $f\colon \mathbb{R} \to \mathbb{R}$. In our approach, we introduce generalized notions of semicontinuity of a function and classify the Lipschitz derivatives with help of these properties. This allows us to easily derive the Borel type of sets $\ell(f)$, $L(f)$, $\mathbb{L}(f)$ of a function f acting between arbitrary metric spaces.

Another interesting question to consider is whether for a given triplet of functions (u, v, w), there exists a continuous function f such that $\operatorname{lip} f = u$, $\operatorname{Lip} f = v$ and $\mathbb{L}\mathrm{ip}\, f = w$. Some advances in this direction were made in [1] and [2]. Our characterization of Lipschitz derivatives in terms of generalized semicontinuity constrains the possible choice of functions (u, v, w). Moreover, we obtain another necessary criterion for a triple (u, v, w) defined on locally convex subset of a normed space: the upper Baire limits functions of u and v are equal to w.

2. Lipschitz derivatives

Let X be a metric space, $a \in X$ and $\varepsilon > 0$. We always denote the metric on X by $|\cdot - \cdot|_X$ and

$$B(a,\varepsilon) = B_X(a,\varepsilon) = \big\{x \in X\colon |x-a|_X < \varepsilon\big\},$$
$$B[a,\varepsilon] = B_X[a,\varepsilon] = \big\{x \in X\colon |x-a|_X \le \varepsilon\big\}.$$

Definition 1. Let X and Y be metric spaces, $f\colon X \to Y$ be a function, $x \in X$. Denote

- $\|f\|_{\mathrm{lip}} = \sup\limits_{u\neq v \in X} \frac{1}{|u-v|_X}\big|f(u)-f(v)\big|_Y$,
- $\mathbb{L}\mathrm{ip}\, f(x) = \limsup\limits_{\substack{(u,v)\to(x,x)\\ u\neq v}} \frac{1}{|u-v|_X}\big|f(u)-f(v)\big|_Y$,
- $\operatorname{Lip} f(x) = \limsup\limits_{u\to x} \frac{1}{|u-x|_X}\big|f(u)-f(x)\big|_Y$,
- $\operatorname{lip} f(x) = \liminf\limits_{r\to 0^+} \sup\limits_{u\in B(x,r)} \frac{1}{r}\big|f(u)-f(x)\big|_Y$;

The number $\|f\|_{\mathrm{lip}}$ is the *Lipschitz constant of* f. The functions $\mathbb{L}\mathrm{ip}\, f$, $\operatorname{Lip} f$ and $\operatorname{lip} f$ are called the *local, big and little Lipschitz derivative* respectively.

We denote by X^d the set of all non-isolated points of X. Throughout the paper, we assume that $\sup\varnothing = 0$. As a consequence of this assumption we have $\mathbb{L}\mathrm{ip}\, f(x) = \operatorname{Lip} f(x) = \operatorname{lip} f(x) = 0$ for any $x \in X \setminus X^d$.

Obviously, if Y is a normed space then $\|\cdot\|_{\mathrm{lip}}$ is an extended seminorm on Y^X in the sense [13]. Moreover, $\|\cdot\|_{\mathrm{lip}}$ is a norm on the space $\mathrm{Lip}_a(X,Y)$ of all Lipschitz functions $f\colon X\to Y$ vanishing at some fixed point $a\in X$.

We introduce some auxiliary notations:

- $\mathbb{L}\mathrm{ip}^r f(x) = \big\|f|_{B(x,r)}\big\|_{\mathrm{lip}} = \sup\limits_{u\neq v\in B(x,r)} \frac{1}{|u-v|_X}\big|f(u)-f(v)\big|_Y$;
- $\mathrm{Lip}^r f(x) = \sup\limits_{u\in B(x,r)} \frac{1}{r}\big|f(u)-f(x)\big|_Y$, $\mathrm{Lip}^r_+ f(x) = \sup\limits_{u\in B[x,r]} \frac{1}{r}\big|f(u)-f(x)\big|_Y$;
- $\mathrm{Lip}_r f(x) = \sup\limits_{0<\varrho<r} \mathrm{Lip}^\varrho f(x)$, $\mathrm{lip}_r f(x) = \inf\limits_{0<\varrho<r} \mathrm{Lip}^\varrho f(x)$.

Therefore, the definitions of the Lipschitz derivatives might be rewritten as follows

$$\mathbb{L}\mathrm{ip}\, f(x) = \inf_{r>0} \mathbb{L}\mathrm{ip}^r f(x), \quad \mathrm{lip}\, f(x) = \liminf_{r\to 0^+} \mathrm{Lip}^r f(x).$$

Some authors (see, for example, [6, 2, 3]) define $\mathrm{Lip}\, f$ and $\mathrm{lip}\, f$ using the function $\mathrm{Lip}^r_+ f$ instead of $\mathrm{Lip}^r f$. In the case where X is a normed space, we have $B[x,r] = \overline{B(x,r)}$. Therefore, $\mathrm{Lip}^r f(x) = \mathrm{Lip}^r_+ f(x)$ for any continuous function f. But the previous equality does not hold for the discrete metric on X, nonconstant f and $r=1$. However, we have the following

Proposition 2.1. *Let X and Y be metric spaces and $f\colon X\to Y$ be a function. Then, for any non-isolated point $x\in X$, the following equalities hold*

$$\mathrm{Lip}\, f(x) = \limsup_{r\to 0^+} \mathrm{Lip}^r f(x) = \limsup_{r\to 0^+} \mathrm{Lip}^r_+ f(x),$$
$$\mathrm{lip}\, f(x) = \liminf_{r\to 0^+} \mathrm{Lip}^r_+ f(x).$$

We start with the following

Lemma 2.2. *Let $\varphi,\psi\colon [0,+\infty)\to[0,+\infty]$ be functions such that*

$$\varphi(\varrho) \le \psi(\varrho) \le \frac{r}{\varrho}\varphi(r)$$

for any $0<\varrho<r$. Then

$$\limsup_{r\to 0^+}\varphi(r) = \limsup_{r\to 0^+}\psi(r), \quad \liminf_{r\to 0^+}\varphi(r) = \liminf_{r\to 0^+}\psi(r).$$

Proof. Denote

$$A = \limsup_{r\to 0^+}\varphi(r), \quad B = \limsup_{r\to 0^+}\psi(r),$$
$$a = \liminf_{r\to 0^+}\varphi(r), \quad b = \liminf_{r\to 0^+}\psi(r).$$

Obviously, $A \le B$ and $a \le b$. Let us show, that $A \ge B$. Choose $\varrho_n \to 0^+$ such that $\psi(\varrho_n) \to B$. Put $r_n = \varrho_n + \varrho_n^2$. Since $r_n > \varrho_n$, $\psi(\varrho_n) \le \frac{r_n}{\varrho_n}\varphi(r_n)$. Therefore,

$$A \ge \limsup_{n\to\infty} \varphi(r_n) \ge \lim_{n\to\infty} \frac{\varrho_n}{r_n}\psi(\varrho_n) = B.$$

Now, we will show that $a \ge b$. Choose $r_n \to 0^+$ such that $0 < r_n < 1$ and $\varphi(r_n) \to a$. Let $\varrho_n = r_n - r_n^2$. Since $0 < \varrho_n < r_n$, $\psi(\varrho_n) \le \frac{r_n}{\varrho_n}\varphi(r_n)$, we have

$$b \le \liminf_{n\to\infty} \psi(\varrho_n) \le \lim_{n\to\infty} \frac{r_n}{\varrho_n}\varphi(r_n) = a. \qquad \square$$

Proof of Proposition 2.1. Fix a non-isolated point $x \in X$. Denote $\varphi(r) = \mathrm{Lip}^r f(x)$, $\psi(r) = \mathrm{Lip}^r_+ f(x)$. Therefore, $r\varphi(r) = \sup\limits_{u\in B(x,r)} |f(u) - f(x)|_Y$ and $r\psi(r) = \sup\limits_{u\in B[x,r]} |f(u) - f(x)|_Y$. Let $0 < \varrho < r$. Then $B(x,\varrho) \subseteq B[x,\varrho] \subseteq B(x,r)$, so

$$\varrho\varphi(\varrho) \le \varrho\psi(\varrho) \le r\varphi(r).$$

Hence, φ and ψ satisfy the condition from Lemma 2.2. Thus, by Lemma 2.2 we conclude that

$$\limsup_{r\to 0^+} \varphi(r) = \limsup_{r\to 0^+} \psi(r) \quad \text{and} \quad \liminf_{r\to 0^+} \varphi(r) = \liminf_{r\to 0^+} \psi(r).$$

By the definition, we have $\mathrm{lip}\, f(x) = \liminf\limits_{r\to 0^+} \varphi(r)$.

It remains to show that $\mathrm{Lip}\, f(x) = \limsup\limits_{r\to 0^+} \psi(r)$. Denote for any $r > 0$

$$\alpha(r) = \sup_{0<\varrho<r} \psi(\varrho) \quad \text{and} \quad \beta(r) = \sup\left\{ \frac{|f(u) - f(x)|_Y}{|u - x|_X} : 0 < |u - x|_X < r \right\}.$$

Therefore,

$$\begin{aligned}
\alpha(r) &= \sup_{0<\rho<r} \sup_{0<|u-x|_X\le\rho} \tfrac{1}{\rho}|f(u) - f(x)|_Y \\
&\le \sup_{0<\rho<r} \sup_{0<|u-x|_X\le\rho} \tfrac{1}{|u-x|_X}|f(u) - f(x)|_Y = \beta(r).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\beta(r) &= \sup_{0<\rho<r} \sup_{|u-x|_X=\rho} \tfrac{1}{\rho}|f(u) - f(x)|_Y \\
&\le \sup_{0<\rho<r} \sup_{0<|u-x|_X\le\rho} \tfrac{1}{\rho}|f(u) - f(x)|_Y \\
&= \sup_{0<\rho<r} \mathrm{Lip}^\rho_+ f(x) = \alpha(r).
\end{aligned}$$

We have shown, that $\alpha(r) = \beta(r)$ for $r > 0$. Hence,

$$\operatorname{Lip} f(x) = \inf_{r>0} \beta(r) = \inf_{r>0} \alpha(r) = \limsup_{r\to 0^+} \psi(r). \qquad \square$$

Note, that

$$\operatorname{Lip}_r f(x) \le \operatorname{Lip}_{r'} f(x) \quad \text{and} \quad \operatorname{lip}_r f(x) \ge \operatorname{lip}_{r'} f(x) \quad \text{if } 0 < r < r'. \tag{2.1}$$

So, the definitions and the previous proposition yield

$$\operatorname{Lip} f(x) = \inf_{r>0} \operatorname{Lip}_r f(x) = \lim_{r\to 0^+} \operatorname{Lip}_r f(x), \tag{2.2}$$

$$\operatorname{lip} f(x) = \sup_{r>0} \operatorname{lip}_r f(x) = \lim_{r\to 0^+} \operatorname{lip}_r f(x). \tag{2.3}$$

Therefore, it is easy to see that the following inequalities hold.

$$\operatorname{lip}_r f(x) \le \operatorname{Lip}_r f(x) \le \mathbb{L}\mathrm{ip}^r f(x) \quad \text{for any } r > 0,$$

$$\operatorname{lip} f(x) \le \operatorname{Lip} f(x) \le \mathbb{L}\mathrm{ip} f(x). \tag{2.4}$$

Definition 2. Let X and Y be metric spaces and $\gamma \ge 0$. A function $f\colon X \to Y$ is called

- *γ-Lipschitz* if $\|f\|_{\mathrm{lip}} \le \gamma$;
- *Lipschitz* if $\|f\|_{\mathrm{lip}} < \infty$;
- *locally Lipschitz* if $\mathbb{L}\mathrm{ip} f < \infty$;
- *pointwise Lipschitz* if $\operatorname{Lip} f < \infty$;
- *weakly pointwise Lipschitz* if $\operatorname{lip} f < \infty$.

Denote

- $\mathbb{L}(f) = \{x \in X : \mathbb{L}\mathrm{ip} f(x) < \infty\}$;
- $\mathbb{L}^\infty(f) = \{x \in X : \mathbb{L}\mathrm{ip} f(x) = \infty\} = X \setminus \mathbb{L}(f)$;
- $L(f) = \{x \in X : \operatorname{Lip} f(x) < \infty\}$;
- $L^\infty(f) = \{x \in X : \operatorname{Lip} f(x) = \infty\} = X \setminus L(f)$;
- $\ell(f) = \{x \in X : \operatorname{lip} f(x) < \infty\}$;
- $\ell^\infty(f) = \{x \in X : \operatorname{lip} f(x) = \infty\} = X \setminus \ell(f)$.

Inequalities (2.4) yield the next assertion.

Proposition 2.3. *Let X and Y be metric spaces, and $f\colon X \to Y$ be a function. Then $\mathbb{L}(f) \subseteq L(f) \subseteq \ell(f)$ and $\ell^\infty(f) \subseteq L^\infty(f) \subseteq \mathbb{L}^\infty(f)$.*

3. Connections of Lipschitz derivatives to classical notion of a derivative

One might ask under what conditions the Lipschitz derivative (of any given type) coincides with one of the “traditional” notions of the derivative of a given function, provided that an appropriate derivative exists. It is obvious that for a real differentiable function $f\colon \mathbb{R} \to \mathbb{R}$, we have $\operatorname{lip} f(x) = \operatorname{Lip} f(x) = |f'(x)|$ at any $x \in \mathbb{R}$.

In [7], the following theorem was proved.

THEOREM 3.1. *Let X and Y be normed spaces, G be an open subset of X, and $f\colon G \to Y$ have a locally bounded Gateaux derivative f'. Then,* $\mathbb{L}\mathrm{ip}\, f(x) = \limsup\limits_{u\to x} \|f'(x)\|$ *for $x \in X$ and so, f is locally Lipschitz. Moreover, if f is C^1 function, then* $\mathbb{L}\mathrm{ip}\, f(x) = \|f'(x)\|$, $x \in X$.

The following result was also stated in [7], but the proof contains a small blunder. Here, we provide the correct proof.

THEOREM 3.2. *Let $f\colon X \to Y$, where X and Y are normed spaces and assume that there exists the Fréchet derivative* $\mathrm{d}f\,(x_0)$ *of f at a point $x_0 \in X$. Then* $\operatorname{lip} f(x_0) = \operatorname{Lip} f(x_0) = \|\mathrm{d}f\,(x_0)\|$.

PROOF. It is enough to consider the case $X \neq \{0\}$. Denote by $A = \mathrm{d}f\,(x_0)$ the Fréchet derivative of f at the point x_0. We have

$$f(x) - f(x_0) = A(x - x_0) + \alpha(x) \quad \text{for } x \in X, \tag{3.1}$$

where α is a function, such that $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{\|x - x_0\|} = 0$. By (3.1) we have

$$\|f(x) - f(x_0)\| \le \|A\|\,\|x - x_0\| + \|\alpha(x)\|\,,$$

hence

$$\frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} \le \|A\| + \left\|\frac{\alpha(x)}{\|x - x_0\|}\right\|.$$

Thus,

$$\begin{aligned}\operatorname{lip} f(x_0) \le \operatorname{Lip} f(x_0) &= \limsup_{x\to x_0} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|}\\ &\le \lim_{x\to x_0}\left(\|A\| + \left\|\frac{\alpha(x)}{\|x - x_0\|}\right\|\right) = \|A\|\,.\end{aligned}$$

We want to prove that $\operatorname{lip} f(x_0) \ge \|A\|$. Fix $\varepsilon > 0$. Then, there exists $e \in X$ such that $\|e\| = 1$ and

$$\|Ae\| \ge \|A\| - \varepsilon.$$

For any $r > 0$, denote $x_r = x_0 + re$. Note that $r = \|x_t - x_0\|$, and $x_r \to x_0$ as $r \to 0$. So, $\frac{\alpha(x_r)}{r} \to 0$ as $r \to 0$. Therefore, by (3.1)

$$f(x_r) - f(x_0) = A(x_r - x_0) + \alpha(x_r) = rAe + \alpha(x_r),$$

so for any $r > 0$

$$\begin{aligned}\operatorname{Lip}^r_+ f(x_0) &= \tfrac{1}{r} \sup_{\|u-x\|\le r} \|f(x) - f(x_0)\| \ge \tfrac{1}{r} \|f(x_r) - f(x_0)\| \\ &= \left\| Ae + \frac{\alpha(x_r)}{r} \right\| \ge \|Ae\| - \left\| \frac{\alpha(x_r)}{r} \right\| \ge \|A\| - \varepsilon - \left\| \frac{\alpha(x_r)}{r} \right\|.\end{aligned}$$

Thus, by Proposition 2.1 we conclude that

$$\operatorname{lip} f(x_0) = \liminf_{r\to 0^+} \operatorname{Lip}^r_+ f(x_0) \ge \lim_{r\to 0^+} \left(\|A\| - \varepsilon - \left\| \frac{\alpha(x_r)}{r} \right\| \right) = \|A\| - \varepsilon.$$

Since the ε was chosen arbitrarily, the proof is finished. □

4. Semicontinuity with respect to a family of sets

In this section we introduce some modification of semicontinuity, which will help us to classify the Lipschitz derivatives.

Let X be a topological space and $\mathcal{A}$ be a family of subsets of X. We denote

$$\begin{aligned}\mathcal{A}_c &= \Big\{ X \setminus A \colon A \in \mathcal{A} \Big\}, \\ \mathcal{A}_\sigma &= \Big\{ \bigcup_{n=1}^{\infty} A_n \colon A_n \in \mathcal{A} \text{ for all } n \in \mathbb{N} \Big\}, \\ \mathcal{A}_\delta &= \Big\{ \bigcap_{n=1}^{\infty} A_n \colon A_n \in \mathcal{A} \text{ for all } n \in \mathbb{N} \Big\}.\end{aligned}$$

We will also combine these symbols. It is easy to check, for example, that $\mathcal{A}_{c\delta c} = \mathcal{A}_\sigma$, $\mathcal{A}_{\sigma c} = \mathcal{A}_{c\delta}$, $\mathcal{A}_{\delta c} = \mathcal{A}_{c\sigma}$ and so on. If $\mathcal{T}$ denotes the topology of X, then applying above notation to the family $\mathcal{A} = \mathcal{T}$, the $\mathcal{T}_\delta$ is the familiar Borel class $\mathcal{G}_\delta$ of G_δ-subsets of X. Complementary, the family $\mathcal{T}_{c\sigma}$ is the Borel class $\mathcal{F}_\sigma$ of F_σ-subsets of X.

We say that $f\colon X \to \overline{\mathbb{R}}$ is an *$\mathcal{A}$-upper ($\mathcal{A}$-lower) semicontinuous function* if $f^{-1}\big([-\infty,\gamma)\big) \in \mathcal{A}$ (resp. $f^{-1}\big((\gamma,+\infty]\big) \in \mathcal{A}$) for any $\gamma \in \mathbb{R}$. If $\mathcal{A} = \mathcal{T}$ is the topology of X, then we omit the symbol $\mathcal{A}$ in the previous definitions. For our purposes, the $\mathcal{F}_\sigma$-upper and lower semicontinuous functions are particularly important.

Proposition 4.1. *Let X be a topological space, $\mathcal{A} \subseteq 2^X$ and $f\colon X \to \overline{\mathbb{R}}$ be an $\mathcal{A}$-upper semicontinuous function. Then*

(i) $f^{-1}\big[[\gamma,+\infty]\big] \in \mathcal{A}_c$ *for any* $\gamma \in \mathbb{R}$*;*

(ii) $f^{-1}\big[[-\infty,+\infty)\big] \in \mathcal{A}_\sigma$ *and, so,* $f^{-1}\big[\{+\infty\}\big] \in \mathcal{A}_{\sigma c}$*;*

(iii) $f^{-1}\big[\{-\infty\}\big] \in \mathcal{A}_\delta$ *and, so,* $f^{-1}\big((-\infty,+\infty]\big) \in \mathcal{A}_{\delta c}$*;*
(iv) f *is* $\mathcal{A}_{c\sigma}$*-lower semicontinuous.*

Proof. (i) For any $\gamma \in \mathbb{R}$ we have that $f^{-1}\big([-\infty,\gamma)\big) \in \mathcal{A}$ and then

$$f^{-1}\big([\gamma,+\infty]\big) = X \setminus f^{-1}\big([-\infty,\gamma)\big) \in \mathcal{A}_c.$$

(ii) Since $f^{-1}\big[[-\infty,n)\big] \in \mathcal{A}$ for any $n \in \mathbb{N}$, we conclude that

$$f^{-1}\big[\,[-\infty,+\infty)\,\big] = \bigcup_{n=1}^{\infty} f^{-1}\big[\,[-\infty,n)\,\big] \in \mathcal{A}_\sigma,$$

and so, $f^{-1}\big[\{+\infty\}\big] = X \setminus f^{-1}\big[\,[-\infty,+\infty)\,\big] \in \mathcal{A}_{\sigma c}$.

(iii) Since $f^{-1}\big[\,[-\infty,-n)\,\big] \in \mathcal{A}$ for any $n \in \mathbb{N}$, we have that

$$f^{-1}\big[\{-\infty\}\big] = \bigcap_{n=1}^{\infty} f^{-1}\big[\,[-\infty,-n)\,\big] \in \mathcal{A}_\delta,$$

and so, $f^{-1}\big[\,(-\infty,+\infty]\,\big] = X \setminus f^{-1}\big[\{-\infty\}\big] \in \mathcal{A}_{\delta c}$.

(iv) Let $\gamma \in \mathbb{R}$ and $\gamma_n \downarrow \gamma$. Since $f^{-1}\big[[\gamma_n,+\infty]\big] \in \mathcal{A}_c$ by (i), we conclude that

$$f^{-1}\big[(\gamma,+\infty]\big] = \bigcup_{n=1}^{\infty} f^{-1}\big[[\gamma_n,+\infty]\big] \in \mathcal{A}_{c\sigma},$$

i.e., f is $\mathcal{A}_{c\sigma}$-lower semicontinuous. □

Proposition 4.2. *Let X be a topological space, $\mathcal{A} \subseteq 2^X$, $f_n\colon X \to \overline{\mathbb{R}}$ be an $\mathcal{A}$-upper semicontinuous function for any $n \in \mathbb{N}$ and $f\colon X \to \overline{\mathbb{R}}$ be a function such that $f(x) = \sup_{n\in\mathbb{N}} f_n(x)$ for any $x \in X$. Then f is an $\mathcal{A}_{c\sigma}$-lower semicontinuous function.*

Proof. Consider $\gamma \in \mathbb{R}$. By Proposition 4.1(iv) the functions f_n are $\mathcal{A}_{c\sigma}$-lower semicontinuous. So, $f_n^{-1}\big[\,(\gamma,+\infty]\,\big] \in \mathcal{A}_{c\sigma}$ for any $n \in \mathbb{N}$. Consequently,

$$f^{-1}\big[\,(\gamma,+\infty]\,\big] = \bigcup_{n=1}^{\infty} f_n^{-1}\big[\,(\gamma,+\infty]\,\big] \in \mathcal{A}_{c\sigma}.$$

Thus, f is an $\mathcal{A}_{c\sigma}$-lower semicontinuous function. □

Observe that f is an $\mathcal{A}$-upper semicontinuous function if and only if $-f$ is $\mathcal{A}$-lower semicontinuous. Therefore, using Proposition 4.1 and 4.2 with $g = -f$ we obtain the following two propositions.

Proposition 4.3. *Let X be a topological space, $\mathcal{A} \subseteq 2^X$ and $f\colon X \to \overline{\mathbb{R}}$ be an $\mathcal{A}$-lower semicontinuous function. Then*
(i) $f^{-1}\big[\,[-\infty,\gamma]\,\big] \in \mathcal{A}_c$ *for any* $\gamma \in \mathbb{R}$*;*
(ii) $f^{-1}\big[\,(-\infty,+\infty]\,\big] \in \mathcal{A}_\sigma$ *and, so,* $f^{-1}\big[\{-\infty\}\big] \in \mathcal{A}_{\sigma c}$*;*
(iii) $f^{-1}\big[\{+\infty\}\big] \in \mathcal{A}_\delta$ *and, so,* $f^{-1}\big[\,[-\infty,+\infty)\,\big] \in \mathcal{A}_{\delta c}$*;*
(iv) f *is* $\mathcal{A}_{c\sigma}$*-upper semicontinuous.*

Proposition 4.4. *Let X be a topological space, $\mathcal{A} \subseteq 2^X$, $f_n\colon X \to \overline{\mathbb{R}}$ be an $\mathcal{A}$-lower semicontinuous function for any $n \in \mathbb{N}$ and $f\colon X \to \overline{\mathbb{R}}$ be a function such that $f(x) = \inf_{n\in\mathbb{N}} f_n(x)$ for any $x \in X$. Then f is an $\mathcal{A}_{c\sigma}$-upper semicontinuous function.*

5. Classification of the Lipschitz derivatives

Now we pass to the investigation of the type of semicontinuity of Lipschitz derivatives of continuous functions. In [3] semicontinuity of Lipschitz derivatives of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ was obtained from the continuity of $\mathrm{Lip}^r f$. But in the general situation, this function need not be continuous. Therefore, we prove semicontinuity of Lipschitz derivatives directly from the definitions.

Lemma 5.1. *Let X and Y be metric spaces, $f\colon X \to Y$ be a continuous function and $r > 0$. Then $\mathrm{lip}_r f\colon X \to [0,+\infty]$ is an upper semicontinuous function.*

Proof. Let $x_0 \in X$ and $\gamma > \mathrm{lip}_r f(x_0)$. Then

$$\inf_{\varrho<r} \mathrm{Lip}^\varrho f(x_0) = \mathrm{lip}_r f(x_0) < \gamma.$$

So, there is positive $\varrho < r$ such that $\mathrm{Lip}^\varrho f(x_0) < \gamma$. Pick γ_1 such that $\mathrm{Lip}^\varrho f(x_0) < \gamma_1 < \gamma$. Then we choose ϱ_1 such that $\frac{\gamma_1}{\gamma}\varrho < \varrho_1 < \varrho$. So, $\gamma\varrho_1 > \gamma_1\varrho$. Therefore,

$$\sup_{u\in B(x_0,\varrho)} \big|f(u) - f(x_0)\big|_Y = \varrho\,\mathrm{Lip}^\varrho f(x_0) < \gamma_1\varrho.$$

Then

$$\big|f(u) - f(x_0)\big|_Y < \gamma_1\varrho \quad \text{for any } u \in B(x_0,\varrho).$$

By the continuity of f at x_0 there exists $\delta > 0$ such that $\varrho_1 + \delta < \varrho$ and

$$\big|f(x) - f(x_0)\big|_Y < \gamma\varrho_1 - \gamma_1\varrho \quad \text{for any } x \in U = B(x_0,\delta).$$

Consider $x \in U$ and $u \in B(x, \varrho_1)$. Then

$$|u - x_0|_X \le |u - x|_X + |x - x_0|_X < \varrho_1 + \delta < \varrho,$$

and so, $u \in B(x_0, \varrho)$. Therefore,

$$\bigl|f(u) - f(x)\bigr|_Y \le \bigl|f(u) - f(x_0)\bigr|_Y + \bigl|f(x_0) - f(x)\bigr|_Y < \gamma_1\varrho + (\gamma\varrho_1 - \gamma_1\varrho) = \gamma\varrho_1.$$

Thus, $\frac{1}{\varrho_1}\bigl|f(u) - f(x)\bigr|_Y \le \gamma$ for any $u \in B(x, \varrho_1)$. Hence, $\mathrm{Lip}^{\varrho_1} f(x) \le \gamma$. But $0 < \varrho_1 < r$. Therefore, $\mathrm{lip}_r f(x) \le \gamma$ for any $x \in U$. Thus, $\mathrm{lip}_r f$ is upper semicontinuous at x_0. □

THEOREM 5.2. *Let X and Y be metric spaces and $f\colon X \to Y$ be a continuous function. Then $\mathrm{lip} f\colon X \to [0, +\infty]$ is a $\mathcal{F}_\sigma$-lower semicontinuous function.*

PROOF. By (2.1) and (2.3) we conclude that $\mathrm{lip} f(x) = \sup_{n\in\mathbb{N}} \mathrm{lip}_{\frac{1}{n}} f(x)$ for any $x \in X$. By Lemma 5.1, the functions $\mathrm{lip}_{\frac{1}{n}} f$ are $\mathcal{T}$-upper semicontinuous, where $\mathcal{T}$ is the topology of X. Therefore, by Proposition 4.2 $\mathrm{lip} f$ is $\mathcal{T}_{c\sigma}$-lower semicontinuous. This means that $\mathrm{lip} f$ is $\mathcal{F}_\sigma$-lower semicontinuous. □

LEMMA 5.3. *Let X and Y be metric spaces, $f\colon X \to Y$ be a continuous function and $r > 0$. Then $\mathrm{Lip}_r f\colon X \to [0, +\infty]$ is a lower semicontinuous function.*

PROOF. Fix $r > 0$. Let $x_0 \in X$ and $\gamma < \mathrm{Lip}_r f(x_0)$. Then

$$\sup_{\varrho<r} \mathrm{Lip}^\varrho f(x_0) = \mathrm{Lip}_r f(x_0) > \gamma.$$

So, there is $\varrho \in (0, r)$ such that $\mathrm{Lip}^\varrho f(x_0) > \gamma$. Pick γ_1 such that

$$\gamma < \gamma_1 < \mathrm{Lip}^\varrho f(x_0).$$

Therefore,

$$\sup_{u\in B(x_0,\varrho)} \bigl|f(u) - f(x_0)\bigr|_Y = \varrho\,\mathrm{Lip}^\varrho f(x_0) > \gamma_1\varrho.$$

Thus, there is $u \in B(x_0, \varrho)$ with

$$\bigl|f(u) - f(x_0)\bigr|_Y > \gamma_1\varrho.$$

Then we choose ϱ_1 such that $\varrho < \varrho_1 < \min\bigl\{r, \frac{\gamma_1}{\gamma}\varrho\bigr\}$. Consequently, $\gamma\varrho_1 < \gamma_1\varrho$. By the continuity of f at x_0 there exists $\delta > 0$ such that $\varrho + \delta < \varrho_1$ and

$$\bigl|f(x) - f(x_0)\bigr|_Y < \gamma_1\varrho - \gamma\varrho_1 \quad \text{for any } x \in U := B(x_0, \delta).$$

Consider $x \in U$. Then

$$|u-x|_X \le |u-x_0|_X + |x_0-x|_X < \varrho + \delta < \varrho_1,$$

and, so, $u \in B(x, \varrho_1)$. Consequently,

$$\big|f(u)-f(x)\big|_Y \ge \big|f(u)-f(x_0)\big|_Y - \big|f(x)-f(x_0)\big|_Y > \gamma_1\varrho-(\gamma_1\varrho-\gamma\varrho_1) = \gamma\varrho_1.$$

Hence, $\operatorname{Lip}^{\varrho_1} f(x) > \gamma$. But $0 < \varrho_1 < r$. Therefore, $\operatorname{Lip}_r f(x) > \gamma$ for any $x \in U$. Thus, $\operatorname{Lip}_r f$ is lower semicontinuous at x_0. □

THEOREM 5.4. *Let X and Y be metric spaces and $f\colon X \to Y$ be a continuous function. Then $\operatorname{Lip} f\colon X \to [0,+\infty]$ is a $\mathcal{F}_\sigma$-upper semicontinuous function.*

PROOF. By (2.1) and (2.2) we conclude that $\operatorname{Lip} f(x) = \inf_{n\in\mathbb{N}} \operatorname{Lip}_{\frac{1}{n}} f(x)$ for any $x \in X$. By Lemma 5.3, the functions $\operatorname{Lip}_{\frac{1}{n}} f$ are $\mathcal{T}$-lower semicontinuous where $\mathcal{T}$ is the topology of X. Therefore, by Proposition 4.4 $\operatorname{Lip} f$ is $\mathcal{T}_{c\sigma}$-upper semicontinuous. This means that $\operatorname{Lip} f$ is $\mathcal{F}_\sigma$-upper semicontinuous. □

THEOREM 5.5. *Let X and Y be metric spaces and $f\colon X \to Y$ be a function. Then $\mathbb{L}\mathrm{ip} f\colon X \to [0,+\infty]$ is an upper semicontinuous function.*

PROOF. Fix $x_0 \in X$ and $\gamma > \mathbb{L}\mathrm{ip} f(x_0)$. Since $\mathbb{L}\mathrm{ip} f(x_0) = \inf\limits_{r>0} \mathbb{L}\mathrm{ip}^r f(x_0)$, there exists $r > 0$ such that $\mathbb{L}\mathrm{ip}^r f(x_0) < \gamma$. Set $\varrho = \frac{r}{2}$ and consider $x \in B(x_0, \varrho)$. Then $B(x,\varrho) \subseteq B(x_0, r)$. Consequently,

$$\begin{aligned}\mathbb{L}\mathrm{ip} f(x) \le \mathbb{L}\mathrm{ip}^\varrho f(x) &= \sup_{u\neq v\in B(x,\varrho)} \tfrac{1}{|u-v|_X}\big|f(u)-f(v)\big|_Y \\ &\le \sup_{u\neq v\in B(x_0,r)} \tfrac{1}{|u-v|_X}\big|f(u)-f(v)\big|_Y = \mathbb{L}\mathrm{ip}^r f(x_0) < \gamma\end{aligned}$$

and, hence, $\mathbb{L}\mathrm{ip} f$ is upper semicontinuous. □

Theorems 5.2, 5.4, 5.5, and Propositions 4.1, 4.3 yield the following assertions.

COROLLARY 5.6. *Let X and Y be metric spaces and $f\colon X \to Y$ be a continuous function. Then*

(i) *$\ell(f)$ is a $G_{\delta\sigma}$-set;*
(ii) *$\ell^\infty(f)$ is an $F_{\sigma\delta}$-set;*
(iii) *$L(f)$ is an F_σ-set;*
(iv) *$L^\infty(f)$ is a G_δ-set.*

Proof. (i) We have

$$\begin{aligned}\ell(f) &= \{x \in X\colon \operatorname{lip} f(x) < \infty\} = (\operatorname{lip} f)^{-1}\big[\,[-\infty, +\infty)\big] \\ &= X \setminus (\operatorname{lip} f)^{-1}\big[\{+\infty\}\big]\end{aligned}$$

and, since $\operatorname{lip} f$ is $\mathcal{F}_\sigma$-lower semicontinuous, by Proposition 4.3(iii)

$$(\operatorname{lip} f)^{-1}\big[\{+\infty\}\big] \in \mathcal{F}_{\sigma\delta},$$

so $\ell(f) = X \setminus (\operatorname{lip} f)^{-1}\big[\{+\infty\}\big] \in \mathcal{G}_{\delta\sigma}$.

(ii) It follows immediately from (i).

(iii) It is easy to see, that $L(f) = \bigcup_{k=1}^{\infty} (\operatorname{Lip} f)^{-1}[[0,k)]$. Since $\operatorname{Lip} f$ is an $\mathcal{F}_\sigma$-upper semicontinuous function, each set $(\operatorname{Lip} f)^{-1}[[0,k)]$ is of F_σ type, hence $L(f)$ is an F_σ-set as a countable sum of F_σ-sets.

(iv) It follows from (iii). □

Corollary 5.7. *Let X and Y be metric spaces and $f\colon X \to Y$ be an arbitrary function. Then*

(i) $\mathbb{L}(f)$ *is an open set;*
(ii) $\mathbb{L}^\infty(f)$ *is a closed set.*

6. Characterization of Lipschitz functions on a convex subset of a normed space

The following lemma was applied by Buczolich, Hanson, Maga and Vértesy in certain investigations of Lipschitz derivatives of the real functions of real variable.

Lemma 6.1 ([2, Lemma 2.2]). *If $E \subseteq \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ such that* $\operatorname{lip} f \le \chi_E$ *then $|f(a) - f(b)| \le \mu([a,b] \cap E)$ for every $a, b \in \mathbb{R}$ (where $a < b$) so, f is Lipschitz and hence absolutely continuous.*

In the above, μ denotes the Lebesgue measure. We will state the following

Corollary 6.2. *Let $\gamma > 0$ and $f\colon [0,1] \to \mathbb{R}$ be a function such that* $\operatorname{lip} f(x) \le \gamma$ *for any $x \in [0,1]$. Then f is γ-Lipschitz.*

Proof. Extend f to $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ by $\tilde{f}(x) = f(0)$ if $x < 0$ and $\tilde{f}(x) = f(1)$ if $x > 1$. Let $g = \frac{1}{\gamma}\tilde{f}$ and $E = [0,1]$. Then by Lemma 6.1 we conclude that $\frac{1}{\gamma}\,|f(x) - f(y)| = |g(x) - g(y)| \le \mu\big([x,y] \cap E\big) = |x - y|$ for any $x, y \in [0,1]$. □

The next result will allow us to apply Lemma 6.1 and Corollary 6.2 for functions defined on normed spaces.

Lemma 6.3. *Let A be a convex subset of the normed space X, $f\colon X \to \mathbb{R}$ be a function, and $a, b \in A$. Moreover, let $T\colon [0,1] \to A$ be an affine function given by $T(u) = a + u(b-a)$ for $0 \le u \le 1$ and $g = f \circ T\colon [0,1] \to \mathbb{R}$. Then,*

$$\operatorname{lip} g \le \|b-a\| \big((\operatorname{lip} f) \circ T\big). \tag{6.1}$$

Proof. It is enough to consider the case where $a \neq b$. Fix $u_0 \in [0,1]$ and observe, that

$$\|T(u) - T(u_0)\| = \|(u-u_0)(b-a)\| = |u-u_0|\,\|b-a\|, \quad 0 \le u \le 1. \tag{6.2}$$

We have

$$\operatorname{lip} g(u_0) = \liminf_{r \to 0^+} \sup_{0<|u-u_0|<r} \frac{|f(T(u)) - f(T(u_0))|}{r}. \tag{6.3}$$

Put $x_0 = T(u_0)$. Substituting $\varrho = r\,\|b-a\|$ and $x = T(u)$ in (6.3) and taking into account (6.2) we obtain that

$$\begin{aligned}
\operatorname{lip} g(u_0) &= \liminf_{\varrho \to 0^+} \sup_{0<|u-u_0|<\frac{\varrho}{\|b-a\|}} \frac{|f(T(u)) - f(T(u_0))|}{\varrho/\,\|b-a\|} \\
&= \|b-a\| \liminf_{\varrho \to 0^+} \sup_{\substack{0<\|x-x_0\|<\varrho \\ x \in T[[0,1]]}} \frac{|f(x)-f(x_0)|}{\varrho} \\
&\le \|b-a\| \liminf_{\varrho \to 0^+} \sup_{0<\|x-x_0\|<\varrho} \frac{|f(x)-f(x_0)|}{\varrho} \\
&= \|b-a\| \operatorname{lip} f(x_0).
\end{aligned}$$

□

As $\big((\operatorname{lip} f) \circ T\big)\,\|b-a\| = \big((\operatorname{lip} f) \circ T\big) \cdot \left\|\frac{\mathrm{d}T}{\mathrm{d}u}\right\|$, the right side of inequality (6.1) is reminiscent of the "chain rule" for the usual derivative. Nevertheless, the inequality can be strict. To see that, take a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x,y) = y$, $(x,y) \in \mathbb{R}^2$ and consider the usual distance on $\mathbb{R}^2$. By Theorem 3.2 we have, $\operatorname{lip} f(x,y) = \|\mathrm{d}f(x,y)\| = \sqrt{\big(\frac{\partial f}{\partial x}(x,y)\big)^2 + \big(\frac{\partial f}{\partial y}(x,y)\big)^2} = 1$. Let $a = (0,0)$, $b = (1,0)$, $T(u) = a + (b-a)u = (u,0)$, $u \in [0,1]$ and $g = f \circ T$. Therefore, $g(u) = f(u,0) = 0$ and so, $\operatorname{lip} g(u) = 0 < 1 = \|b-a\| \operatorname{lip} f(T(u))$ for any u.

Theorem 6.4. *Let D be a convex subset of a normed space X, $f\colon D \to \mathbb{R}$ be a function and $\gamma \ge 0$. Then f is γ-Lipschitz if and only if $\operatorname{lip} f(x) \le \gamma$ for any $x \in D$.*

Proof. Fix $a, b \in D$. Define $T\colon [0,1] \to D$ as

$$T(u) = a + u(b-a) \quad \text{for } u \in [0,1]$$

and put $g = f \circ T\colon [0,1] \to \mathbb{R}$. Applying Lemma 6.3 we get

$$\operatorname{lip} g(u) \le \|b-a\| \operatorname{lip} f\big(T(u)\big) \le \|b-a\|\,\gamma,$$

for any $u \in [0,1]$. Therefore, Corollary 6.2 implies that g is Lipschitz with the constant $\gamma_1 = \|b-a\|\,\gamma$. Thus,

$$|f(a) - f(b)| = |g(0) - g(1)| \le \gamma_1\,|0-1| = \gamma\,\|a-b\|\,.$$

So, f is γ-Lipschitz on D. □

By $\|\cdot\|_\infty$ we denote standard norm on space of bounded real functions $B(D)$ defined on a set D, i.e., $\|h\|_\infty = \sup_{x\in D} |h(x)|$ for any $h\colon D \to \mathbb{R}$.

Corollary 6.5. *Let $f\colon D \to \mathbb{R}$ be a continuous function, where D is a convex subset of some normed space X. Then, $\|f\|_{\mathrm{lip}} = \|\operatorname{lip} f\|_\infty$.*

Proof. We simply check, that if $\|f\|_{\mathrm{lip}} < \infty$ or $\|\operatorname{lip} f\|_\infty < \infty$, then

$$\begin{aligned}\|f\|_{\mathrm{lip}} &= \inf\{\gamma > 0\colon f \text{ is } \gamma\text{-Lipschitz}\}\\ &= \inf\{\gamma > 0\colon \operatorname{lip} f(x) \le \gamma \ \text{ for any } \ x \in D\}\\ &= \sup_{x\in D} \operatorname{lip} f(x) = \|\operatorname{lip} f\|_\infty\,,\end{aligned}$$

where the second equality follows from Theorem 6.4. □

Therefore, lip is an isometric injection of the normed space $\operatorname{Lip}_a(D,\mathbb{R})$ with some $a \in X$, into the space $B(D)$.

7. Baire limit functions of Lipschitz derivatives

For a given function $f\colon X \to \overline{\mathbb{R}}$, defined on a metric space X, its *upper Baire function* $f^\vee$ is defined by

$$f^\vee(x) = \inf_{U\in\mathcal{U}(x)} \sup_{u\in U} f(u), \quad x \in X,$$

and its *lower Baire function* $f^\wedge$ is defined by

$$f^\wedge(x) = \sup_{U\in\mathcal{U}(x)} \inf_{u\in U} f(u), \quad x \in X,$$

where $\mathcal{U}(x)$ is the family of all the neighborhoods of x in X. (See, for example, [10].) The upper Baire function $f^\vee$ is upper semicontinuous and the lower Baire function $f^\wedge$ is lower semicontinuous.

A subset D of a normed space X is called *locally convex* if for any point $x \in D$ and any neighborhood U of x in D there is a convex neighborhood V of x in D such that $V \subseteq U$. For example, every convex set and every open set in X is locally convex.

THEOREM 7.1. *Let D be a locally convex subset of a normed space X and let $f \colon D \to \mathbb{R}$ be a function. Then*

$$(\operatorname{lip} f)^{\vee} = (\operatorname{Lip} f)^{\vee} = \mathbb{L}\mathrm{ip}\, f.$$

PROOF. Since $\operatorname{lip} f \le \operatorname{Lip} f \le \mathbb{L}\mathrm{ip}\, f$ and $\mathbb{L}\mathrm{ip}\, f$ is upper semicontinuous by Theorem 5.5, we have

$$(\operatorname{lip} f)^{\vee} \le (\operatorname{Lip} f)^{\vee} \le (\mathbb{L}\mathrm{ip}\, f)^{\vee} = \mathbb{L}\mathrm{ip}\, f.$$

Therefore, it is enough to prove that $\mathbb{L}\mathrm{ip}\, f \le (\operatorname{lip} f)^{\vee}$. Fix $x_0 \in D$. The case where $(\operatorname{lip} f)^{\vee}(x_0) = \infty$ is obvious. So, we suppose that $(\operatorname{lip} f)^{\vee}(x_0) < \infty$. Let $\gamma > (\operatorname{lip} f)^{\vee}(x_0)$. Then, there exists a convex neighborhood U of x_0, such that

$$\operatorname{lip} f(x) < \gamma \quad \text{for any } x \in U.$$

By Theorem 6.4, the function f is γ-Lipschitz on U. Hence,

$$\begin{aligned} \mathbb{L}\mathrm{ip}\, f(x_0) &= \inf_{r>0} \left\| f\big|_{B(x_0,r)} \right\|_{\mathrm{lip}} \\ &\le \left\| f\big|_{U} \right\|_{\mathrm{lip}} \le \gamma, \end{aligned}$$

where $B(x_0, r)$ means the ball in the metric subspace D. Passing to the limit with $\gamma \to (\operatorname{lip} f)^{\vee}(x_0)$ we obtain the desired inequality. □

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