Annales Mathematicae Silesianae (2026),
DOI: 10.2478/amsil-2026-0004

A PEXIDERIZED COSINE FUNCTIONAL EQUATION

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Abstract. Let S be a semigroup and K be a field. In a recent article we introduced a new cosine functional equation $g(xyz) - g(x)g(yz) - g(y)g(xz) - g(z)g(xy) + 2g(x)g(y)g(z) = 0$ for an unknown function $g\colon S \to K$. It was shown that this equation is closely connected to the sine addition formula, and for $K = \mathbb{C}$ its solutions are expressible in terms of multiplicative functions. Here we solve the more general functional equation $f(xyz)+g(x)g(yz)+g(y)g(xz)+g(z)g(xy) + h(x)h(y)h(z) = 0$ for three unknown functions $f, g, h\colon S \to \mathbb{C}$, where S is a monoid. The solutions are linear combinations of two multiplicative functions.

1. Introduction

In a recent paper [3] we introduced a new cosine functional equation

$$g(xyz) - [g(x)g(yz) + g(y)g(xz) + g(z)g(xy)] + 2g(x)g(y)g(z) = 0, \quad x, y, z \in S, \tag{1.1}$$

for unknown $g\colon S \to K$, where S is a semigroup and K is a field. (We called it a "new" cosine equation since d'Alembert's equation was previously called a cosine functional equation.) One of the main results of that paper showed that if g is a solution of (1.1) then there exists a function $f\colon S \to K$ such that the pair (f, g) is a solution of the sine addition formula

$$f(xy) = f(x)g(y) + g(x)f(y), \quad x, y \in S. \tag{1.2}$$

Conversely if the pair (f, g) satisfies (1.2) and $f \neq 0$, then g satisfies (1.1). (A simpler proof of this result has been found by Stetkaer [5].)

Received: 04.11.2025. Accepted: 08.02.2026.
(2020) Mathematics Subject Classification: 39B52.
Key words and phrases: cosine functional equation, sine addition formula, multiplicative function, semigroup, monoid.

The other main result in [3] gave the general solution of (1.1) for the case $K = \mathbb{C}$, and that forms part of the foundation for the present paper. If X is a topological space, let $C(X)$ denote the algebra of continuous functions mapping X into $\mathbb{C}$. A function $m\colon S \to \mathbb{C}$ is *multiplicative* if $m(xy) = m(x)m(y)$ for all $x, y \in S$. The following is [3, Corollary 3.3].

Proposition 1.1. *Let S be a topological semigroup. A function $g \in C(S)$ is a solution of* (1.1) *if and only if there exist multiplicative functions $m_1, m_2 \in C(S)$ such that*

$$g = \frac{m_1 + m_2}{2}.$$

Here we consider the more general functional equation

$$f(xyz) + g(x)g(yz) + g(y)g(xz) + g(z)g(xy) + h(x)h(y)h(z) = 0, \quad x, y, z \in S, \tag{1.3}$$

for three unknown functions $f, g, h\colon S \to \mathbb{C}$, where S is a monoid.

The next section contains some terminology, notation, and other preliminaries. There we describe the solution form of a special case of (1.2) that arises in the preparations for solving (1.3). In the third section we prove some preparatory results about a simple Levi-Civita equation and an extension of Proposition 1.1. The main result is Theorem 4.1, which describes the complex-valued solutions of (1.3) on a monoid. It shows that all such solutions are specified linear combinations of at most two multiplicative functions.

All results are stated for topological semigroups, but the discrete topology is allowed.

2. Preliminaries

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

A function $A\colon S \to \mathbb{C}$ is said to be *additive* if $A(xy) = A(x) + A(y)$ for all $x, y \in S$.

If $m\colon S \to \mathbb{C}$ is multiplicative and $m \neq 0$ then we call m an *exponential*.

For any multiplicative $m\colon S \to \mathbb{C}$ we define the sets

$$I_m := \{x \in S \mid m(x) = 0\}$$

and

$$P_m := \{p \in I_m \setminus I_m I_m \mid up, pv, upv \in I_m \setminus I_m I_m \text{ for all } u, v \in S \setminus I_m\}.$$

It follows that if $p \in P_m$ and $u, v \in S \setminus I_m$, then $up, pv, upv \in P_m$ also.

A *monoid* is a semigroup with an identity element that we denote e. If S is a monoid and $m\colon S \to K$ is multiplicative, then $m(e) = m(e^2) = m(e)^2$ so $m(e) \in \{0,1\}$. If $m(e) = 0$ then $m(x) = m(xe) = m(x)m(e) = 0$ for all $x \in S$, so $m = 0$. Therefore if $m \neq 0$ then $m(e) = 1$.

The general solution of (1.2) for $f, g\colon S \to \mathbb{C}$ is given in [1, Theorem 3.1], but here we need only the following corollary describing the solutions in the special case that g is an exponential and f is not the zero function.

Proposition 2.1. *Let S be a topological semigroup. Suppose $f, m \in C(S)$ satisfy*

$$f(xy) = f(x)m(y) + m(x)f(y), \quad x, y \in S, \tag{2.1}$$

where $f \neq 0$ and m is an exponential. Then there exists an additive function $A \in C(S \setminus I_m)$ and a function $f_P \in C(P_m)$ such that

(a)
$$f(x) = \begin{cases} A(x)m(x) & \text{for } x \in S \setminus I_m, \\ f_P(x) & \text{for } x \in P_m, \\ 0 & \text{for } x \in I_m \setminus P_m, \end{cases}$$

(b) *$f(qt) = f(tq) = 0$ for all $q \in I_m \setminus P_m$ and $t \in S \setminus I_m$, and*

(c) *for $p \in P_m$ and $u, v \in S \setminus I_m$ we have $f_P(up) = f_P(p)m(u)$, $f_P(pv) = f_P(p)m(v)$, and $f_P(upv) = f_P(p)m(uv)$.*

Conversely, for any exponential m the function f described above satisfies (2.1).

Proof. This is an immediate consequence of [1, Theorem 3.1]. □

Definition 2.2. For f, m as described in Proposition 2.1 we refer to the pair (f, m) as a *sine-exponential pair*.

A function γ on S is said to be *central* if $\gamma(xy) = \gamma(yx)$ for all $x, y \in S$.

3. Preparatory results

We will use the following, which is [2, Theorem 4.4].

Proposition 3.1. *Let S be a topological monoid, and suppose $f, g_1, g_2, h_1, h_2 \in C(S)$ satisfy the Levi-Civita equation*

$$f(xy) = g_1(x)h_1(y) + g_2(x)h_2(y), \quad x, y \in S,$$

with f central, and with $\{g_1, g_2\}$ and $\{h_1, h_2\}$ linearly independent. Then the solutions are given by the following families, where $m, m_1, m_2 \in C(S)$ are exponentials with $m_1 \neq m_2$, and (ϕ, m) is a continuous sine-exponential pair.

(a) *There exist constants* $a_i, b_i, c_i, d_i, e_i \in \mathbb{C}$ *satisfying*

$$\begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ e_1 & e_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

such that

$$f = a_1 m_1 + a_2 m_2, \quad g_1 = b_1 m_1 + b_2 m_2, \quad g_2 = d_1 m_1 + d_2 m_2,$$
$$h_1 = c_1 m_1 + c_2 m_2, \quad h_2 = e_1 m_1 + e_2 m_2.$$

(b) *There exist constants* $a_i, b_i, c_i, d_i, e_i \in \mathbb{C}$ *satisfying*

$$\begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ e_1 & e_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & 0 \end{pmatrix}$$

such that

$$f = a_1 m + a_2 \phi, \quad g_1 = b_1 m + b_2 \phi, \quad g_2 = d_1 m + d_2 \phi,$$
$$h_1 = c_1 m + c_2 \phi, \quad h_2 = e_1 m + e_2 \phi.$$

Moreover all matrices above are invertible.

We also need some simple linear independence results.

Lemma 3.2. *Let* S *be a semigroup.*

(i) *Any set of distinct exponentials on* S *into* $\mathbb{C}$ *is linearly independent.*
(ii) *If* (ϕ, m) *is a sine-exponential pair on* S *into* $\mathbb{C}$*, then* $\{m, \phi\}$ *is linearly independent.*

Proof. Part (i) is [4, Theorem 3.18(b)]. Part (ii) is [1, Lemma 5.1(b)]. □

The next step is to prove the following consequence of Proposition 3.1.

Corollary 3.3. *Let* S *be a topological monoid. The functions* $F, G, H \in C(S)$ *satisfy*

$$F(xy) = G(x)G(y) + H(x)H(y), \quad x, y \in S, \tag{3.1}$$

if and only if they belong to one of the following families, where $m \in C(S)$ *is multiplicative,* $m_1, m_2 \in C(S)$ *are exponentials with* $m_1 \neq m_2$*,* (ϕ, m) *is a continuous sine-exponential pair, and* $a, b, c \in \mathbb{C}$*.*

(i) $F = c^2 m$*,* $G = cm$*, and* $H = 0$*.*
(ii) $F = 0$ *and* $G = \pm iH$*, where* H *is an arbitrary nonzero function.*
(iii) *For* $c \neq 0$*,* $b \neq \pm i$*, and* $m \neq 0$ *we have*

$$F = (b^2 + 1)c^2 m, \quad G = bcm, \quad H = cm.$$

(iv) *For* $bc(a^2+c^2) \neq 0$ *we have*

$$F = (a^2+c^2)m_1 + b^2\Big(1+\frac{a^2}{c^2}\Big)m_2, \quad G = am_1 + bm_2, \quad H = cm_1 - \frac{ab}{c}m_2.$$

(v) *For* $b(a+ic) \neq 0$ *we have*

$$F = (a^2+c^2)m + b(a+ic)\phi, \quad G = am + b\phi, \quad H = \pm(cm + ib\phi).$$

PROOF. It is easily checked that the triples (F, G, H) in each case (i)–(v) satisfy (3.1), using the identity $\phi(xy) = \phi(x)m(y) + m(x)\phi(y)$ in case (v).

For the converse suppose $F, G, H \in C(S)$ satisfy (3.1). First we consider the case that $\{G, H\}$ is linearly dependent. If $H = 0$ then (3.1) reduces to

$$F(xy) = G(x)G(y), \quad x, y \in S,$$

which with $x = e$ yields $F = G(e)G$. Thus we have $G(e)G(xy) = G(x)G(y)$ for all $x, y \in S$. If $G(e) = 0$ it follows that $G = 0$, so $F = 0$ and we are in solution family (i) with $c = 0$. If $G(e) \neq 0$, then $m := G(e)^{-1}G$ is multiplicative. Defining $c := G(e) \in \mathbb{C}^*$ we are in family (i) with $c \neq 0$.

If $H \neq 0$ then by linear dependence we have $G = bH$ for some $b \in \mathbb{C}$, and (3.1) reduces to

$$F(xy) = (b^2+1)H(x)H(y), \quad x, y \in S.$$

With $x = e$ we get $F = (b^2+1)H(e)H$, so

$$(3.2) \qquad (b^2+1)H(e)H(xy) = (b^2+1)H(x)H(y), \quad x, y \in S.$$

If $b = \pm i$ then $F = 0$ and we are in family (ii). If $b \neq \pm i$ then we get $H(e) \neq 0$ from (3.2) since $H \neq 0$. Defining $c := H(e)$ and proceeding as in the previous paragraph, we are in family (iii).

Henceforth we assume that $\{G, H\}$ is linearly independent. Since the right hand side of (3.1) is symmetric in x and y, the function F is central. Therefore we may apply Proposition 3.1 with $f := F$, $g_1 = h_1 := G$, and $g_2 = h_2 := H$.

The result in case (a) is

$$F = a_1m_1 + a_2m_2, \quad G = b_1m_1 + b_2m_2, \quad H = d_1m_1 + d_2m_2,$$

where $m_1, m_2 \in C(S)$ are exponentials with $m_1 \neq m_2$, and the coefficients satisfy

$$\begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Since the matrices are non-singular, we have $a_1a_2 \neq 0$, $b_1d_2 - b_2d_1 \neq 0$, $b_j^2 + d_j^2 = a_j$ for $j \in \{1, 2\}$, and $b_1b_2 + d_1d_2 = 0$. There are several sub-cases to consider. If $b_1 = 0$, then $b_2d_1 \neq 0$ and $d_2 = 0$. Defining $c := d_1 \in \mathbb{C}^*$ and

$b := b_2 \in \mathbb{C}^*$, we are in solution family (iv) with $a = 0$. Similarly, if $b_2 = 0$ (or $d_1 = 0$ or $d_2 = 0$) we are again in solution family (iv) (switching the roles of m_1 and m_2 if needed) with $a = 0$. Finally, suppose $b_1 b_2 d_1 d_2 \neq 0$. Defining $a := b_1$, $b := b_2$, and $c := d_1$ we have $d_2 = -ab/c$, and we arrive at solution family (iv) with $a \neq 0$.

From case (b) of Proposition 3.1 we get

$$F = a_1 m + a_2 \phi, \quad G = b_1 m + b_2 \phi, \quad H = d_1 m + d_2 \phi,$$

where $m \in C(S)$ is an exponential, (ϕ, m) is a continuous sine-exponential pair, and the coefficients satisfy

$$\begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & 0 \end{pmatrix}$$

where the matrices are invertible. Thus we have $a_2 \neq 0$, $b_1 d_2 - b_2 d_1 \neq 0$, $b_1^2 + d_1^2 = a_1$, $b_1 b_2 + d_1 d_2 = a_2$, and $b_2^2 + d_2^2 = 0$. Defining $a := b_1$, $b := b_2$, and $c := d_1$, we have $d_2 = \pm ib$, $a_1 = a^2 + c^2$, and $a_2 = (a \pm ic)b$. If $d_2 = ib$ then $a_2 = (a + ic)b \neq 0$ implies that $a \neq -ic$ and we are in solution family (v) with the plus sign in the formula for H. If $d_2 = -ib$ we arrive by similar calculations at

$$F = (a^2 + c^2)m + b(a - ic)\phi, \quad G = am + b\phi, \quad H = cm - ib\phi.$$

Replacing c by $-c$ we arrive at family (v) with the minus sign in the formula for H. □

We also need the following small extension of Proposition 1.1.

Corollary 3.4. *Let S be a topological monoid. A function $g \in C(S)$ is a solution of*

$$\text{(3.3)} \quad g(e)^2 g(xyz) - g(e)[g(x)g(yz) + g(y)g(xz) + g(z)g(xy)] + 2g(x)g(y)g(z) = 0, \quad x, y, z \in S,$$

if and only if there exist multiplicative functions $m_1, m_2 \in C(S)$ such that

$$\text{(3.4)} \qquad g = g(e)\frac{m_1 + m_2}{2}.$$

Proof. Suppose $g \in C(S)$ satisfies (3.3). If $g(e) = 0$, then (3.3) reduces to $g(x)g(y)g(z) = 0$ for all $x, y, z \in S$, so $g = 0$ and we have (3.4).

If $g(e) \neq 0$, then multiplying (3.3) by $g(e)^{-3}$ and defining $g' \colon S \to \mathbb{C}$ by

$$g' := g(e)^{-1} g$$

we find that g' is a solution of (1.1). Since $g' \in C(S)$ we get (3.4) by Proposition 1.1.

The converse is straightforward. □

4. The main result

Recall that (1.3) is the functional equation

$$f(xyz) + g(x)g(yz) + g(y)g(xz) + g(z)g(xy) + h(x)h(y)h(z) = 0, \quad x, y, z \in S.$$

THEOREM 4.1. *Let S be a topological monoid. The functions $f, g, h \in C(S)$ satisfy (1.3) if and only if they belong to one of the following families, where $m \in C(S)$ is multiplicative, $m_1, m_2 \in C(S)$ are exponentials with $m_1 \neq m_2$, and $a, b, c \in \mathbb{C}$.*

(a) $f = -(3b^2 + c^3)m$, $g = bm$, *and* $h = cm$.

(b) *For some* $a \in \mathbb{C}^*$ *we have*

$$f = \frac{a^6}{4}(m_1 + m_2), \quad g = \pm\frac{ia^3}{2\sqrt{2}}(m_1 + m_2), \quad h = \frac{a^2}{2}(m_1 + m_2).$$

(c) *For* $b, c \in \mathbb{C}^*$ *we have*

$$f = -3b^2 m_1 - c^3 m_2, \quad g = bm_1, \quad h = cm_2.$$

(d) *For* $b, c \in \mathbb{C}^*$ *we have*

$$f = -(3b^2 + c^3)m_1 - c^3\Big(\frac{3c^3}{b^2} + 1\Big)m_2, \quad g = bm_1 - \frac{c^3}{b}m_2, \quad h = c(m_1 + m_2).$$

PROOF. It is easy to check that each family (a)–(d) is a solution of (1.3).

Conversely, suppose (f, g, h) is a (continuous) solution of (1.3). Putting $z = e$ we find that

$$-f(xy) - g(e)g(xy) = 2g(x)g(y) + h(e)h(x)h(y), \quad x, y \in S. \tag{4.1}$$

Choose $a \in \mathbb{C}$ such that $a^2 = h(e)$, and define $F, G, H \in C(S)$ by

$$F := -f - g(e)g, \quad G := \sqrt{2}g, \quad H := ah. \tag{4.2}$$

Then (4.1) becomes (3.1):

$$F(xy) = G(x)G(y) + H(x)H(y), \quad x, y \in S,$$

so by Corollary 3.3 we have solution families (i)–(v) for the triple (F, G, H).

Case 1: Suppose (F, G, H) are given by (i) or (iii). Then $F, G, H \in span\{m\}$ for some multiplicative $m \in C(S)$. By (4.2) this means $f = dm$ and $g = bm$ for some constants $b, d \in \mathbb{C}$. Putting these forms into (1.3) we get

$$(d + 3b^2)m(x)m(y)m(z) + h(x)h(y)h(z) = 0, \quad x, y, z \in S. \tag{4.3}$$

If $m = 0$ then we have $f = g = h = 0$, which is included in family (a) by taking $m = 0$. Now suppose $m \neq 0$, so $m(e) = 1$. With $y = z = e$ in (4.3) we see that

$$(d + 3b^2)m(x) + h(e)^2 h(x) = 0, \quad x \in S.$$

If $h(e) = 0$ then $d + 3b^2 = 0$, so by (4.3) we have $h = 0$. This solution is the case $c = 0$ and $d = -3b^2$ of family (a). On the other hand if $h(e) \neq 0$ then we have $h = cm$, where $c = h(e) \neq 0$. Now (4.3) yields

$$(d + 3b^2 + c^3)m(x)m(y)m(z) = 0, \quad x, y, z \in S,$$

and since $m \neq 0$ we have $d = -3b^2 - c^3$. That is again in family (a), and that concludes Case 1.

Case 2: Suppose (F, G, H) are given by (ii), that is $F = 0$ and $G = \pm iH$ for arbitrary $H \neq 0$. By (4.2) we have $h \neq 0$ and

$$g = \pm \frac{iah}{\sqrt{2}}, \quad f = \frac{h(e)^2 h}{2},$$

where $a^2 = h(e) \neq 0$. Inserting these forms into (1.3) we find that

$$\frac{h(e)^2}{2} h(xyz) - \frac{h(e)}{2}[h(x)h(yz) + h(y)h(xz) + h(z)h(xy)] + h(x)h(y)h(z) = 0$$

for all $x, y, z \in S$. That is, h is a solution of (3.3). Therefore by Corollary 3.4 there exist multiplicative functions $m_1, m_2 \in C(S)$ such that

$$h = h(e)\frac{m_1 + m_2}{2}.$$

Since $h \neq 0$, at least one of m_1 or m_2 must be an exponential.

If either $m_1 = m_2 \neq 0$, or $m_2 = 0 \neq m_1$, or $m_1 = 0 \neq m_2$, then we have $(F, G, H) \in span\{m_1\}$ or $(F, G, H) \in span\{m_2\}$. Hence in these cases we are back in Case 1, where $m := m_1$ or $m := m_2$.

If m_1 and m_2 are distinct exponentials, then the solution is in family (b) (see Case 3).

For the rest of the proof (Cases 3 and 4) we have $h(e) \neq 0$, since by (4.2) we have $ah = H \neq 0$ and $h(e) = a^2$.

Case 3: Suppose (F, G, H) are given by (iv). Then $F, G, H \in span\{m_1, m_2\}$ for distinct exponentials $m_1, m_2 \in C(S)$. By (4.2) the same is true for f, g, h, since $h(e) \neq 0$. Let

$$f = a_1 m_1 + a_2 m_2, \quad g = b_1 m_1 + b_2 m_2, \quad h = c_1 m_1 + c_2 m_2$$

for some $a_j, b_j, c_j \in \mathbb{C}$. Inserting these forms into (1.3) we have

$$\begin{aligned} 0 =& a_1 m_1(xyz) + a_2 m_2(xyz) + [b_1 m_1(x) + b_2 m_2(x)][b_1 m_1(yz) + b_2 m_2(yz)] \\ &+ [b_1 m_1(y) + b_2 m_2(y)][b_1 m_1(xz) + b_2 m_2(xz)] \\ &+ [b_1 m_1(z) + b_2 m_2(z)][b_1 m_1(xy) + b_2 m_2(xy)] \\ &+ [c_1 m_1(x) + c_2 m_2(x)][c_1 m_1(y) + c_2 m_2(y)][c_1 m_1(z) + c_2 m_2(z)] \\ =& (a_1 + 3b_1^2 + c_1^3) m_1(x) m_1(y) m_1(z) + (a_2 + 3b_2^2 + c_2^3) m_2(x) m_2(y) m_2(z) \\ &+ (b_1 b_2 + c_1 c_2^2)[m_1(x) m_2(y) m_2(z) + m_2(x) m_1(y) m_2(z) \\ &+ m_2(x) m_2(y) m_1(z)] + (b_1 b_2 + c_1^2 c_2)[m_2(x) m_1(y) m_1(z) \\ &+ m_1(x) m_2(y) m_1(z) + m_1(x) m_1(y) m_2(z)] \end{aligned}$$

for all $x, y, z \in S$. By Lemma 3.2 all of the coefficients must vanish, that is

$$a_j + 3b_j^2 + c_j^3 = 0, \quad j \in \{1, 2\}, \quad \text{and} \quad b_1 b_2 + c_1 c_2^2 = 0 = b_1 b_2 + c_1^2 c_2.$$

It follows that $c_1 c_2 (c_2 - c_1) = 0$.

If $c_1 = 0$ then $b_1 b_2 = 0$. In this case if $b_1 = 0$ then $a_1 = 0$, so we have $f, g, h \in span\{m_2\}$ and are back Case 1. If $b_1 \neq 0$ then $b_2 = 0$ so we have $a_1 = -3b_1^2$ and $a_2 = -c_2^3$, thus

$$f = -3b_1^2 m_1 - c_2^3 m_2, \quad g = b_1 m_1, \quad h = c_2 m_2.$$

If $c_2 = 0$ too, then again we are back to family (a). If not, then we are in solution family (c) with $b := b_1$ and $c := c_2$.

The case $c_1 \neq 0$ and $c_2 = 0$ is similar to the preceding one and leads to the same results by interchanging m_1 and m_2.

Lastly, if $c_1 c_2 \neq 0$ then we have $c_2 = c_1 =: c \in \mathbb{C}^*$. Then $b_1 b_2 = -c^3 \neq 0$ and $a_j = -3b_j^2 - c^3$ for $j = 1, 2$. Defining $b := b_1 \in \mathbb{C}^*$ we are in solution family (d).

Case 4: Suppose (F, G, H) are given by (v). Then $F, G, H \in span\{M, \phi\}$ for a continuous sine-exponential pair (ϕ, M). As in the preceding case, the same is true for f, g, h. It is helpful to recall that (ϕ, M) satisfy the sine addition formula $\phi(xy) = \phi(x)M(y) + M(x)\phi(y)$. So for $x = y = e$ we have $\phi(e) = 2\phi(e)$ since $M(e) = 1$, therefore $\phi(e) = 0$. Let

$$f = a_1 M + a_2 \phi, \quad g = b_1 M + b_2 \phi, \quad h = c_1 M + c_2 \phi,$$

where $a_j, b_j, c_j \in \mathbb{C}$. Inserting these forms into (1.3) we have

$$
\begin{aligned}
0 =& a_1 M(xyz) + a_2\left(\phi(x)M(yz) + M(x)\phi(y)M(z) + M(x)M(y)\phi(z)\right)\\
&+ [b_1 M(x) + b_2\phi(x)][b_1 M(yz) + b_2(\phi(y)M(z) + M(y)\phi(z))]\\
&+ [b_1 M(y) + b_2\phi(y)][b_1 M(xz) + b_2(\phi(x)M(z) + M(x)\phi(z))]\\
&+ [b_1 M(z) + b_2\phi(z)][b_1 M(xy) + b_2(\phi(x)M(y) + M(x)\phi(y))]\\
&+ [c_1 M(x) + c_2\phi(x)][c_1 M(y) + c_2\phi(y)][c_1 M(z) + c_2\phi(z)]\\
=&(a_1 + 3b_1^2 + c_1^3)M(x)M(y)M(z) + c_2^3\phi(x)\phi(y)\phi(z)\\
&+ (a_2 + 3b_1 b_2 + c_2 c_1^2)[\phi(x)M(y)M(z) + M(x)\phi(y)M(z) + M(x)M(y)\phi(z)]\\
&+ (2b_2^2 + c_1 c_2^2)[\phi(x)\phi(y)M(z) + \phi(x)M(y)\phi(z) + M(x)\phi(y)\phi(z)]
\end{aligned}
$$

for all $x, y, z \in S$. By Lemma 3.2 we see that all coefficients vanish, so in particular $c_2 = 0$, and for the rest we have $a_1 + 3b_1^2 + c_1^3 = a_2 + 3b_1 b_2 = 2b_2^2 = 0$. Thus $a_2 = b_2 = 0$ also. Now we have $f, g, h \in \mathit{span}\{M\}$ and revert to family (a).

This completes the proof. □

Acknowledgement. The author is grateful to a referee for a correction and a helpful suggestion which simplified a proof.

Declarations. Funding - none.
Conflicts of interest/Competing interests - none.
Availability of data and material - not applicable.
Code availability - not applicable.
Authors' contributions - This article is the work of the author alone.

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