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JENSEN-TYPE INEQUALITIES FOR DIVIDED DIFFERENCES VIA GENERALIZED CONVEX FUNCTIONS

Gorana Aras-Gazić, Julije Jakšetić, Josip Pečarić

Abstract. In this paper, we use $(m+4)$-convex functions to derive an estimate for Jensen's inequality in the context of divided differences. In addition, we extend these results for $(h, g; \alpha - n)$-convex functions. Finally, we present some results for g-convex functions, (h, g)-convex functions and provide a discussion and examples concerning h-convex functions.

1. Introduction

The divided difference of order m for a function $f\colon [a,b] \to \mathbb{R}$ at the distinct points $x_0, \dots, x_m \in [a,b]$ is defined recursively, as shown in [2], as follows:

$$f[x_i] = f(x_i), \ (i = 0, \dots, m)$$

and

$$f[x_0, \dots, x_m] = \frac{f[x_1, \dots, x_m] - f[x_0, \dots, x_{m-1}]}{x_m - x_0}.$$

The concept of m-convex functions was originally introduced by Popoviciu in [7], providing a foundational framework for the study of generalized convexity. We now present the definition of m-convex function following approach outlined in [6, 16 pp.] and [8, 238 pp.].

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Definition 1. A function $f\colon [a,b] \to \mathbb{R}$ is said to be *m-convex*, $m \geq 0$ on $[a,b]$ iff for all choices of $(m+1)$ distinct points in $[a,b]$,

$$f\left[x_0,\dots,x_m\right] \geq 0.$$

The integral representation for the $m-$th divided difference over the m-dimensional simplex is well-known and is given as follows (refer to [2]):

$$f[x_0,\dots,x_m] = \int_{\triangle_m} f^{(m)}\Big(\sum_{j=0}^{m} u_j x_j\Big) du_0 \dots du_{m-1}, \tag{1.1}$$

where the simplex $\triangle_m$ is defined as

$$\triangle_m = \left\{ (u_0,\dots,u_{m-1}) : u_j \geq 0, \sum_{j=0}^{m-1} u_j \leq 1 \right\}, \quad u_m = 1 - \sum_{j=0}^{m-1} u_j,$$

assuming the function f has continuous m-th derivative on the interval $[a,b]$. The formula (1.1) remains valid if some of the nodes $x_0,\dots,x_m$ are repeated.

From the relation (1.1), it follows that for every function f with a continuous m-th derivative on the interval $\langle a,b\rangle$,

$$f \text{ is } m\text{−convex} \;\Leftrightarrow\; f^{(m)} \geqslant 0.$$

The following result, derived using the Schur polynomial and the Vandermonde determinant (extended with a logarithmic function), holds true.

Proposition 1. *For monomial function $h(x) = x^{m+k}$, where $k \geq 1$ is an integer, the following holds*

$$h[x_0,\dots,x_m] = \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \cdots \sum_{j_k=0}^{j_{k-1}} x_{j_1} x_{j_2} \dots x_{j_k}.$$

Example 1. In the following sections, we will utilize the integral representation of the divided difference and derive the corresponding integrals. These integrals will be evaluated here using Proposition 1 and basic calculus techniques.

$$\int_{\Delta_m} h^{(m)}\left(\sum_{j=0}^{m} u_j x_j\right) du_0 \dots du_{m-1}$$
$$= \begin{cases} \int_{\Delta_m} du_0 \dots du_{m-1} = \frac{1}{m!}, & \text{for} \quad h(x) = \frac{x^m}{m!}, \\ \int_{\Delta_m} \sum_{j=0}^{m} u_j x_j du_0 \dots du_{m-1} = \frac{\sum_{j=0}^{m} x_j}{(m+1)!}, & \text{for} \quad h(x) = \frac{x^{m+1}}{(m+1)!}. \end{cases}$$

The following Farwig and Zwick's result in [3] offers an important generalization of Jensen's inequality specifically tailored for divided differences. This generalization extends the classical form of Jensen's inequality, adapting it to the context of higher-order differences.

THEOREM 1. *Let f be $(m+2)$-convex on $\langle a,b\rangle$. Then*

$$G(\mathbf{x}) = f[x_0,\dots,x_m]$$

is a convex function of the vector $\mathbf{x}=(x_0,\dots,x_m)$. Consequently,

$$f\Big[\sum_{i=0}^{l} a_i x_0^i,\dots,\sum_{i=0}^{l} a_i x_m^i\Big] \le \sum_{i=0}^{l} a_i f[x_0^i,\dots,x_m^i] \quad (i \text{ is an upper index})$$

holds for all $a_i \ge 0, i\in\{1,\dots,l\}$, such that $\sum\limits_{i=0}^{l} a_i = 1$.

In [4], the authors developed a refinement of Jensen's inequality for 4-convex functions. In the next section, we will apply their result to the case of positive weights. For convenience, their result for positive weights is outlined below:

THEOREM 2. *Let $f\in C^2[\rho_1,\rho_2]$ be a 4-convex function and $s_j\in[\rho_1,\rho_2]$, $u_j\ge 0$ for $j=1,2,\dots,m$ with $U_m := \sum_{j=1}^m u_j \ne 0$ and $\frac{1}{U_m}\sum_{j=1}^m u_j s_j \in [\rho_1,\rho_2]$. Then we have*

$$\begin{aligned}(1.2)\quad & \frac{1}{U_m}\sum_{j=1}^{m} u_j f(s_j) - f\Big(\frac{1}{U_m}\sum_{j=1}^{m} u_j s_j\Big)\\ &\le \frac{f''(\rho_2)-f''(\rho_1)}{6(\rho_2-\rho_1)}\Big(\frac{1}{U_m}\sum_{j=1}^{m} u_j s_j^3 - \Big(\frac{1}{U_m}\sum_{j=1}^{m} u_j s_j\Big)^3\Big)\\ &\quad + \frac{\rho_2 f''(\rho_1)-\rho_1 f''(\rho_2)}{2(\rho_2-\rho_1)}\Big(\frac{1}{U_m}\sum_{j=1}^{m} u_j s_j^2 - \Big(\frac{1}{U_m}\sum_{j=1}^{m} u_j s_j\Big)^2\Big).\end{aligned}$$

If f is 4-concave function, then the reverse inequality holds in (1.2).

In the final section, we provide some generalizations for $(h,g;\alpha-n)$-convex functions, and therefore, we introduce this concept of generalized convexity.

DEFINITION 2. Let h be a non-negative function on $J\subset\mathbb{R}$, $\langle 0,1\rangle\subset J$, $h\ne 0$ and let g be a positive function on $I\subset\mathbb{R}$ and $\alpha,n\in\langle 0,1]$. A function $f\colon I\to\mathbb{R}$ is said to be *$(h,g;\alpha-n)$-convex* if it is non-negative and satisfy the following inequality

$$f(\lambda x + n(1-\lambda)y) \le h(\lambda^\alpha) f(x)g(x) + nh(1-\lambda^\alpha) f(y)g(y) \tag{1.3}$$

for all $\lambda\in\langle 0,1\rangle$ and $x,y\in I$. If (1.3) holds in the reverse sense, then f is said to be *$(h,g;\alpha-n)$-concave* function.

This definition extends the concept of an h-convex function, as outlined in the following definition (see [9]).

Definition 3. Let $h\colon J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f\colon I \to \mathbb{R}$ is *h-convex* function if f is non-negative and for all $x, y \in I$, $\lambda \in \langle 0, 1\rangle$ we have

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda) f(x) + h(1-\lambda) f(y). \tag{1.4}$$

If inequality (1.4) is reversed, then f is said to be *h-concave.*

2. Extensions of Jensen's inequality using $(m+4)$-convex functions and divided differences

In this paper, we establish an evaluation of Jensen's inequality for divided differences by utilizing $(m+4)$-convex functions. In this way, we provide generalizations of the existing results for divided differences. In Theorem 3, we extend the result from Farwig and Zwick's Theorem 1, and in Theorem 4, we generalize Theorem 2.62 from [6].

Theorem 3. *Let $f^{(m)} \in C^2[\rho_1, \rho_2]$ be a $(m+4)$-convex function, $\mathbf{s}^i = (s_0^i, \ldots, s_m^i) \in [\rho_1, \rho_2]^{m+1}$ and let $a_i \geq 0$, $i \in \{0, \ldots, l\}$ be such that $\sum_{i=0}^{l} a_i = 1$, $\overline{s}_j = \sum_{i=0}^{l} a_i s_j^i$, $j \in \{0, \ldots, m\}$, $\overline{s}_{j_1 j_2} = \sum_{i=0}^{l} a_i s_{j_1}^i s_{j_2}^i$, and $\overline{s}_{j_1 j_2 j_3} = \sum_{i=0}^{l} a_i s_{j_1}^i s_{j_2}^i s_{j_3}^i$. Then we have*

$$\begin{aligned}
&\sum_{i=0}^{l} a_i f[s_0^i, \ldots, s_m^i] - f[\overline{s}_0, \ldots, \overline{s}_m] \\
&\quad \leq \frac{f^{(m+2)}(\rho_2) - f^{(m+2)}(\rho_1)}{\rho_2 - \rho_1} \left(\sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} \left(\overline{s}_{j_1 j_2 j_3} - \overline{s}_{j_1} \overline{s}_{j_2} \overline{s}_{j_3} \right) \right) \\
&\qquad + \frac{\rho_2 f^{(m+2)}(\rho_1) - \rho_1 f^{(m+2)}(\rho_2)}{\rho_2 - \rho_1} \left(\sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \left(\overline{s}_{j_1 j_2} - \overline{s}_{j_1} \overline{s}_{j_2} \right) \right).
\end{aligned}$$

Proof. We will begin by utilizing the integral representation of the divided difference, followed by applying inequality (1.2) for the $(m+4)$-convex function $f^{(m)}$, and then perform the integration over the simplex. So we have

$$\sum_{i=0}^{l} a_i f[s_0^i, \ldots, s_m^i] - f[\overline{s}_0, \ldots, \overline{s}_m] = \sum_{i=0}^{l} a_i \int_{\triangle_m} f^{(m)}\Big(\sum_{j=0}^{m} u_j s_j^i \Big) du_0 \ldots du_{m-1}$$

$$
\begin{aligned}
&-\int_{\triangle_m} f^{(m)}\Big(\sum_{j=0}^{m} u_j \sum_{i=0}^{l} a_i s_j^i\Big) du_0 \dots du_{m-1} \\
=&\int_{\triangle_m} \left[\sum_{i=0}^{l} a_i f^{(m)}\Big(\sum_{j=0}^{m} u_j s_j^i\Big) - f^{(m)}\Big(\sum_{i=0}^{l} a_i \sum_{j=0}^{m} u_j s_j^i\Big)\right] du_0 \dots du_{m-1} \\
\leq &\int_{\triangle_m} \left[\frac{f^{(m+2)}(\rho_2) - f^{(m+2)}(\rho_1)}{6(\rho_2-\rho_1)}\left(\sum_{i=0}^{l} a_i \Big(\sum_{j=0}^{m} u_j s_j^i\Big)^3 - \Big(\sum_{i=0}^{l} a_i \sum_{j=0}^{m} u_j s_j^i\Big)^3\right)\right. \\
&+ \frac{\rho_2 f^{(m+2)}(\rho_1) - \rho_1 f^{(m+2)}(\rho_2)}{2(\rho_2-\rho_1)}\left(\sum_{i=0}^{l} a_i \Big(\sum_{j=0}^{m} u_j s_j^i\Big)^2\right. \\
&\qquad\qquad \left.\left. - \Big(\sum_{i=0}^{l} a_i \sum_{j=0}^{m} u_j s_j^i\Big)^2\right)\right] du_0 \dots du_{m-1}.
\end{aligned}
$$

By setting $h(x) = \frac{x^{m+2}}{(m+2)!}$ in Proposition 1, we compute

$$
\int_{\triangle_m} \sum_{i=0}^{l} a_i \Big(\sum_{j=0}^{m} u_j s_j^i\Big)^2 du_0 \dots du_{m-1} \text{ and } \int_{\triangle_m} \Big(\sum_{i=0}^{l} a_i \sum_{j=0}^{m} u_j s_j^i\Big)^2 du_0 \dots du_{m-1}.
$$

Similarly, by setting $h(x) = \frac{x^{m+3}}{(m+3)!}$ in Proposition 1 we calculate integrals

$$
\int_{\triangle_m} \sum_{i=0}^{l} a_i \Big(\sum_{j=0}^{m} u_j s_j^i\Big)^3 du_0 \dots du_{m-1}
$$

$$
\text{and} \quad \int_{\triangle_m} \Big(\sum_{i=0}^{l} a_i \sum_{j=0}^{m} u_j s_j^i\Big)^3 du_0 \dots du_{m-1}.
$$

So we have

$$
\begin{aligned}
&\sum_{i=0}^{l} a_i f[s_0^i, \dots, s_m^i] - f[\overline{s}_0, \dots, \overline{s}_m] \\
&\qquad \leq \frac{f^{(m+2)}(\rho_2) - f^{(m+2)}(\rho_1)}{(\rho_2-\rho_1)}\left(\sum_{i=0}^{l} a_i \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} s_{j_1}^i s_{j_2}^i s_{j_3}^i \right. \\
&\qquad\qquad \left. - \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} \overline{s}_{j_1} \overline{s}_{j_2} \overline{s}_{j_3}\right) \\
&+ \frac{\rho_2 f^{(m+2)}(\rho_1) - \rho_1 f^{(m+2)}(\rho_2)}{(\rho_2-\rho_1)}\left(\sum_{i=0}^{l} a_i \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} s_{j_1}^i s_{j_2}^i - \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \overline{s}_{j_1} \overline{s}_{j_2}\right)
\end{aligned}
$$

$$= \frac{f^{(m+2)}(\rho_2) - f^{(m+2)}(\rho_1)}{\rho_2 - \rho_1} \Bigg(\sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} \left(\overline{s}_{j_1 j_2 j_3} - \overline{s}_{j_1} \overline{s}_{j_2} \overline{s}_{j_3} \right) \Bigg)$$

$$+ \frac{\rho_2 f^{(m+2)}(\rho_1) - \rho_1 f^{(m+2)}(\rho_2)}{\rho_2 - \rho_1} \Bigg(\sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \left(\overline{s}_{j_1 j_2} - \overline{s}_{j_1} \overline{s}_{j_2} \right) \Bigg). \quad \square$$

The following theorem presents a generalization of the result from Theorem 2.62 in [6]:

THEOREM 4. *Let $f^{(m)} \in C^2[\rho_1, \rho_2]$ be a $(m+4)$-convex function and $s_k \in [\rho_1, \rho_2]$. Then we have*

$$\frac{\sum_{j=0}^{m} f^{(m)}(s_j)}{(m+1)!} - f[s_0, \ldots, s_m]$$

$$\leq \frac{f^{(m+2)}(\rho_2) - f^{(m+2)}(\rho_1)}{6(\rho_2 - \rho_1)} \left(\frac{\sum_{j=0}^{m} s_j^3}{(m+1)!} - 6 \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} s_{j_1} s_{j_2} s_{j_3} \right)$$

$$+ \frac{\rho_2 f^{(m+2)}(\rho_1) - \rho_1 f^{(m+2)}(\rho_2)}{2(\rho_2 - \rho_1)} \left(\frac{\sum_{j=0}^{m} s_j^2}{(m+1)!} - 2 \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} s_{j_1} s_{j_2} \right).$$

PROOF. Similarly as in Theorem 3, we have:

$$\int_{\triangle_m} \sum_{j=0}^{m} u_j f^{(m)}(s_j) du_0 \ldots du_{m-1} - \int_{\triangle_m} f^{(m)} \Big(\sum_{j=0}^{m} u_j s_j \Big) du_0 \ldots du_{m-1}$$

$$\leq \frac{f^{(m+2)}(\rho_2) - f^{(m+2)}(\rho_1)}{6(\rho_2 - \rho_1)} \Bigg(\int_{\triangle_m} \sum_{j=0}^{m} u_j s_j^3 du_0 \ldots du_{m-1}$$

$$- \int_{\triangle_m} \Big(\sum_{j=0}^{m} u_j s_j \Big)^3 du_0 \ldots du_{m-1} \Bigg)$$

$$+ \frac{\rho_2 f^{(m+2)}(\rho_1) - \rho_1 f^{(m+2)}(\rho_2)}{2(\rho_2 - \rho_1)} \Bigg(\int_{\triangle_m} \sum_{j=0}^{m} u_j s_j^2 du_0 \ldots du_{m-1}$$

$$- \int_{\triangle_m} \Big(\sum_{j=0}^{m} u_j s_j \Big)^2 du_0 \ldots du_{m-1} \Bigg).$$

We apply Example 1 to compute the following integrals:

$$\int_{\triangle_m} \sum_{j=0}^{m} u_j f^{(m)}(s_j) du_0 \ldots du_{m-1}, \quad \int_{\triangle_m} \sum_{j=0}^{m} u_j s_j^3 du_0 \ldots du_{m-1},$$

$$\text{and} \quad \int_{\triangle_m} \sum_{j=0}^{m} u_j s_j^2 du_0 \ldots du_{m-1}.$$

Next, by setting

$$h(x) = \frac{x^{m+2}}{(m+2)!} \quad \text{and} \quad h(x) = \frac{x^{m+3}}{(m+3)!}$$

in Proposition 1, we compute the corresponding expression

$$\int_{\triangle_m} \Big(\sum_{j=0}^{m} u_j s_j\Big)^2 du_0 \ldots du_{m-1} \quad \text{and} \quad \int_{\triangle_m} \Big(\sum_{j=0}^{m} u_j s_j\Big)^3 du_0 \ldots du_{m-1}.$$

So we have

$$\begin{aligned}
&\frac{\sum_{j=0}^{m} f^{(m)}(s_j)}{(m+1)!} - f\big[s_0, \ldots, s_m\big] \\
&\quad \le \frac{f^{(m+2)}(\rho_2) - f^{(m+2)}(\rho_1)}{6(\rho_2 - \rho_1)} \bigg(\frac{\sum_{j=0}^{m} s_j^3}{(m+1)!} - 6 \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} s_{j_1} s_{j_2} s_{j_3}\bigg) \\
&\quad + \frac{\rho_2 f^{(m+2)}(\rho_1) - \rho_1 f^{(m+2)}(\rho_2)}{2(\rho_2 - \rho_1)} \bigg(\frac{\sum_{j=0}^{m} s_j^2}{(m+1)!} - 2 \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} s_{j_1} s_{j_2}\bigg). \quad \square
\end{aligned}$$

The following theorem extends inequalities related to divided differences for $(h, g; \alpha - n)$-convex functions.

THEOREM 5. *Let $f^{(m)}$ be a nonnegative $(h, g; \alpha - n)$-convex function on $[0, \infty\rangle$ where h is a nonnegative function on $J \subset \mathbb{R}$, $h \neq 0$, g is positive function on $[0, \infty\rangle$, $\alpha, n \in \langle 0, 1]$, $0 \le \rho_1 < \rho_2 < \infty$, $f^{(m)}, g, h \in L_1[\rho_1, \rho_2]$ and $a_i \ge 0$, $i \in \{0, \ldots, l\}$ such that $\sum_{i=0}^{l} a_i = 1$. Then the following inequality holds*

$$\begin{aligned}
(2.1) \quad &\sum_{i=0}^{l} a_i f[s_0^i, \ldots, s_m^i] \\
&\le \sum_{i=0}^{l} a_i \int_{\triangle_m} \min\bigg\{\bigg[h\Big(\big(\frac{\rho_2 n - \sum_{j=0}^{m} u_j s_j^i}{\rho_2 n - \rho_1}\big)^{\alpha}\Big) f^{(m)}(\rho_1) g(\rho_1) \\
&\quad + nh\Big(1 - \big(\frac{\rho_2 n - \sum_{j=0}^{m} u_j s_j^i}{\rho_2 n - \rho_1}\big)^{\alpha}\Big) f^{(m)}(\rho_2) g(\rho_2)\bigg], \\
&\quad \bigg[h\Big(\big(\frac{\sum_{j=0}^{m} u_j s_j^i - n\rho_1}{\rho_2 - n\rho_1}\big)^{\alpha}\Big) f^{(m)}(\rho_2) g(\rho_2) \\
&\quad + nh\Big(1 - \big(\frac{\sum_{j=0}^{m} u_j s_j^i - n\rho_1}{\rho_2 - n\rho_1}\big)^{\alpha}\Big) f^{(m)}(\rho_1) g(\rho_1)\bigg]\bigg\} du_0 \ldots du_{m-1}
\end{aligned}$$

$$\leq \sum_{i=0}^{l} a_i \min\Bigg\{ \int_{\triangle_m} \Bigg[h\Bigg(\Big(\frac{\rho_2 n - \sum_{j=0}^{m} u_j s_j^i}{\rho_2 n - \rho_1}\Big)^{\alpha}\Bigg) f^{(m)}(\rho_1) g(\rho_1)$$
$$+ nh\Bigg(1 - \Big(\frac{\rho_2 n - \sum_{j=0}^{m} u_j s_j^i}{\rho_2 n - \rho_1}\Big)^{\alpha}\Bigg) f^{(m)}(\rho_2) g(\rho_2)\Bigg] du_0 \ldots du_{m-1},$$
$$\int_{\triangle_m} \Bigg[h\Bigg(\Big(\frac{\sum_{j=0}^{m} u_j s_j^i - n\rho_1}{\rho_2 - n\rho_1}\Big)^{\alpha}\Bigg) f^{(m)}(\rho_2) g(\rho_2)$$
$$+ nh\Bigg(1 - \Big(\frac{\sum_{j=0}^{m} u_j s_j^i - n\rho_1}{\rho_2 - n\rho_1}\Big)^{\alpha}\Bigg) f^{(m)}(\rho_1) g(\rho_1)\Bigg] du_0 \ldots du_{m-1}\Bigg\}.$$

Proof. First, we express the divided difference in its integral form. Next, we will prove an auxiliary result using Definition 2. To do so, we apply Definition 2 and set

$$u = \lambda x + n(1-\lambda)y,$$

and then solve for

$$\lambda = \frac{u - ny}{x - ny}.$$

Substituting this into the inequality (1.3), we obtain

$$f(u) \leq h\Bigg(\Big(\frac{u-ny}{x-ny}\Big)^{\alpha}\Bigg) f(x)g(x) + nh\Bigg(1 - \Big(\frac{u-ny}{x-ny}\Big)^{\alpha}\Bigg) f(y)g(y).$$

By swapping x and y, we obtain

$$\lambda = \frac{u - nx}{y - nx},$$

which leads to

$$f(u) \leq h\left(\left(\frac{u-nx}{y-nx}\right)^{\alpha}\right) f(y)g(y) + nh\left(1 - \left(\frac{u-nx}{y-nx}\right)^{\alpha}\right) f(x)g(x).$$

Thus, we derive the following inequality:

$$f(u) \leq \min\Bigg\{ h\Bigg(\Big(\frac{u-ny}{x-ny}\Big)^{\alpha}\Bigg) f(x)g(x) + nh\Bigg(1 - \Big(\frac{u-ny}{x-ny}\Big)^{\alpha}\Bigg) f(y)g(y),$$
$$h\Bigg(\Big(\frac{u-nx}{y-nx}\Big)^{\alpha}\Bigg) f(y)g(y) + nh\Bigg(1 - \Big(\frac{u-nx}{y-nx}\Big)^{\alpha}\Bigg) f(x)g(x)\Bigg\}. \tag{2.2}$$

We apply (2.2) to the function $f^{(m)}$, and integrating over the simplex, while utilizing the simple fact that

$$\int_{\triangle_m} \min\{f, g\} \leq \min\Bigg\{ \int_{\triangle_m} f, \int_{\triangle_m} g \Bigg\},$$

we obtain the desired result (2.1) □

3. Applications

In the special case of Theorem 5 when $h(x) = x$, $\alpha = 1$, $n = 1$, we obtain the following result for what we will refer to as g-convex functions:

COROLLARY 1. *Let $f^{(m)}$ be a nonnegative function on $[0,\infty\rangle$, g is positive function on $[0,\infty\rangle$, $0 \le \rho_1 < \rho_2 < \infty$ and $f^{(m)}, g \in L_1[\rho_1,\rho_2]$, $a_i \ge 0$, $i \in \{0,\dots,l\}$ such that $\sum_{i=0}^{l} a_i = 1$ and $\overline{s}_j = \sum_{i=0}^{l} a_i s_j^i$. Then the following inequality holds*

$$m!\sum_{i=0}^{l} a_i f[s_0^i,\dots,s_m^i] \le \frac{f^{(m)}(\rho_1)g(\rho_1)}{\rho_2-\rho_1}\left(\rho_2 - \frac{1}{m+1}\sum_{j=0}^{m}\overline{s}_j\right) + \frac{f^{(m)}(\rho_2)g(\rho_2)}{\rho_2-\rho_1}\left(\frac{1}{m+1}\sum_{j=0}^{m}\overline{s}_j - \rho_1\right). \tag{3.1}$$

PROOF. We derive the following from (2.1):

$$\begin{aligned}
&\sum_{i=0}^{l} a_i f[s_0^i,\dots,s_m^i] = \sum_{i=0}^{l} a_i \int_{\triangle_m} f^{(m)}\left(\sum_{j=0}^{m} u_j s_j^i\right) du_0\dots du_{m-1}\\
&\le \sum_{i=0}^{l} a_i \min\left\{\int_{\triangle_m}\left[\frac{\rho_2-\sum_{j=0}^{m}u_j s_j^i}{\rho_2-\rho_1} f^{(m)}(\rho_1)g(\rho_1) + \frac{\sum_{j=0}^{m}u_j s_j^i-\rho_1}{\rho_2-\rho_1}\right.\right.\\
&\quad\left.\times f^{(m)}(\rho_2)g(\rho_2)\right] du_0\dots du_{m-1}, \int_{\triangle_m}\left[\frac{\sum_{j=0}^{m}u_j s_j^i-\rho_1}{\rho_2-\rho_1} f^{(m)}(\rho_2)g(\rho_2)\right.\\
&\quad\left.\left.+\frac{\rho_2-\sum_{j=0}^{m}u_j s_j^i}{\rho_2-\rho_1} f^{(m)}(\rho_1)g(\rho_1)\right] du_0\dots du_{m-1}\right\}\\
&= \sum_{i=0}^{l} a_i \min\left\{\frac{(m+1)\rho_2-\sum_{j=0}^{m}s_j^i}{(m+1)!(\rho_2-\rho_1)} f^{(m)}(\rho_1)g(\rho_1)\right.\\
&\quad+\frac{\sum_{j=0}^{m}s_j^i-\rho_1(m+1)}{(m+1)!(\rho_2-\rho_1)} f^{(m)}(\rho_2)g(\rho_2), \frac{\sum_{j=0}^{m}s_j^i-\rho_1(m+1)}{(m+1)!(\rho_2-\rho_1)} f^{(m)}(\rho_2)g(\rho_2)\\
&\quad\left.+\frac{(m+1)\rho_2-\sum_{j=0}^{m}s_j^i}{(m+1)!(\rho_2-\rho_1)} f^{(m)}(\rho_1)g(\rho_1)\right\}\\
&= \frac{(m+1)\rho_2-\sum_{j=0}^{m}\overline{s}_j}{(m+1)!(\rho_2-\rho_1)} f^{(m)}(\rho_1)g(\rho_1) + \frac{\sum_{j=0}^{m}\overline{s}_j-\rho_1(m+1)}{(m+1)!(\rho_2-\rho_1)} f^{(m)}(\rho_2)g(\rho_2).
\end{aligned}$$

With some elementary calculations and rearrangement, we arrive at (3.1). □

REMARK 1. In the special case where $h(x) = x$, $\alpha = 1$, $n = 1$, $g \equiv 1$, we obtain the result for convex functions, and the following Lah-Ribarič inequality holds, as proven [1, Theorem 10].

$$\begin{aligned}
&\sum_{i=0}^{l} a_i f[s_0^i, \ldots, s_m^i] = \sum_{i=0}^{l} a_i \int_{\triangle_m} f^{(m)}\Big(\sum_{j=0}^{m} u_j s_j^i\Big) du_0 \ldots du_{m-1} \\
&\leq \sum_{i=0}^{l} a_i \min\Big\{ \int_{\triangle_m} \Big[\frac{\rho_2 - \sum_{j=0}^{m} u_j s_j^i}{\rho_2 - \rho_1} f^{(m)}(\rho_1) \\
&\quad + \frac{\sum_{j=0}^{m} u_j s_j^i - \rho_1}{\rho_2 - \rho_1} f^{(m)}(\rho_2)\Big] du_0 \ldots du_{m-1}, \int_{\triangle_m} \Big[\frac{\sum_{j=0}^{m} u_j s_j^i - \rho_1}{\rho_2 - \rho_1} f^{(m)}(\rho_2) \\
&\quad + \frac{\rho_2 - \sum_{j=0}^{m} u_j s_j^i}{\rho_2 - \rho_1} f^{(m)}(\rho_1)\Big] du_0 \ldots du_{m-1}\Big\} \\
&= \frac{(m+1)\rho_2 - \sum_{j=0}^{m} \bar{s}_j}{(m+1)!(\rho_2 - \rho_1)} f^{(m)}(\rho_1) + \frac{\sum_{j=0}^{m} \bar{s}_j - \rho_1(m+1)}{(m+1)!(\rho_2 - \rho_1)} f^{(m)}(\rho_2).
\end{aligned}$$

In the following theorem, we apply the results obtained to the function F of two variables, as described by Pečarić and Beesack in [5] (see also [6]).

THEOREM 6. *Let $f^{(m)}$ be a nonnegative function on $[0, \infty\rangle$, g is positive function on $[0, \infty\rangle$, $0 \leq \rho_1 < \rho_2 < \infty$ and $f^{(m)}, g \in L_1[\rho_1, \rho_2]$, $a_i \geq 0$, $i \in \{0, \ldots, l\}$ such that $\sum_{i=0}^{l} a_i = 1$ and $\bar{s}_j = \sum_{i=0}^{l} a_i s_j^i$ and let J be an interval such that $J \supset f^{(m)}(I)$. If $F\colon J \times J \to \mathbb{R}$ is a function defined such that $u \mapsto F(u, v)$ is increasing for any $v \in J$, then for every $\xi = \frac{1}{m+1} \sum_{j=0}^{m} \bar{s}_j$ we have*

$$\begin{aligned}
&F\Big(m! \sum_{i=0}^{l} a_i f[s_0^i, \ldots, s_m^i], f^{(m)}(\xi)\Big) \\
&\leq F\Big(\frac{f^{(m)}(\rho_1) g(\rho_1)}{\rho_2 - \rho_1} (\rho_2 - \xi) + \frac{f^{(m)}(\rho_2) g(\rho_2)}{\rho_2 - \rho_1} (\xi - \rho_1), f^{(m)}(\xi)\Big) \\
&\leq \max_{\xi \in [\rho_1, \rho_2]} F\Big(\frac{f^{(m)}(\rho_1) g(\rho_1)}{\rho_2 - \rho_1} (\rho_2 - \xi) + \frac{f^{(m)}(\rho_2) g(\rho_2)}{\rho_2 - \rho_1} (\xi - \rho_1), f^{(m)}(\xi)\Big).
\end{aligned}$$

In the special case of Theorem 5 when $\alpha = 1$, $n = 1$, we obtain the following result for (h, g)-convex functions:

COROLLARY 2. *Let $f^{(m)}$ be a nonnegative (h, g)-convex function on $[0, \infty\rangle$ where h is a nonnegative concave function on $J \subset \mathbb{R}$, $h \neq 0$, g is positive function on $[0, \infty\rangle$, $0 \leq \rho_1 < \rho_2 < \infty$, $f^{(m)}, g, h \in L_1[\rho_1, \rho_2]$ and $a_i \geq 0$,*

$i \in \{0,\dots,l\}$ *such that* $\sum_{i=0}^{l} a_i = 1$. *Then the following inequality holds*

$$m!\sum_{i=0}^{l} a_i f[s_0^i,\dots,s_m^i] \le \sum_{i=0}^{l} a_i\Big[f^{(m)}(\rho_1)g(\rho_1)h\Big(\frac{\rho_2-\frac{1}{m+1}\sum_{j=0}^{m}s_j^i}{\rho_2-\rho_1}\Big) + f^{(m)}(\rho_2)g(\rho_2)h\Big(\frac{\frac{1}{m+1}\sum_{j=0}^{m}s_j^i-\rho_1}{\rho_2-\rho_1}\Big)\Big].$$

PROOF.

$$\begin{aligned}
\sum_{i=0}^{l} a_i f[s_0^i,\dots,s_m^i] &\le \sum_{i=0}^{l} a_i \int_{\triangle_m} \min\Big\{\Big[h\Big(\frac{\rho_2-\sum_{j=0}^{m}u_js_j^i}{\rho_2-\rho_1}\Big)f^{(m)}(\rho_1)g(\rho_1) \\
&\quad + h\Big(\frac{\sum_{j=0}^{m}u_js_j^i-\rho_1}{\rho_2-\rho_1}\Big)f^{(m)}(\rho_2)g(\rho_2)\Big], \Big[h\Big(\frac{\sum_{j=0}^{m}u_js_j^i-\rho_1}{\rho_2-\rho_1}\Big)f^{(m)}(\rho_2)g(\rho_2) \\
&\quad + h\Big(\frac{\rho_2-\sum_{j=0}^{m}u_js_j^i}{\rho_2-\rho_1}\Big)f^{(m)}(\rho_1)g(\rho_1)\Big]\Big\}du_0\dots du_{m-1} \\
&\le \sum_{i=0}^{l} a_i \min\Big\{\int_{\triangle_m}\Big[h\Big(\frac{\rho_2-\sum_{j=0}^{m}u_js_j^i}{\rho_2-\rho_1}\Big)f^{(m)}(\rho_1)g(\rho_1) \\
&\quad + h\Big(\frac{\sum_{j=0}^{m}u_js_j^i-\rho_1}{\rho_2-\rho_1}\Big)f^{(m)}(\rho_2)g(\rho_2)\Big]du_0\dots du_{m-1}, \\
&\quad \int_{\triangle_m}\Big[h\Big(\frac{\sum_{j=0}^{m}u_js_j^i-\rho_1}{\rho_2-\rho_1}\Big)f^{(m)}(\rho_2)g(\rho_2) \\
&\quad + h\Big(\frac{\rho_2-\sum_{j=0}^{m}u_js_j^i}{\rho_2-\rho_1}\Big)f^{(m)}(\rho_1)g(\rho_1)\Big]du_0\dots du_{m-1}\Big\} \\
&\le \sum_{i=0}^{l} a_i \min\Big\{\Big[\frac{f^{(m)}(\rho_1)g(\rho_1)}{m!}h\Big(m!\int_{\triangle_m}\frac{\rho_2-\sum_{j=0}^{m}u_js_j^i}{\rho_2-\rho_1}du_0\dots du_{m-1}\Big) \\
&\quad + \frac{f^{(m)}(\rho_2)g(\rho_2)}{m!}h\Big(m!\int_{\triangle_m}\frac{\sum_{j=0}^{m}u_js_j^i-\rho_1}{\rho_2-\rho_1}du_0\dots du_{m-1}\Big)\Big], \\
&\quad \Big[\frac{f^{(m)}(\rho_2)g(\rho_2)}{m!}h\Big(m!\int_{\triangle_m}\frac{\sum_{j=0}^{m}u_js_j^i-\rho_1}{\rho_2-\rho_1}du_0\dots du_{m-1}\Big) \\
&\quad + \frac{f^{(m)}(\rho_1)g(\rho_1)}{m!}h\Big(m!\int_{\triangle_m}\frac{\rho_2-\sum_{j=0}^{m}u_js_j^i}{\rho_2-\rho_1}du_0\dots du_{m-1}\Big)\Big]\Big\} \\
&= \sum_{i=0}^{l} a_i \min\Big\{\Big[\frac{f^{(m)}(\rho_1)g(\rho_1)}{m!}h\Big(\frac{\rho_2-\frac{1}{m+1}\sum_{j=0}^{m}s_j^i}{\rho_2-\rho_1}\Big) \\
&\quad + \frac{f^{(m)}(\rho_2)g(\rho_2)}{m!}h\Big(\frac{\frac{1}{m+1}\sum_{j=0}^{m}s_j^i-\rho_1}{\rho_2-\rho_1}\Big)\Big],
\end{aligned}$$

$$\left[\frac{f^{(m)}(\rho_2)g(\rho_2)}{m!}h\Big(\frac{\frac{1}{m+1}\sum_{j=0}^m s_j^i-\rho_1}{\rho_2-\rho_1}\Big)+\frac{f^{(m)}(\rho_1)g(\rho_1)}{m!}h\Big(\frac{\rho_2-\frac{1}{m+1}\sum_{j=0}^m s_j^i}{\rho_2-\rho_1}\Big)\right]\Bigg\}$$

$$=\sum_{i=0}^{l}a_i\Bigg[\frac{f^{(m)}(\rho_1)g(\rho_1)}{m!}h\Big(\frac{\rho_2-\frac{1}{m+1}\sum_{j=0}^m s_j^i}{\rho_2-\rho_1}\Big)$$
$$+\frac{f^{(m)}(\rho_2)g(\rho_2)}{m!}h\Big(\frac{\frac{1}{m+1}\sum_{j=0}^m s_j^i-\rho_1}{\rho_2-\rho_1}\Big)\Bigg]. \qquad \square$$

THEOREM 7. *Let $f^{(m)}$ be a nonnegative (h,g)-convex function on $[0,\infty\rangle$ where h is a nonnegative concave function on $J\subset\mathbb{R}$, $h\neq 0$, g is positive function on $[0,\infty\rangle$, $0\le\rho_1<\rho_2<\infty$ and $f^{(m)},g,h\in L_1[\rho_1,\rho_2]$, $a_i\ge 0$, $i\in\{0,\dots,l\}$ such that $\sum_{i=0}^l a_i=1$ and let J be an interval such that $J\supset f^{(m)}(I)$. If $F\colon J\times J\to\mathbb{R}$ is a function defined such that $u\mapsto F(u,v)$ is increasing for any $v\in J$, then for every $\xi^i=\frac{1}{m+1}\sum_{j=0}^m s_j^i$ we have*

$$F\Big(m!\sum_{i=0}^{l}a_i f[s_0^i,\dots,s_m^i],f^{(m)}(\xi^i)\Big)$$
$$\le F\Big[\sum_{i=0}^{l}a_i\Big(f^{(m)}(\rho_1)g(\rho_1)h\Big(\frac{\rho_2-\xi^i}{\rho_2-\rho_1}\Big)+f^{(m)}(\rho_2)g(\rho_2)h\Big(\frac{\xi^i-\rho_1}{\rho_2-\rho_1}\Big)\Big),f^{(m)}(\xi^i)\Big]$$
$$\le\max_{\xi^i\in[\rho_1,\rho_2]}F\Big[\sum_{i=0}^{l}a_i\Big(f^{(m)}(\rho_1)g(\rho_1)h\Big(\frac{\rho_2-\xi^i}{\rho_2-\rho_1}\Big)$$
$$+f^{(m)}(\rho_2)g(\rho_2)h\Big(\frac{\xi^i-\rho_1}{\rho_2-\rho_1}\Big)\Big),f^{(m)}(\xi^i)\Big].$$

In the last part of the section, we will provide a discussion on h-convex functions.

REMARK 2. Let $f\colon[\rho_1,\rho_2]\to\mathbb{R}$ be a h-convex function. We apply inequality (1.4) on the concave function h by setting $x=\rho_1$, $y=\rho_2$,

$$u=\lambda\rho_1+(1-\lambda)\rho_2$$

and then proceed to solve for

$$\lambda=\frac{u-\rho_2}{\rho_1-\rho_2}.$$

Substituting this into the inequality (1.4), we obtain

$$f(u)\le h\left(\frac{\rho_2-u}{\rho_2-\rho_1}\right)f(\rho_1)+h\left(\frac{u-\rho_1}{\rho_2-\rho_1}\right)f(\rho_2). \tag{3.2}$$

By substituting $u = x_j$, where $j \in \{0, \dots, m\}$, into (3.2) and multiplying each inequality by λ_j, $j \in \{0, \dots, m\}$, such that $\sum_{j=0}^{m} \lambda_j = 1$, we obtain the following result

$$\lambda_j f(x_j) \leq \lambda_j h\left(\frac{\rho_2 - x_j}{\rho_2 - \rho_1}\right) f(\rho_1) + \lambda_j h\left(\frac{x_j - \rho_1}{\rho_2 - \rho_1}\right) f(\rho_2), \quad j \in \{0, \dots, m\}.$$

By summing these inequalities from $j = 0$ to $j = m$, we obtain

$$\sum_{j=0}^{m} \lambda_j f(x_j) \leq \sum_{j=0}^{m} \lambda_j h\Big(\frac{\rho_2 - x_j}{\rho_2 - \rho_1}\Big) f(\rho_1) + \sum_{j=0}^{m} \lambda_j h\Big(\frac{x_j - \rho_1}{\rho_2 - \rho_1}\Big) f(\rho_2).$$

Since h is a concave function and f is assumed to be a nonnegative, we have, by denoting $\overline{x} = \sum_{j=0}^{m} \lambda_j x_j$, the following result:

$$\begin{aligned} \sum_{j=0}^{m} \lambda_j f(x_j) &\leq \sum_{j=0}^{m} \lambda_j h\Big(\frac{\rho_2 - x_j}{\rho_2 - \rho_1}\Big) f(\rho_1) + \sum_{j=0}^{m} \lambda_j h\Big(\frac{x_j - \rho_1}{\rho_2 - \rho_1}\Big) f(\rho_2) \\ &\leq h\Big(\frac{\rho_2 - \overline{x}}{\rho_2 - \rho_1}\Big) f(\rho_1) + h\Big(\frac{\overline{x} - \rho_1}{\rho_2 - \rho_1}\Big) f(\rho_2). \end{aligned}$$

Now we apply the obtained result to the function F of two variables, which is monotonic in its first argument (compare [5] and [6]), and get

$$\begin{aligned} F\Big(\sum_{j=0}^{m} \lambda_j f(x_j), f(\overline{x})\Big) &\leq F\Big(h\Big(\frac{\rho_2 - \overline{x}}{\rho_2 - \rho_1}\Big) f(\rho_1) + h\Big(\frac{\overline{x} - \rho_1}{\rho_2 - \rho_1}\Big) f(\rho_2), f(\overline{x})\Big) \\ &\leq \max_{\xi \in [\rho_1, \rho_2]} F\Big(h\Big(\frac{\rho_2 - \xi}{\rho_2 - \rho_1}\Big) f(\rho_1) + h\Big(\frac{\xi - \rho_1}{\rho_2 - \rho_1}\Big) f(\rho_2), f(\xi)\Big). \end{aligned}$$

If we for the function F set $F(u, v) = u - v$, we denote

$$\Phi(\xi) = h\Big(\frac{\rho_2 - \xi}{\rho_2 - \rho_1}\Big) f(\rho_1) + h\Big(\frac{\xi - \rho_1}{\rho_2 - \rho_1}\Big) f(\rho_2) - f(\xi).$$

Now, we will use this result to determine the conversion of inequality (1.4) i.e. we need to find the constant μ such that

$$h(\lambda) f(x) + h(1 - \lambda) f(y) \leq f(\lambda x + (1 - \lambda) y) + \mu$$

is valid, where

$$\mu = \max_{\xi \in [\rho_1, \rho_2]} \Phi(\xi).$$

According to the Bolzano–Weierstrass theorem, determining the maximum value of the function Φ requires evaluating it at key points. The potential candidates for the global maximum are the boundary points $\xi = \rho_1$ and $\xi = \rho_2$, along with any critical points where the derivative satisfies $\Phi'(\xi) = 0$, hence to identify the global maximum, we will compute Φ at each of these points and compare their values:

1° $\Phi(\rho_1) = h(1)f(\rho_1) + h(0)f(\rho_2) - f(\rho_1)$,
2° $\Phi(\rho_2) = h(0)f(\rho_1) + h(1)f(\rho_2) - f(\rho_2)$,
3° $\Phi(\xi_0)$ where $\Phi'(\xi_0) = 0$ i.e.

$$0 = -h'\Big(\frac{\rho_2 - \xi_0}{\rho_2 - \rho_1}\Big)\frac{f(\rho_1)}{\rho_2 - \rho_1} + h'\Big(\frac{\xi_0 - \rho_1}{\rho_2 - \rho_1}\Big)\frac{f(\rho_2)}{\rho_2 - \rho_1} - f'(\xi_0).$$

Now, we need to analyze the second derivative of the function Φ

$$\Phi''(\xi) = \Big(h''\Big(\frac{\rho_2 - \xi}{\rho_2 - \rho_1}\Big)f(\rho_1) + h''\Big(\frac{\xi - \rho_1}{\rho_2 - \rho_1}\Big)f(\rho_2)\Big)\frac{1}{(\rho_2 - \rho_1)^2} - f''(\xi),$$

to determine type of local extrema. For a function f that is h-convex, determining the maximum of the function Φ is not a straightforward task. The complexity arises from the dependence on the (classical) properties of both functions f and h, which influence the analytical behavior of Φ in the given domain. For example, consider the functions f and h_k defined as $h_k(x) = x^k$, $f(x) = x^\lambda$, $x > 0$, $k, \lambda \in \mathbb{R}$. From [9], we know that the function f is h_k-convex if:

(i) $\lambda \in \langle -\infty, 0] \cup [1, \infty\rangle$ and $k \le 1$;
(ii) $\lambda \in \langle 0, 1\rangle$ and $k \le \lambda$.

In the first case, the function f is both h_k-convex and convex in the classical sense, while h_k is concave in the classical sense. In the second case, the function f is h_k-convex and concave in the classical sense, while h_k is concave for $0 < k < \lambda$ and convex for $k < 0$ (in the classical sense). We consider here the case (i).

$$\Phi(\rho_1) = h(1)f(\rho_1) + h(0)f(\rho_2) - f(\rho_1) = 1 \cdot f(\rho_1) + 0 \cdot f(\rho_2) - f(\rho_1) = 0,$$

and

$$\Phi(\rho_2) = h(0)f(\rho_1) + h(1)f(\rho_2) - f(\rho_2) = 0 \cdot f(\rho_1) + 1 \cdot f(\rho_2) - f(\rho_2) = 0.$$

Also, according to inequality (1.4), we know that the function Φ is nonnegative, $\Phi'' \le 0$, and we can conclude that Φ achieves a global maximum at some interior point ξ_0, where $\Phi'(\xi_0) = 0$.

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Gorana Aras-Gazić
University of Zagreb
Faculty Architecture
Kačićeva 26
10000 Zagreb
Croatia
e-mail: gorana.aras-gazic@arhitekt.unizg.hr

Julije Jakšetić
University of Zagreb
Faculty of Food Technology and Biotechnology, Mathematics department
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: julije.jaksetic@pbf.unizg.hr

Josip Pečarić
Department of Mathematical, Physical and Chemical Sciences
Croatian Academy of Sciences and Arts
Zrinski trg 11
10000 Zagreb
Croatia
e-mail: jopecaric@gmail.com