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POWER INEQUALITIES FOR THE NUMERICAL RADIUS IN TERMS OF GENERALIZED ALUTHGE TRANSFORM OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. In this paper we show among others that, if $S, V \in \mathcal{B}(H)$, $r \geq 1$, $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ and $pr, qr \geq 2$, then

$$\omega^{2r}(SV) \leq \frac{1}{2}\left(\left\|\frac{1}{p}|S^*|^{2pr} + \frac{1}{q}|V|^{2qr}\right\| + \omega^r\left(|V|^2|S^*|^2\right)\right).$$

Moreover, if $T = U|T|$ is the *polar decomposition* of the bounded linear operator T with U a partial isometry, then

$$\omega^{2r}(\Delta_t(T)) \leq \frac{1}{2}\left(\left\|\frac{1}{p}|T|^{2tpr} + \frac{1}{q}|T|^{2(1-t)qr}\right\| + \|T\|^{2r}\right)$$

where $\Delta_t(T) := |T|^t U |T|^{1-t}$, $t \in [0,1]$ is the *generalized Aluthge transform.*

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2.$$

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It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\| \tag{1.2}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [11], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.2):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.3}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [12] improved the inequality (1.2) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [9]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{1.4}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha |T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{1.5}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.4) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\|$$

and from (1.5) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|.$$

For more related results, see the recent books on inequalities for numerical radii [5] and [8].

Let $T = U|T|$ be the *polar decomposition* of the bounded linear operator T with U a partial isometry. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T} := |T|^{1/2} U |T|^{1/2}$, see [2].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\|\widetilde{T}\right\| \leq \|T\|$,

(ii) $\omega\left(\widetilde{T}\right) \leq \omega(T)$,

(iii) $r\left(\widetilde{T}\right) = \omega(T)$,

(iv) $\omega\left(\widetilde{T}\right) \leq \left\|T^2\right\|^{1/2} \left(\leq \|T\|\right)$, [14].

Utilizing this transform T. Yamazaki, [14] obtained in 2007 the following refinement of Kittaneh's inequality (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widetilde{T}\right)\right) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right) \tag{1.6}$$

for any operator $T \in B(H)$.

We remark that if $\widetilde{T} = 0$, then obviously $w(T) = \frac{1}{2}\|T\|$.

For $t \in (0,1)$

$$\Delta_t(T) := |T|^t U |T|^{1-t}$$

is the *generalized Aluthge transform* introduced in by Cho and Tanahashi in [7]. Recall also that *Dougal transform* is defined as $\widehat{T} := |T| U$.

Abu-Omar and Kittaneh [1] improved on inequality (1.6) using generalized Aluthge transform to prove that

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \min_{t\in\left[0,1\right]} \omega\left(\Delta_t(T)\right)\right).$$

For $t = 1$ this also gives the following result for the *Dougal transform*

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widehat{T}\right)\right).$$

Also, if we put $|T|^0 = I$, then $\Delta_0(T) := U|T| = T$.

In [4] Bunia et al. also proved that

$$\omega(T) \leq \min_{t\in[0,1]} \left\{\frac{1}{2}\omega\left(\Delta_t(T)\right) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2(1-t)}\right)\right\}.$$

For $t = 1/2$, it gives the first inequality of (1.6) as well.

Motivated by the above results, in this paper we show among others that, if S, $V \in \mathcal{B}(H)$, $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then

$$\omega^{2r}(SV) \leq \frac{1}{2}\left(\left\|\frac{1}{p}|S^*|^{2pr} + \frac{1}{q}|V|^{2qr}\right\| + \omega^r\left(|V|^2|S^*|^2\right)\right).$$

Moreover, we have

$$\omega^{2r}\left(\Delta_t\left(T\right)\right) \leq \frac{1}{2}\left(\left\|\frac{1}{p}\left|T\right|^{2tpr}+\frac{1}{q}\left|T\right|^{2(1-t)qr}\right\|+\left\|T\right\|^{2r}\right)$$

for $t \in [0,1]$

2. Main results

We recall the following vector inequality for positive operators $A \geq 0$, obtained by C.A. McCarthy in [13]

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \quad p \geq 1$$

for $x \in H$, $\|x\| = 1$ and Buzano's inequality [6],

$$|\langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{2}\left[\|x\| \|y\| + |\langle x, y\rangle|\right]$$

that holds for any x, y, $e \in H$ with $\|e\| = 1$.

Also recall the following result for operator matrices obtained by F. Kittaneh in [10]:

LEMMA 1. *Let A, B, $C \in \mathcal{B}(H)$ with A, $B \geq 0$. Then the operator matrix*

$$\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in \mathcal{B}(H \oplus H)$$

is positive, if and only if

$$|\langle Cx, y\rangle|^2 \leq \langle Ax, x\rangle \langle By, y\rangle$$

for all x, $y \in H$.

We need the following results that are of interest in themselves:

LEMMA 2. *Let A, B, $C \in \mathcal{B}(H)$ with A, $B \geq 0$. If the operator matrix*

$$\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in B(H \oplus H)$$

is positive then for $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$,

$$\omega^{2r}(C) \leq \left\|\frac{1}{p}A^{pr} + \frac{1}{q}B^{qr}\right\|. \tag{2.1}$$

If $r \geq 1$, then

$$\omega^{2r}(C) \leq \frac{1}{2}\left[\|A\|^r \|B\|^r + \omega^r(BA)\right]. \tag{2.2}$$

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then also

$$\omega^{2r}(C) \leq \frac{1}{2}\left(\left\|\frac{1}{p}A^{pr} + \frac{1}{q}B^{qr}\right\| + \omega^r(BA)\right). \tag{2.3}$$

Proof. Let $x \in H$ with $\|x\| = 1$. Then by Lemma 1 we get

$$|\langle Cx, x\rangle|^2 \leq \langle Ax, x\rangle \langle Bx, x\rangle.$$

If we take the power $r > 0$, we get, by Young and McCarthy inequalities that

$$\begin{aligned}|\langle Cx, x\rangle|^{2r} &\leq \langle Ax, x\rangle^r \langle Bx, x\rangle^r \leq \frac{1}{p}\langle Ax, x\rangle^{pr} + \frac{1}{q}\langle Bx, x\rangle^{qr} \\ &\leq \frac{1}{p}\langle A^{pr}x, x\rangle + \frac{1}{q}\langle B^{qr}x, x\rangle = \left\langle \left(\frac{1}{p}A^{pr} + \frac{1}{q}B^{qr}\right)x, x\right\rangle\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$\begin{aligned}\omega^{2r}(C) &= \sup_{\|x\|=1} |\langle Cx, x\rangle|^{2r} \leq \sup_{\|x\|=1} \left\langle \left(\frac{1}{p}A^{pr} + \frac{1}{q}B^{qr}\right)x, x\right\rangle \\ &= \left\|\frac{1}{p}A^{pr} + \frac{1}{q}B^{qr}\right\|,\end{aligned}$$

which proves (2.1).

Further, using Buzano's inequality, we have

$$|\langle Cx, x\rangle|^2 \leq \langle Ax, x\rangle \langle Bx, x\rangle \leq \frac{\|Ax\| \|Bx\| + |\langle Ax, Bx\rangle|}{2}.$$

By taking the power $r \geq 1$ and using the convexity of the power function, we get

$$\begin{aligned}|\langle Cx, x\rangle|^{2r} &\leq \left(\frac{\|Ax\| \|Bx\| + |\langle Ax, Bx\rangle|}{2}\right)^r \\ &\leq \frac{\|Ax\|^r \|Bx\|^r + |\langle Ax, Bx\rangle|^r}{2} = \frac{\|Ax\|^r \|Bx\|^r + |\langle BAx, x\rangle|^r}{2}\end{aligned} \tag{2.4}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$\begin{aligned}
\omega^{2r}(C) &= \sup_{\|x\|=1} |\langle Cx, x\rangle|^{2r} \leq \sup_{\|x\|=1} \left(\frac{\|Ax\|^r \|Bx\|^r + |\langle BAx, x\rangle|^r}{2} \right) \\
&\leq \frac{1}{2} \left(\sup_{\|x\|=1} \{\|Ax\|^r \|Bx\|^r\} + \sup_{\|x\|=1} |\langle BAx, x\rangle|^r \right) \\
&\leq \frac{1}{2} \left(\sup_{\|x\|=1} \|Ax\|^r \sup_{\|x\|=1} \|Bx\|^r + \sup_{\|x\|=1} |\langle BAx, x\rangle|^r \right) \\
&= \frac{1}{2} \left(\|A\|^r \|B\|^r + \omega^r(BA) \right)
\end{aligned}$$

and the inequality (2.2) is proved.

From (2.4) we also have

$$\begin{aligned}
|\langle Cx, x\rangle|^{2r} &\leq \frac{1}{2} \left(\|Ax\|^r \|Bx\|^r + |\langle Ax, Bx\rangle|^r \right) \\
&\leq \frac{1}{2} \left(\frac{1}{p} \|Ax\|^{pr} + \frac{1}{q} \|Bx\|^{qr} + |\langle Ax, Bx\rangle|^r \right) \\
&= \frac{1}{2} \left(\frac{1}{p} \|Ax\|^{2\frac{pr}{2}} + \frac{1}{q} \|Bx\|^{2\frac{qr}{2}} + |\langle Ax, Bx\rangle|^r \right) \\
&= \frac{1}{2} \left(\frac{1}{p} \left\langle A^2x, x \right\rangle^{\frac{pr}{2}} + \frac{1}{q} \left\langle B^2x, x \right\rangle^{\frac{qr}{2}} + |\langle Ax, Bx\rangle|^r \right) \\
&\leq \frac{1}{2} \left(\frac{1}{p} \langle A^{pr}x, x\rangle + \frac{1}{q} \langle B^{qr}x, x\rangle + |\langle Ax, Bx\rangle|^r \right) \\
&= \frac{1}{2} \left(\left\langle \left(\frac{1}{p} A^{pr} + \frac{1}{q} B^{qr} \right) x, x \right\rangle + |\langle Ax, Bx\rangle|^r \right)
\end{aligned}$$

and by taking the supremum over $\|x\| = 1$, we derive (2.3). □

Remark 1. With the assumptions in Lemma 2, if we take $p = q = 2$ and assume that $r \geq \frac{1}{2}$, then from (2.1) we get

$$\omega^{2r}(C) \leq \frac{1}{2} \left\| A^{2r} + B^{2r} \right\|,$$

which for $r = \frac{1}{2}$ gives

$$\omega(C) \leq \frac{1}{2} \|A + B\|,$$

while for $r = 1$ gives the result from [3]

$$\omega^2(C) \leq \frac{1}{2} \left\| A^2 + B^2 \right\|.$$

If we take in (2.1) $r=1$, then we get

$$\omega^2(C) \leq \left\| \frac{1}{p}A^p + \frac{1}{q}B^q \right\|$$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If we take $r=1$ in (2.2) then we get the result from [3]

$$\omega^2(C) \leq \frac{1}{2}\left[\|A\| \|B\| + \omega(BA) \right],$$

while for $r=2$,

$$\omega^4(C) \leq \frac{1}{2}\left[\|A\|^2 \|B\|^2 + \omega^2(BA) \right].$$

Also, if we take $p=q=2$ and $r \geq 1$ in (2.3), then we get

$$\omega^{2r}(C) \leq \frac{1}{2}\left(\frac{1}{2}\left\| A^{2r} + B^{2r} \right\| + \omega^r(BA) \right).$$

In particular, for $r=1$ we derive the result obtained in [3]

$$\omega^2(C) \leq \frac{1}{2}\left(\frac{1}{2}\left\| A^2 + B^2 \right\| + \omega(BA) \right).$$

Moreover, if p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and take $r=2$ in (2.3), then we derive the inequality

$$\omega^4(C) \leq \frac{1}{2}\left(\left\| \frac{1}{p}A^{2p} + \frac{1}{q}B^{2q} \right\| + \omega^2(BA) \right),$$

which for $p=q=2$ provides

$$\omega^4(C) \leq \frac{1}{2}\left(\frac{1}{2}\left\| A^4 + B^4 \right\| + \omega^2(BA) \right).$$

We also have:

THEOREM 1. *Let S, $V \in \mathcal{B}(H)$, then for $r>0$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr \geq 1$,*

$$\omega^{2r}(SV) \leq \left\| \frac{1}{p}|S^*|^{2pr} + \frac{1}{q}|V|^{2qr} \right\|. \tag{2.5}$$

If $r \geq 1$, then

$$\omega^{2r}(SV) \leq \frac{1}{2}\left[\|S\|^{2r} \|V\|^{2r} + \omega^r\left(|V|^2 |S^*|^2 \right) \right]. \tag{2.6}$$

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$\omega^{2r}(SV) \leq \frac{1}{2}\left(\left\|\frac{1}{p}|S^*|^{2pr} + \frac{1}{q}|V|^{2qr}\right\| + \omega^r\left(|V|^2|S^*|^2\right)\right). \tag{2.7}$$

Proof. Observe that the operator matrix

$$\begin{bmatrix} SS^* & SV \\ V^*S^* & V^*V \end{bmatrix} \in \mathcal{B}(H \oplus H)$$

is positive. Then by Lemma 2 for $A = |S^*|^2$, $B = |V|^2$ and $C = V^*S^*$ we get the desired inequalities (2.5)–(2.7). □

Remark 2. With the assumptions of Theorem 1, if we take $p = q = 2$ and assume that $r \geq \frac{1}{2}$, then from (2.5) we get

$$\omega^{2r}(SV) \leq \frac{1}{2}\left\||S^*|^{4r} + |V|^{4r}\right\|.$$

If we take $r = \frac{1}{2}$, then we obtain the known result

$$\omega(SV) \leq \frac{1}{2}\left\||S^*|^2 + |V|^2\right\|,$$

while for $r = 1$, the result from [3]

$$\omega^2(SV) \leq \frac{1}{2}\left\||S^*|^4 + |V|^4\right\|.$$

If we take in (2.5) $r = 1$, then we get

$$\omega^2(SV) \leq \left\|\frac{1}{p}|S^*|^{2p} + \frac{1}{q}|V|^{2q}\right\|,$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

If we take $r = 1$ in (2.6), then we get, see also [3]

$$\omega^2(SV) \leq \frac{1}{2}\left[\|S\|^2\|V\|^2 + \omega\left(|V|^2|S^*|^2\right)\right],$$

while for $r = 2$,

$$\omega^4(SV) \leq \frac{1}{2}\left[\|S\|^4\|V\|^4 + \omega^2\left(|V|^2|S^*|^2\right)\right].$$

Also, if we take $p = q = 2$ and $r \geq 1$ in (2.7), then we get

$$\omega^{2r}(SV) \leq \frac{1}{2}\left(\frac{1}{2}\left\||S^*|^{4r} + |V|^{4r}\right\| + \omega^r\left(|V|^2|S^*|^2\right)\right).$$

In particular, for $r=1$ we derive the result obtained in [3]

$$\omega^2(SV) \leq \frac{1}{2}\left(\frac{1}{2}\left\||S^*|^4+|V|^4\right\|+\omega\left(|V|^2|S^*|^2\right)\right).$$

Moreover, if p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and take $r=2$ in (2.7), then we derive the inequality

$$\omega^4(SV) \leq \frac{1}{2}\left(\left\|\frac{1}{p}|S^*|^{4p}+\frac{1}{q}|V|^{4q}\right\|+\omega^2\left(|V|^2|S^*|^2\right)\right),$$

which for $p=q=2$ provides

$$\omega^4(SV) \leq \frac{1}{2}\left(\frac{1}{2}\left\||S^*|^8+\frac{1}{q}|V|^8\right\|+\omega^2\left(|V|^2|S^*|^2\right)\right).$$

We also have:

LEMMA 3. *Let A, B, $C\in\mathcal{B}(H)$ with A, $B\geq 0$. If the operator matrix*

$$\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in B(H\oplus H)$$

is positive then for $\alpha\in[0,1]$ and $r\geq 1$,

$$\omega^2(C) \leq \left\|(1-\alpha)A^r+\alpha B^r\right\|^{1/r}\|A\|^{\alpha}\|B\|^{1-\alpha} \tag{2.8}$$

and

$$\omega^2(C) \leq \left\|(1-\alpha)A^r+\alpha B^r\right\|^{1/r}\left\|\alpha A^r+(1-\alpha)B^r\right\|^{1/r}. \tag{2.9}$$

PROOF. From Lemma 1 we have for $\alpha\in[0,1]$ that

$$\begin{aligned}|\langle Cx,x\rangle|^2 &\leq \langle Ax,x\rangle\langle Bx,x\rangle = \langle Ax,x\rangle^{1-\alpha}\langle Bx,x\rangle^{\alpha}\langle Ax,x\rangle^{\alpha}\langle Bx,x\rangle^{1-\alpha}\\ &\leq \left[(1-\alpha)\langle Ax,x\rangle+\alpha\langle Bx,x\rangle\right]\langle Ax,x\rangle^{\alpha}\langle Bx,x\rangle^{1-\alpha}\end{aligned}$$

for all $x\in H$, $\|x\|=1$.

If we take the power $r\geq 1$, then we get by the convexity of power r

$$\begin{aligned}|\langle Cx,x\rangle|^{2r} &\leq \left[(1-\alpha)\langle Ax,x\rangle+\alpha\langle Bx,x\rangle\right]^r\langle Ax,x\rangle^{r\alpha}\langle Bx,x\rangle^{r(1-\alpha)}\\ &\leq \left[(1-\alpha)\langle Ax,x\rangle^r+\alpha\langle Bx,x\rangle^r\right]\langle Ax,x\rangle^{r\alpha}\langle Bx,x\rangle^{r(1-\alpha)}.\end{aligned} \tag{2.10}$$

If we use McCarthy inequality for power $r\geq 1$, then we get

$$\begin{aligned}(1-\alpha)\langle Ax,x\rangle^r+\alpha\langle Bx,x\rangle^r &\leq (1-\alpha)\langle A^rx,x\rangle+\alpha\langle B^rx,x\rangle\\ &= \langle\left[(1-\alpha)A^r+\alpha B^r\right]x,x\rangle\end{aligned}$$

and by (2.10), we obtain

$$|\langle Cx,x\rangle|^{2r} \leq [(1-\alpha)\langle Ax,x\rangle + \alpha\langle Bx,x\rangle]^r \langle Ax,x\rangle^{r\alpha}\langle Bx,x\rangle^{r(1-\alpha)}$$
$$\leq \langle [(1-\alpha)A^r + \alpha B^r]x,x\rangle \langle Ax,x\rangle^{r\alpha}\langle Bx,x\rangle^{r(1-\alpha)}$$

for all $x \in H$, $\|x\| = 1$.

If we take the supremum over $\|x\| = 1$, then we get

$$\omega^{2r}(C) = \sup_{\|x\|=1} |\langle Cx,x\rangle|^{2r}$$
$$\leq \sup_{\|x\|=1} \left\{ \langle [(1-\alpha)A^r + \alpha B^r]x,x\rangle \langle Ax,x\rangle^{r\alpha}\langle Bx,x\rangle^{r(1-\alpha)} \right\}$$
$$\leq \sup_{\|x\|=1} \{\langle [(1-\alpha)A^r + \alpha B^r]x,x\rangle\} \sup_{\|x\|=1} \langle Ax,x\rangle^{r\alpha} \sup_{\|x\|=1} \langle Bx,x\rangle^{r(1-\alpha)}$$
$$= \|(1-\alpha)A^r + \alpha B^r\| \|A\|^{r\alpha}\|B\|^{r(1-\alpha)},$$

which is equivalent to (2.8).

Similarly

$$|\langle Cx,x\rangle|^2 \leq \langle Ax,x\rangle\langle Bx,x\rangle = \langle Ax,x\rangle^{1-\alpha}\langle Bx,x\rangle^{\alpha}\langle Ax,x\rangle^{\alpha}\langle Bx,x\rangle^{1-\alpha}$$
$$\leq [(1-\alpha)\langle Ax,x\rangle + \alpha\langle Bx,x\rangle][\alpha\langle Ax,x\rangle + (1-\alpha)\langle Bx,x\rangle],$$

which gives that

$$|\langle Cx,x\rangle|^{2r} \leq \langle [(1-\alpha)A^r + \alpha B^r]x,x\rangle \langle [\alpha A^r + (1-\alpha)B^r]x,x\rangle$$

for all $x \in H$, $\|x\| = 1$. This proves (2.9). □

Corollary 1. *With the assumption of Lemma 3, we have*

$$\omega^2(C) \leq \frac{1}{2^{1/r}} \|A^r + B^r\|^{1/r} \|A\|^{1/2}\|B\|^{1/2}$$

and

$$\omega(C) \leq \frac{1}{2^{1/r}} \|A^r + B^r\|^{1/r}.$$

We have:

Theorem 2. *Let S, $V \in \mathcal{B}(H)$, then for $\alpha \in [0,1]$ and $r \geq 1$,*

$$\omega^2(SV) \leq \left\| (1-\alpha)|S^*|^{2r} + \alpha|V|^{2r} \right\|^{1/r} \|S\|^{2\alpha}\|V\|^{2(1-\alpha)} \tag{2.11}$$

and

$$\omega^2(SV) \leq \left\| (1-\alpha)|S^*|^{2r} + \alpha|V|^{2r} \right\|^{1/r} \left\| \alpha|S^*|^{2r} + (1-\alpha)|V|^{2r} \right\|^{1/r}.$$

In particular,

$$\omega^2(SV) \leq \frac{1}{2^{1/r}} \left\| |S^*|^{2r} + |V|^{2r} \right\|^{1/r} \|S\| \|V\|$$

and

$$\omega(SV) \leq \frac{1}{2^{1/r}} \left\| |S^*|^{2r} + |V|^{2r} \right\|^{1/r}.$$

PROOF. We take in Lemma 3 $A = |S^*|^2$, $B = |V|^2$ and $C = V^*S^*$ to get the desired inequalities. □

3. Applications for generalized Aluthge transform

We have the following inequalities for one operator:

THEOREM 3. *Let $T \in \mathcal{B}(H)$ and $t \in [0,1]$, then*

$$\omega^{2r}(T) \leq \left\| \frac{1}{p} |T^*|^{2tpr} + \frac{1}{q} |T|^{2(1-t)qr} \right\|, \tag{3.1}$$

for $r > 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 1$.

Also, for $r \geq 1$,

$$\omega^{2r}(T) \leq \frac{1}{2} \left[\|T\|^{2r} + \omega^r \left(|T|^{2(1-t)} |T^*|^{2t} \right) \right]. \tag{3.2}$$

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then

$$\omega^{2r}(T) \leq \frac{1}{2} \left(\left\| \frac{1}{p} |T^*|^{2tpr} + \frac{1}{q} |T|^{2(1-t)qr} \right\| + \omega^r \left(|T|^{2(1-t)} |T^*|^{2t} \right) \right). \tag{3.3}$$

Moreover, if $\alpha \in [0,1]$ and $r \geq 1$, then

$$\omega^2(T) \leq \left\| (1-\alpha) |T^*|^{2tr} + \alpha |T|^{2(1-t)r} \right\|^{1/r} \|T\|^{2[t\alpha+(1-t)(1-\alpha)]}. \tag{3.4}$$

PROOF. If we take $S = U|T|^t$ and $V = |T|^{1-t}$ in (2.5) and observe that $SV = U|T| = T$,

$$|S^*|^2 = SS^* = U|T|^t |T|^t U^* = U|T|^{2t} U^* = |T^*|^{2t},$$

then

$$\omega^{2r}(T) \leq \left\| \frac{1}{p} |T^*|^{2tpr} + \frac{1}{q} |T|^{2(1-t)qr} \right\|,$$

which proves (3.1).

The same choice of S and V in (2.6) gives

$$\omega^{2r}(T) \leq \frac{1}{2}\left[\left\|U|T|^t\right\|^{2r}\|T\|^{2(1-t)r} + \omega^r\left(|T|^{2(1-t)}|T^*|^{2t}\right)\right]. \tag{3.5}$$

Observe that

$$|S|^2 = S^*S = |T|^t U^*U|T|^t = |T|^t|T|^t = |T|^{2t}$$

since U is an isometry on $\operatorname{ran}(|T|)$. Then $\|U|T|^t\|^{2r} = \|T\|^{2rt}$ and by (3.5) we get (3.2).

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then by (2.7) we get (3.3).

Further, if we use (2.11) for $S = U|T|^t$ and $V = |T|^{1-t}$ we also get for $\alpha \in [0,1]$ and $r \geq 1$ that

$$\begin{aligned}\omega^2(T) &\leq \left\|(1-\alpha)|T^*|^{2tr} + \alpha|T|^{2(1-t)r}\right\|^{1/r}\|T\|^{2t\alpha}\|T\|^{2(1-t)(1-\alpha)} \\ &= \left\|(1-\alpha)|T^*|^{2tr} + \alpha|T|^{2(1-t)r}\right\|^{1/r}\|T\|^{2[t\alpha+(1-t)(1-\alpha)]},\end{aligned}$$

which proves (3.4). □

Corollary 2. *Let $T \in \mathcal{B}(H)$, then*

$$\omega^{2r}(T) \leq \left\|\frac{1}{p}|T^*|^{pr} + \frac{1}{q}|T|^{qr}\right\|, \tag{3.6}$$

for $r > 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 1$.

Also, for $r \geq 1$

$$\omega^{2r}(T) \leq \frac{1}{2}\left[\|T\|^{2r} + \omega^r(|T||T^*|)\right].$$

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then

$$\omega^{2r}(T) \leq \frac{1}{2}\left(\left\|\frac{1}{p}|T^*|^{pr} + \frac{1}{q}|T|^{qr}\right\| + \omega^r(|T||T^*|)\right). \tag{3.7}$$

Moreover, if $\alpha \in [0,1]$ and $r \geq 1$, then

$$\omega^2(T) \leq \left\|(1-\alpha)|T^*|^r + \alpha|T|^r\right\|^{1/r}\|T\|.$$

Remark 3. If we take $r = 1$ in (3.6), then we get

$$\omega^2(T) \leq \left\|\frac{1}{p}|T^*|^p + \frac{1}{q}|T|^q\right\|,$$

for $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. In particular, for $p=q=2$ we get

$$\omega^2(T)\leq\frac{1}{2}\left\|\,|T^*|^2+|T|^2\,\right\|.$$

If we take $r=1$ in (3.2), then we get

$$\omega^2(T)\leq\frac{1}{2}\left[\|T\|^2+\omega\left(|T|^{2(1-t)}\,|T^*|^{2t}\right)\right].$$

If $r=1$ and $p=q=2$, then by (3.7) we get

$$\omega^2(T)\leq\frac{1}{2}\left(\frac{1}{2}\left\|\,|T^*|^2+|T|^2\,\right\|+\omega\left(|T|\,|T^*|\right)\right).$$

If we take $r=2$ and $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ in (3.7), then we obtain

$$\omega^4(T)\leq\frac{1}{2}\left(\left\|\frac{1}{p}\,|T^*|^{2p}+\frac{1}{q}\,|T|^{2q}\right\|+\omega^2\left(|T|\,|T^*|\right)\right),$$

which for $p=q=2$ gives

$$\omega^4(T)\leq\frac{1}{2}\left(\frac{1}{2}\left\|\,|T^*|^4+|T|^4\,\right\|+\omega^2\left(|T|\,|T^*|\right)\right).$$

We also have:

THEOREM 4. *Let $T\in\mathcal{B}(H)$ and $t\in[0,1]$, then*

$$\omega^{2r}\left(\Delta_t(T)\right)\leq\left\|\frac{1}{p}\left|U^*\,|T|^t\right|^{2pr}+\frac{1}{q}\,|T|^{2(1-t)qr}\right\|$$

for $r>0$, $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\, qr\geq1$.

Also

$$\omega^{2r}\left(\Delta_t(T)\right)\leq\frac{1}{2}\left[\|T\|^{2r}+\omega^r\left(|T|^{2-t}\,UU^*\,|T|^t\right)\right] \tag{3.8}$$

for $r\geq1$.

Moreover,

$$\begin{aligned}&\omega^{2r}\left(\Delta_t(T)\right)\\&\quad\leq\frac{1}{2}\left(\left\|\frac{1}{p}\left|U^*\,|T|^t\right|^{2pr}+\frac{1}{q}\,|T|^{2(1-t)qr}\right\|+\omega^r\left(|T|^{2-t}\,UU^*\,|T|^t\right)\right)\end{aligned} \tag{3.9}$$

for $r\geq1$, $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\, qr\geq2$.

Also, for $\alpha\in[0,1]$ and $r\geq1$,

$$\omega^2(\Delta_t(T))\leq\left\|(1-\alpha)|U^*|T|^t|^{2pr}+\alpha|T|^{2(1-t)qr}\right\|^{1/r}\|T\|^{2[t\alpha+(1-t)(1-\alpha)]}. \tag{3.10}$$

PROOF. If we take $S = |T|^t U$ and $V = |T|^{1-t}$ in (2.5) and observe that $SV = |T|^t U |T|^{1-t} = \Delta_t(T)$, then we get

$$\omega^{2r}(\Delta_t(T)) \leq \left\| \frac{1}{p} \left| U^* |T|^t \right|^{2pr} + \frac{1}{q} |T|^{2(1-t)qr} \right\|.$$

With the same choice and by using (2.6) we derive

$$\begin{aligned}
\omega^{2r}(\Delta_t(T)) &\leq \frac{1}{2} \left[\left\| |T|^t U \right\|^{2r} \|T\|^{2(1-t)r} + \omega^r \left(|T|^{2(1-t)} |T|^t UU^* |T|^t \right) \right] \\
&= \frac{1}{2} \left[\left\| |T|^t U \right\|^{2r} \|T\|^{2(1-t)r} + \omega^r \left(|T|^{2-t} UU^* |T|^t \right) \right] \\
&\leq \frac{1}{2} \left[\|T\|^{2tr} \|T\|^{2(1-t)r} + \omega^r \left(|T|^{2-t} UU^* |T|^t \right) \right] \\
&= \frac{1}{2} \left[\|T\|^{2r} + \omega^r \left(|T|^{2-t} UU^* |T|^t \right) \right],
\end{aligned}$$

which proves (3.8).

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then from (2.7) we get (3.9).

If we take $S = |T|^t U$ and $V = |T|^{1-t}$ in (2.11) then for $\alpha \in [0,1]$ and $r \geq 1$,

$$\begin{aligned}
\omega^2(\Delta_t(T)) &\leq \left\| (1-\alpha) \left| U^* |T|^t \right|^{2pr} + \alpha |T|^{2(1-t)qr} \right\|^{1/r} \left\| |T|^t U \right\|^{2\alpha} \|T\|^{2(1-t)(1-\alpha)} \\
&\leq \left\| (1-\alpha) \left| U^* |T|^t \right|^{2pr} + \alpha |T|^{2(1-t)qr} \right\|^{1/r} \|T\|^{2t\alpha} \|T\|^{2(1-t)(1-\alpha)} \\
&= \left\| (1-\alpha) \left| U^* |T|^t \right|^{2pr} + \alpha |T|^{2(1-t)qr} \right\|^{1/r} \|T\|^{2[t\alpha + (1-t)(1-\alpha)]},
\end{aligned}$$

which proves (3.10). □

For $t = 1/2$ we obtain the following inequalities for the Aluthge transform $\widetilde{T}$.

COROLLARY 3. *Let $T \in \mathcal{B}(H)$, then*

$$\omega^{2r}\left(\widetilde{T}\right) \leq \left\| \frac{1}{p} \left| U^* |T|^{1/2} \right|^{2pr} + \frac{1}{q} |T|^{qr} \right\|$$

for $r > 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 1$.

Also

$$\omega^{2r}\left(\widetilde{T}\right) \leq \frac{1}{2} \left[\|T\|^{2r} + \omega^r \left(|T|^{3/2} UU^* |T|^{1/2} \right) \right]$$

for $r \geq 1$.

Moreover,

$$\omega^{2r}\left(\widetilde{T}\right) \leq \frac{1}{2}\left(\left\|\frac{1}{p}\left|U^{*}\left|T\right|^{1/2}\right|^{2pr}+\frac{1}{q}\left|T\right|^{qr}\right\|+\omega^{r}\left(\left|T\right|^{3/2}UU^{*}\left|T\right|^{1/2}\right)\right)$$

for $r \geq 1$, p, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr \geq 2$.

Also, for $\alpha \in [0,1]$ *and* $r \geq 1$,

$$\omega^{2}\left(\widetilde{T}\right) \leq \left\|\left(1-\alpha\right)\left|U^{*}\left|T\right|^{1/2}\right|^{2pr}+\alpha\left|T\right|^{qr}\right\|^{1/r}\left\|T\right\|.$$

For $t=1$ we also obtain the following results for the Dougal transform $\widehat{T}$.

Corollary 4. *Let* $T \in \mathcal{B}\left(H\right)$, *then*

$$\omega^{2r}\left(\widehat{T}\right) \leq \left\|\frac{1}{p}\left|\left(\widehat{T}\right)^{*}\right|^{2pr}+\frac{1}{q}I\right\|$$

for $r > 0$, p, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr \geq 1$.

Also

$$\omega^{2r}\left(\widehat{T}\right) \leq \frac{1}{2}\left[\left\|T\right\|^{2r}+\left\|\widehat{T}\right\|^{2r}\right]$$

for $r \geq 1$.

Moreover,

$$\omega^{2r}\left(\widehat{T}\right) \leq \frac{1}{2}\left(\left\|\frac{1}{p}\left|\left(\widehat{T}\right)^{*}\right|^{2pr}+\frac{1}{q}I\right\|+\left\|\widehat{T}\right\|^{2r}\right)$$

for $r \geq 1$, p, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr \geq 2$.

Also, for $\alpha \in [0,1]$ *and* $r \geq 1$,

$$\omega^{2}\left(\widehat{T}\right) \leq \left\|\left(1-\alpha\right)\left|\left(\widehat{T}\right)^{*}\right|^{2pr}+\alpha I\right\|^{1/r}\left\|T\right\|^{2\alpha}.$$

For $t=0$ we also get:

Corollary 5. *Let* $T \in \mathcal{B}\left(H\right)$, *then*

$$\omega^{2r}\left(T\right) \leq \left\|\frac{1}{p}\left|U^{*}\right|^{2pr}+\frac{1}{q}\left|T\right|^{2qr}\right\|$$

for $r > 0$, p, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr \geq 1$.

Also

$$\omega^{2r}\left(T\right) \leq \frac{1}{2}\left[\left\|T\right\|^{2r}+\omega^{r}\left(\left|T\right|^{2}UU^{*}\right)\right]$$

for $r \geq 1$.

Moreover,

$$\omega^{2r}(T) \leq \frac{1}{2}\left(\left\|\frac{1}{p}|U^*|^{2pr} + \frac{1}{q}|T|^{2qr}\right\| + \omega^r\left(|T|^2 UU^*\right)\right)$$

for $r \geq 1$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr, qr \geq 2$.

Also, for $\alpha \in [0,1]$ *and* $r \geq 1$,

$$\omega^2(T) \leq \left\|(1-\alpha)|U^*|^{2pr} + \alpha|T|^{2qr}\right\|^{1/r} \|T\|^{2(1-\alpha)}.$$

We also have the following upper bounds for the numerical radius of generalized Aluthge transform:

THEOREM 5. *Let* $T \in \mathcal{B}(H)$ *and* $t \in [0,1]$, *then*

$$\omega^{2r}(\Delta_t(T)) \leq \left\|\frac{1}{p}|T|^{2tpr} + \frac{1}{q}|T|^{2(1-t)qr}\right\| \tag{3.11}$$

for $r > 0$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr, qr \geq 1$.

If $r \geq 1$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr, qr \geq 2$, *then*

$$\omega^{2r}(\Delta_t(T)) \leq \frac{1}{2}\left(\left\|\frac{1}{p}|T|^{2tpr} + \frac{1}{q}|T|^{2(1-t)qr}\right\| + \|T\|^{2r}\right). \tag{3.12}$$

Also, for $\alpha \in [0,1]$ *and* $r \geq 1$,

$$\omega^2(\Delta_t(T)) \leq \left\|(1-\alpha)|T|^{2tr} + \alpha|T|^{2(1-t)r}\right\|^{1/r} \|T\|^{2[t\alpha+(1-t)(1-\alpha)]}. \tag{3.13}$$

PROOF. If we take $S = |T|^t$ and $V = U|T|^{1-t}$ and observe that $SV = |T|^t U|T|^{1-t} = \Delta_t(T)$,

$$|V|^{2qr} = (V^*V)^{qr} = \left(|T|^{1-t}U^*U|T|^{1-t}\right)^{qr} = |T|^{2(1-t)qr}$$

then by (2.5) we get (3.11).

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then by the same choices in (2.7) we obtain

$$\begin{aligned}\omega^{2r}(\Delta_t(T)) &\leq \frac{1}{2}\left(\left\|\frac{1}{p}|T|^{2tpr} + \frac{1}{q}|T|^{2(1-t)qr}\right\| + \omega^r\left(|T|^{2(1-t)}|T|^{2t}\right)\right) \\ &= \frac{1}{2}\left(\left\|\frac{1}{p}|T|^{2tpr} + \frac{1}{q}|T|^{2(1-t)qr}\right\| + \|T\|^{2r}\right),\end{aligned}$$

which proves (3.12).

For $\alpha \in [0,1]$ and $r \geq 1$, then by (2.11) we get

$$\begin{aligned}\omega^2\left(\Delta_t\left(T\right)\right) &\leq \left\| \left(1-\alpha\right)\left|T\right|^{2tr} + \alpha\left|T\right|^{2(1-t)r} \right\|^{1/r} \left\|T\right\|^{2t\alpha} \left\|T\right\|^{2(1-t)(1-\alpha)} \\ &= \left\| \left(1-\alpha\right)\left|T\right|^{2tr} + \alpha\left|T\right|^{2(1-t)r} \right\|^{1/r} \left\|T\right\|^{2[t\alpha+(1-t)(1-\alpha)]},\end{aligned}$$

which proves (3.13). □

Remark 4. If we take $t = 1/2$ in (3.11) then we get

$$\omega^{2r}\left(\widetilde{T}\right) \leq \left\| \frac{1}{p}\left|T\right|^{pr} + \frac{1}{q}\left|T\right|^{qr} \right\|$$

for $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then by (3.12) we obtain

$$\omega^{2r}\left(\widetilde{T}\right) \leq \frac{1}{2}\left(\left\| \frac{1}{p}\left|T\right|^{pr} + \frac{1}{q}\left|T\right|^{qr} \right\| + \left\|T\right\|^{2r} \right).$$

Similar results can be stated for the Dougal transform $\widehat{T}$, however the details are not presented here.

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