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ON THE ARITHMETIC PROPERTIES OF A TRUNCATED POWER-RESIDUE FUNCTION AND RELATED DIOPHANTINE EQUATIONS

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Abstract. We study the arithmetic function $\kappa_m(n)$, defined as the largest m-power-free divisor of n. We establish its fundamental properties and solve several Diophantine equations involving $\kappa_m(n)$ and Euler's totient function $\phi(n)$. We prove that $\phi(n) = \kappa_m(n)$ has exactly $m+1$ solutions, characterize all solutions to $\phi(n) \mid \kappa_m(n)$, and provide a complete parametric description of solutions to $\kappa_m(n) = d$. Our results highlight the utility of the maximal divisor characterization in Diophantine analysis.

1. Introduction

Let n be a positive integer with canonical prime factorization

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $k \geq 0$, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers, and $p_1, p_2, \ldots, p_k$ are distinct primes. The Euler totient function $\phi(n)$ counts the number of positive integers up to n that are coprime to n and is given by the well-known formula:

$$\phi(n) = p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)\cdots p_k^{\alpha_k-1}(p_k-1).$$

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For a fixed integer $m \geq 2$, and for any integer $n \geq 1$, we define the arithmetic function $\kappa_m(n)$ by the product formula

$$\kappa_m(n) = \prod_{p^m \nmid n} p^{v_p(n)} \times \prod_{p^m \| n} p^{m-1}, \tag{1}$$

where the first product runs over all primes p such that p^m does not divide n, and the second product runs over all primes p such that p^m exactly divides n. Here, $v_p(n)$ denotes the p-adic valuation of n. This function extracts from n its largest divisor that is free of m-th powers. In particular, $\kappa_2(n)$ coincides with the classical square-free kernel (or radical) of n, often denoted $\mathrm{rad}(n) = \prod_{p|n} p$.

Many mathematicians have investigated Diophantine equations involving arithmetic functions, particularly those combining $\phi(n)$ with other number-theoretic functions. For example, Sándor and Bhattacharjee [8] studied certain equations and inequalities involving the arithmetical functions $\phi(n)$ and $d(n)$. In [5], Ford, Luca, and Pomerance examined common values of the arithmetic functions ϕ and σ. The equation $\phi(n) = \phi(n+1)$ was investigated by Kinlaw, Kobayashi, and Pomerance [7], who established new results on this equation. More recently, Bellaouar, Togbé, and Jakimczuk [2] studied equations of the form $d(n) = d(\phi(n))$ and established several new results on the distribution of solutions.

The function $\kappa_m(n)$ belongs to a broader class of arithmetic functions defined through constraints on prime power exponents. Related functions have been studied in the context of power-free integers [1], k-free numbers [9], and divisor problems [6]. However, the specific properties of $\kappa_m(n)$ as defined above, particularly its interaction with Euler's totient function, have not been systematically investigated.

The structural characterization of $\kappa_m(n)$ as the largest m-power-free divisor plays a central role in simplifying the Diophantine analysis.

The outline of our work is as follows. First, we provide equivalent definitions of $\kappa_m(n)$ and establish its fundamental properties, in particular multiplicativity and the characterisation of the greatest divisor. Second, we completely solve the Diophantine equation $\phi(n) = \kappa_m(n)$, thus studying the divisibility relation $\phi(n) \mid \kappa_m(n)$ and classifying all solutions. Finally, we study the equation $\kappa_m(n) = d$ for fixed d and derive the structural properties of the set of solutions.

Throughout this paper, we use standard notation from number theory: $\mathbb{N}^*$ denotes the set of positive integers, $\mathcal{P}$ denotes the set of prime numbers, $v_p(n)$ denotes the p-adic valuation of n. The symbol $p^\alpha \| n$ means that $p^\alpha \mid n$ but $p^{\alpha+1} \nmid n$. In this work, we focus on a specific truncation function that captures the largest divisor of an integer free of m-th powers.

2. Preliminaries

We now establish equivalent formulations, which are often more convenient for calculations, and thus we deal with some Diophantine equations that essentially use the function κ_m.

LEMMA 2.1. *Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ be the canonical prime factorization of n, with $\alpha_i \geq 1$ for all $i \in \{1, \ldots, k\}$. Then the following identity holds:*

$$\kappa_m(n) = \prod_{i=1}^{k} p_i^{\min\{m-1, \alpha_i\}}. \tag{2}$$

PROOF. Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ with $\alpha_i \geq 1$. For each prime p_i appearing in the factorization of n, we have $v_{p_i}(n) = \alpha_i$. Moreover, we observe the elementary equivalence:

$$p_i^m \| n \iff m \leq \alpha_i, \quad p_i^m \nmid n \iff \alpha_i < m.$$

Consequently, the set $\{p_1, \ldots, p_k\}$ admits the disjoint partition:

$$\{p_1, \ldots, p_k\} = \{p_i : \alpha_i < m\} \sqcup \{p_i : \alpha_i \geq m\}.$$

Now, let us consider the product $\prod_{i=1}^{k} p_i^{\min\{m-1, \alpha_i\}}$. We distinguish two cases:

i) If $\alpha_i < m$, then $\alpha_i \leq m-1$, whence $\min\{m-1, \alpha_i\} = \alpha_i = v_{p_i}(n)$. In this situation, $p_i^m \nmid n$ by the above equivalence.
ii) If $\alpha_i \geq m$, then $m - 1 < \alpha_i$, whence $\min\{m-1, \alpha_i\} = m-1$. In this situation, $p_i^m \| n$.

Therefore, we may rewrite

$$\prod_{i=1}^{k} p_i^{\min\{m-1, \alpha_i\}} = \left(\prod_{i:\alpha_i < m} p_i^{\alpha_i} \right) \times \left(\prod_{i:\alpha_i \geq m} p_i^{m-1} \right)$$
$$= \left(\prod_{p^m \nmid n} p^{v_p(n)} \right) \times \left(\prod_{p^m \| n} p^{m-1} \right),$$

which establishes the desired identity. □

The following lemma provides a structural interpretation of $\kappa_m(n)$ in terms of divisor theory.

LEMMA 2.2. *Let $n \in \mathbb{N}^*$ be a positive integer with canonical factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. The function $\kappa_m(n)$ defined by*

$$\kappa_m(n) = \prod_{i=1}^{k} p_i^{\min\{m-1,\alpha_i\}}$$

coincides with the largest divisor of n that is free of m-th powers. More precisely

$$\kappa_m(n) = \max\{d \in \mathbb{N}^* : d \mid n \text{ and } \forall p \in \mathcal{P}, p^m \nmid d\}. \tag{3}$$

PROOF. Denote by

$$E_n = \{d \in \mathbb{N}^* : d \mid n \text{ and } \forall p \in \mathcal{P}, p^m \nmid d\}$$

the set of divisors of n that are m-power-free. We will prove that $\kappa_m(n) = \max(E_n)$.

By Lemma 2.1, the expression (2) is equivalent to the partitioned product (1). However, for the maximality argument, the form $\prod_{i=1}^{k} p_i^{\beta_i}$ is more convenient.

i) Verification that $\kappa_m(n) \in E_n$. For each $i \in \{1, \ldots, k\}$, we have by definition $\beta_i \leq \alpha_i$. This implies that $\kappa_m(n)$ divides n. Furthermore, again by definition, $\beta_i \leq m - 1 < m$. Thus, for every prime p, the valuation $v_p(\kappa_m(n))$ is strictly less than m. Consequently, $p^m \nmid \kappa_m(n)$ for all primes p, which means that $\kappa_m(n)$ is m-power-free. Hence $\kappa_m(n) \in E_n$.

ii) Proof of maximality. Let $d \in E_n$ be an arbitrary divisor of n that is m-power-free. Write $d = \prod_{i=1}^{k} p_i^{\gamma_i}$ for its prime factorization (where some γ_i may be zero). The condition $d \mid n$ implies

$$0 \leq \gamma_i \leq \alpha_i, \quad \forall i \in \{1, \ldots, k\}. \tag{4}$$

The m-power-free condition implies that for each i, $v_{p_i}(d) < m$, i.e.

$$\gamma_i \leq m - 1, \quad \forall i \in \{1, \ldots, k\}. \tag{5}$$

Combining inequalities (4) and (5), we obtain for each index i:

$$\gamma_i \leq \min\{\alpha_i, m - 1\} = \beta_i.$$

This implies that d divides $\kappa_m(n)$ and consequently $d \leq \kappa_m(n)$. Since $\kappa_m(n)$ belongs to E_n and dominates every element of E_n, we conclude that $\kappa_m(n) = \max(E_n)$, as claimed. □

REMARK 2.3. The characterization (3) reveals that $\kappa_m(n)$ may be viewed as a natural generalization of the square-free kernel (or radical) of n, which

corresponds to the special case $m = 2$. Indeed, when $m = 2$, we recover $\kappa_2(n) = \prod_{p|n} p$, the classical radical function.

COROLLARY 2.4. *For any prime power p^α with $\alpha \geq 1$, we have:*

$$\kappa_m(p^\alpha) = \begin{cases} p^\alpha & \text{if } \alpha < m, \\ p^{m-1} & \text{if } \alpha \geq m. \end{cases} \tag{6}$$

PROOF. Formula (6) is a direct specialization of (2) to the case of a single prime factor. □

PROPOSITION 2.5. *The arithmetic function κ_m is multiplicative. That is, $\kappa_m(1) = 1$ and for any coprime positive integers a and b (i.e., $\gcd(a, b) = 1$), we have:*

$$\kappa_m(ab) = \kappa_m(a)\kappa_m(b).$$

However, κ_m is not completely multiplicative in general.

PROOF. We first verify the normalization condition. For $n = 1$, the prime factorization is empty, and the empty product in (2) yields $\kappa_m(1) = 1$.

Now let $a, b \in \mathbb{N}^*$ with $\gcd(a, b) = 1$. Write their canonical prime factorizations as:

$$a = \prod_{i=1}^{r} p_i^{\alpha_i}, \quad b = \prod_{j=1}^{s} q_j^{\beta_j},$$

where $\{p_1, \ldots, p_r\} \cap \{q_1, \ldots, q_s\} = \emptyset$ since a and b are coprime. Then the factorization of the product is:

$$ab = \prod_{i=1}^{r} p_i^{\alpha_i} \times \prod_{j=1}^{s} q_j^{\beta_j}.$$

Applying the definition given by (2) to ab, we obtain

$$\kappa_m(ab) = \prod_{i=1}^{r} p_i^{\min\{m-1,\alpha_i\}} \times \prod_{j=1}^{s} q_j^{\min\{m-1,\beta_j\}} = \kappa_m(a)\kappa_m(b),$$

which establishes the multiplicativity of κ_m. Multiplicativity follows immediately from the definition (1), as the condition $p^m \| n$ and the valuation $v_p(n)$ depend only on the local behavior at each prime.

To see that κ_m is not completely multiplicative, let us give a counterexample. Fix $m \geq 3$ and let p be any prime. Take $a = b = p^{m-1}$. Then, we get $\kappa_m(a) = \kappa_m(p^{m-1}) = p^{m-1}$, $\kappa_m(b) = p^{m-1}$, so $\kappa_m(a)\kappa_m(b) = p^{2(m-1)}$. On the other hand, $ab = p^{2(m-1)}$ with $2(m-1) \geq m$ (since $m \geq 3$). Hence, one

can see that $\kappa_m(ab) = \kappa_m(p^{2(m-1)}) = p^{m-1} \neq p^{2(m-1)} = \kappa_m(a)\kappa_m(b)$. Since $p^{m-1} \neq p^{2(m-1)}$ for $m \geq 3$, we conclude that $\kappa_m(ab) \neq \kappa_m(a)\kappa_m(b)$ when a and b are not coprime. Therefore, κ_m is multiplicative but not completely multiplicative. □

REMARK 2.6. The multiplicativity of κ_m is a direct consequence of its definition via local factors at each prime, as seen in (1). This property is essential for deriving Dirichlet series representations and mean value estimates, which remains a direction for future work.

PROPOSITION 2.7. *Let $2 \leq m < r$ be fixed integers. The equation*

$$\kappa_m(n) = \kappa_r(n)$$

holds if and only if n is m-power-free (i.e., $\forall p \in \mathcal{P}, p^m \nmid n$). In this case, $\kappa_m(n) = \kappa_r(n) = n$.

PROOF. We begin by assuming that n is m-power-free. Then for every prime p, we have $v_p(n) \leq m - 1 < r - 1$. By Lemma 2.1 we have

$$\kappa_m(n) = \prod_p p^{\min\{m-1, v_p(n)\}} = \prod_p p^{v_p(n)} = n,$$
$$\kappa_r(n) = \prod_p p^{\min\{r-1, v_p(n)\}} = \prod_p p^{v_p(n)} = n.$$

Therefore, $\kappa_m(n) = \kappa_r(n) = n$.

Now suppose that $\kappa_m(n) = \kappa_r(n)$. Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ be the canonical factorization. By Lemma 2.2, $\kappa_m(n)$ is the largest m-power-free divisor of n, and $\kappa_r(n)$ is the largest r-power-free divisor of n. Since $m < r$, every m-power-free divisor is also r-power-free. Therefore, one can see that $\kappa_m(n) \mid \kappa_r(n)$.

The equality $\kappa_m(n) = \kappa_r(n)$ implies that the largest r-power-free divisor of n is actually m-power-free. We prove this forces n to be m-power-free. Suppose, for contradiction, that n is not m-power-free. Then there exists a prime p such that $v_p(n) = \alpha \geq m$. We have:

$$v_p(\kappa_m(n)) = \min\{m-1, \alpha\} = m - 1, \quad v_p(\kappa_r(n)) = \min\{r-1, \alpha\}.$$

– If $m \leq \alpha \leq r - 1$, then $v_p(\kappa_r(n)) = \alpha \geq m > m - 1 = v_p(\kappa_m(n))$, contradiction.
– If $\alpha \geq r$, then $v_p(\kappa_r(n)) = r - 1 > m - 1 = v_p(\kappa_m(n))$ (since $r > m$), contradiction.

Therefore, no such prime exists, and n must be m-power-free. □

3. Statements and proofs of the main results

In this section, we will prove three main results related to Diophantine equations involving $\kappa_m(n)$. Here is the first theorem.

THEOREM 3.1. *Let $m \geq 2$ be a fixed integer. Then the equation*

$$\phi(n) = \kappa_m(n) \tag{7}$$

has exactly $m+1$ solutions, namely $n = 1$, $n = 2^m$, and $n = 2^\alpha \cdot 3^m$ for $\alpha = 1, 2, \ldots, m-1$, where ϕ denotes Euler's totient function.

PROOF. Recall that Euler's totient function $\phi(n)$ is defined for any integer $n \geq 1$ as the number of positive integers not exceeding n that are relatively prime to n. If $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ denotes the canonical prime factorization of n, then $\phi(n)$ admits the explicit product formula

$$\phi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1) = p_1^{\alpha_1 - 1}(p_1 - 1) \cdots p_k^{\alpha_k - 1}(p_k - 1). \tag{8}$$

This classical result is fundamental in elementary number theory (see, for instance, Apostol [1] or Hardy and Wright [6]).

According to definition (2) in Lemma 2.1, we have

$$\kappa_m(n) = \prod_{i=1}^{k} p_i^{\min\{m-1, \alpha_i\}}. \tag{9}$$

It is clear that $n = 1$ is a solution to equation (7) since $\phi(1) = 1 = \kappa_m(1)$. Let us now assume that $n > 1$ and consider its prime factorisation $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ with $\alpha_i \geq 1$ for all i. We will distinguish between four cases.

Case (i): If there exists an index i such that $\alpha_i > m$, then $\alpha_i - 1 \geq m$ while $\max_{1 \leq j \leq k}\{\min\{m-1, \alpha_j\}\} \leq m-1$. In this case, we have the valuation inequality:

$$v_{p_i}(\phi(n)) \geq \alpha_i - 1 + \sum_{j \neq i} v_{p_i}(p_j - 1) \geq \alpha_i - 1 \geq m,$$

whereas

$$v_{p_i}(\kappa_m(n)) = m - 1.$$

Thus equality is impossible. By comparing (8) and (9), we see that $\phi(n) \neq \kappa_m(n)$. Thus, there is no solution in this case.

Case (ii): If $\alpha_1 = \alpha_2 = \cdots = \alpha_k = m$, so according to (9) we get $\kappa_m(n) = p_1^{m-1} p_2^{m-1} \cdots p_k^{m-1}$, and according to (8) we have $\phi(n) = p_1^{m-1}(p_1 - 1) p_2^{m-1}(p_2 - 1) \cdots p_k^{m-1}(p_k - 1)$. Therefore, the equality $\phi(n) = \kappa_m(n)$ implies $(p_1 - 1)(p_2 - 1) \cdots (p_k - 1) = 1$, which is only possible if $k = 1$ and $p_1 = 2$. Therefore, the only solution in this case is $n = 2^m$.

Case (iii): If $\max_{1\leq i\leq k}\{\alpha_i\} < m$, for all i such that $\alpha_i \leq m-1$, then for all i, we have $\min\{m-1, \alpha_i\} = \alpha_i$. Thus, one obtains $\kappa_m(n) = \prod_{i=1}^{k} p_i^{\alpha_i} = n$, whereas $\phi(n) < n$ for all $n > 1$. Therefore, $\phi(n) \neq \kappa_m(n)$ for all n in this case and equation (7) has no solution.

Case (iv): We assume that among the exponents of the prime factorization of n, at least one is equal to m and at least one is strictly less than m. To understand this delicate situation well, we will raise four basic partial assumptions.

a) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $p_1 < p_2 < \cdots < p_k$. We claim that $p_1 = 2$. Let us reason by contradiction. Suppose that all prime factors of n are odd (i.e., $p_i \geq 3$ for all i). Then:

– $\phi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$ is even, since each factor $p_i - 1$ is even.
– $\kappa_m(n) = \prod_{i=1}^{k} p_i^{\min\{m-1, \alpha_i\}}$ is odd, as it is a product of odd primes.

This contradicts $\phi(n) = \kappa_m(n)$. Therefore, we must have $p_1 = 2$.

b) We now determine the possible value of α_1. There are two possibilities:

– If $\alpha_1 = m$ and all other exponents also equal m, then we are in Case (ii), which yields the solution $n = 2^m$. However, Case (iv) requires at least one exponent to be strictly less than m. If $\alpha_1 = m$ but some $\alpha_k < m$ for $k \geq 2$, then comparing p_k-adic valuations $v_{p_k}(\phi(n)) = \alpha_k - 1$, $v_{p_k}(\kappa_m(n)) = \min\{m-1, \alpha_k\} = \alpha_k$ (since $\alpha_k < m$). This gives $\alpha_k - 1 = \alpha_k$, a contradiction. Hence $\alpha_1 \neq m$.
– Let $\alpha_1 < m$. This is consistent with the assumptions in Case (iv). Under this assumption, we compute the 2-adic valuations of both sides of the equation $\phi(n) = \kappa_m(n)$: $v_2(\phi(n)) = \alpha_1 - 1$, $v_2(\kappa_m(n)) = \min\{m-1, \alpha_1\} = \alpha_1$ (since $\alpha_1 < m$). Consequently, $\kappa_m(n)$ contains exactly one more factor of 2 than $\phi(n)$ from the power of 2 alone. This deficit must be compensated by the factors $(p_i - 1)$ arising from the odd prime divisors p_i of n, each of which contributes at least one factor of 2 to $\phi(n)$. This observation will be crucial in restricting the number of odd prime factors of n. Therefore, in Case (iv), we must have $\alpha_1 < m$.

c) Since $\alpha_1 < m$, there must exist at least one index $k \geq 2$ such that $\alpha_k = m$ (by the hypothesis of Case (iv)). Without loss of generality, after relabeling, we may assume $\alpha_k = m$.

The equation $\phi(n) = \kappa_m(n)$ becomes

$$2^{\alpha_1 - 1} \cdot p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_k^{m-1}(p_k - 1) = 2^{\alpha_1} \cdot p_2^{\min\{m-1, \alpha_2\}} \cdots p_{k-1}^{\min\{m-1, \alpha_{k-1}\}} \cdot p_k^{m-1}.$$

Canceling $2^{\alpha_1 - 1}$ from both sides:

$$p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_k^{m-1}(p_k - 1) = 2 \cdot p_2^{\min\{m-1, \alpha_2\}} \cdots p_{k-1}^{\min\{m-1, \alpha_{k-1}\}} \cdot p_k^{m-1}. \tag{10}$$

We perform a 2-adic evaluation analysis where we see that each factor $(p_i - 1)$ for $i \geq 2$ is even (since p_i is odd). The right-hand side of (10) contains exactly one factor of 2. Therefore, there can be only one factor of the form $(p_i - 1)$ on the left-hand side. This implies $k = 2$, meaning n has exactly two distinct prime factors $n = 2^\alpha \cdot p^m$, where $\alpha = \alpha_1 < m$.

d) Substituting $n = 2^\alpha p^m$ into $\phi(n) = \kappa_m(n)$:

$$\phi(2^\alpha p^m) = 2^{\alpha-1} \cdot p^{m-1}(p-1), \quad \kappa_m(2^\alpha p^m) = 2^{\min\{m-1,\alpha\}} \cdot p^{m-1}.$$

The equality becomes $2^{\alpha-1} \cdot p^{m-1}(p-1) = 2^{\min\{m-1,\alpha\}} \cdot p^{m-1}$. Canceling p^{m-1} and using $\alpha < m$ (so $\min\{m-1, \alpha\} = \alpha$):

$$2^{\alpha-1}(p-1) = 2^\alpha \implies p - 1 = 2 \implies p = 3.$$

If we summarize the study conducted in Case (iv), we find that the solutions in this case are exactly as follows:

$$n = 2^\alpha \cdot 3^m \quad \text{with } \alpha \in \{1, 2, \ldots, m-1\}.$$

This yields exactly $m - 1$ distinct solutions. By combining the above four cases, we immediately obtain all $m+1$ solutions to the equation $\phi(n) = \kappa_m(n)$, namely $n = 1$, $n = 2^m$, and $n = 2^\alpha \cdot 3^m$ for $\alpha = 1, 2, \ldots, m-1$. This completes the proof of the theorem. □

The second result involves the divisibility of $\kappa_m(n)$ by $\phi(n)$.

THEOREM 3.2. *Let $m \geq 2$ be a fixed integer. The divisibility $\phi(n) \mid \kappa_m(n)$ holds if and only if n is of one of the following conditions:*

i) $n = 1$;
ii) $n = 2^\alpha$ *with* $1 \leq \alpha \leq m$;
iii) $n = 2^\alpha \cdot 3^\beta$ *with* $1 \leq \alpha \leq m-1$ *and* $1 \leq \beta \leq m$.

Consequently, the equation $\phi(n) \mid \kappa_m(n)$ has exactly $m^2 + 1$ positive integer solutions.

PROOF. Recall that for $n = \prod_{i=1}^k p_i^{\alpha_i}$,

$$\phi(n) = \prod_{i=1}^k p_i^{\alpha_i - 1}(p_i - 1), \quad \kappa_m(n) = \prod_{i=1}^k p_i^{\min\{m-1,\alpha_i\}}.$$

The condition $\phi(n) \mid \kappa_m(n)$ is equivalent to, for every prime p, we have $v_p(\phi(n)) \leq v_p(\kappa_m(n))$.

i) If $n = 1$, then we have $\phi(1) = 1$ and $\kappa_m(1) = 1$. Clearly, $1 \mid 1$, so $n = 1$ is a solution.

ii) If $n = p^\alpha$, then one can see that $\phi(p^\alpha) = p^{\alpha-1}(p-1)$, $\kappa_m(p^\alpha) = p^{\min\{m-1,\alpha\}}$. If $p^{\alpha-1}(p-1) \mid p^{\min\{m-1,\alpha\}}$, we need to consider two conditions:

– Condition A: $\alpha - 1 \leq \min\{m-1, \alpha\}$.
– Condition B: $(p-1) \mid p^{\min\{m-1,\alpha\}-(\alpha-1)}$.

According to Condition B, if $p > 2$, then $p-1$ is even and $p-1 \geq 2$. But p^k is odd, for all $k \geq 0$. Therefore, $(p-1) \nmid p^k$, for any k. We deduce that $p = 2$.

For $p = 2$, Condition A becomes $\alpha - 1 \leq \min\{m-1, \alpha\}$.

– If $\alpha \leq m-1$, then $\min\{m-1, \alpha\} = \alpha$, so $\alpha - 1 \leq \alpha$. This is always true.
– If $\alpha \geq m$, then $\min\{m-1, \alpha\} = m-1$. So, we get $\alpha - 1 \leq m-1$, which gives $\alpha \leq m$.

Therefore, we must have $1 \leq \alpha \leq m$. So, we see that all the solutions must be of the form $n = 2^\alpha$ with $1 \leq \alpha \leq m$.

iii) If n has at least two distinct prime factors, then we write $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with $k \geq 2$ and $p_1 < p_2 < \cdots < p_k$. Thus, we have $\phi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$.

In order for us to continue, we have this key observation: For each i, the factor $p_i - 1$ must divide $\kappa_m(n)$. Since $\kappa_m(n) = \prod_{j=1}^{k} p_j^{\min\{m-1, \alpha_j\}}$, any prime divisor q of $p_i - 1$ must divide $\kappa_m(n)$, which means $q \in \{p_1, p_2, \ldots, p_k\}$.

Now consider the smallest prime factor p_1 of n. All prime divisors of $p_1 - 1$ are strictly less than p_1 (since $p_1 - 1 < p_1$). But these divisors must belong to $\{p_1, p_2, \ldots, p_k\}$, all of which are $\geq p_1$. Therefore, the only possibility is $p_1 - 1 = 1$, i.e. $p_1 = 2$.

a) If $n = 2^\alpha \cdot p^\beta$, with p an odd prime, then we have

$$\phi(n) = 2^{\alpha-1} \cdot p^{\beta-1}(p-1), \quad \kappa_m(n) = 2^{\min\{m-1,\alpha\}} \cdot p^{\min\{m-1,\beta\}}.$$

Since $(p-1) \mid 2^{\min\{m-1,\alpha\}} \cdot p^{\min\{m-1,\beta\}}$ and $\gcd(p-1, p) = 1$ (as p is odd), we must have $(p-1) \mid 2^{\min\{m-1,\alpha\}}$. Thus, $p-1$ is a power of 2. Furthermore, we have $v_2(\phi(n)) = (\alpha - 1) + v_2(p-1)$, $v_2(\kappa_m(n)) = \min\{m-1, \alpha\}$. Therefore, we obtain $\alpha - 1 + v_2(p-1) \leq \min\{m-1, \alpha\}$.

This results in the following two sub-cases.

– If $\alpha \leq m-1$, then $\min\{m-1, \alpha\} = \alpha$. So, one can see that $\alpha - 1 + v_2(p-1) \leq \alpha \implies v_2(p-1) \leq 1$. Thus, $p-1 \in \{1, 2\}$, giving $p \in \{2, 3\}$. Since p is odd, we must have $p = 3$.
– If $\alpha \geq m$, then $\min\{m-1, \alpha\} = m-1$. So, we get $\alpha - 1 + v_2(p-1) \leq m-1 \implies v_2(p-1) \leq m - \alpha$. Since $\alpha \geq m$, we have $m - \alpha \leq 0$. Hence, $v_2(p-1) = 0$, which gives $p = 2$. This contradicts the fact that p is odd. In conclusion, we must have $\alpha \leq m-1$ and $p = 3$.

To obtain a condition on 3-adic valuations (for $p = 3$), we have $v_3(\phi(n)) = \beta - 1$, $v_3(\kappa_m(n)) = \min\{m-1, \beta\}$. Thus, we see that $\beta - 1 \leq \min\{m-1, \beta\}$.

– If $\beta \leq m-1$, then $\beta - 1 \leq \beta$, which is always true.
– If $\beta \geq m$, then $\beta - 1 \leq m-1$, so $\beta \leq m$. Therefore, $1 \leq \beta \leq m$.

So, in this case the solutions are integers of the form $n = 2^\alpha \cdot 3^\beta$, with $1 \leq \alpha \leq m-1$ and $1 \leq \beta \leq m$.

b) Suppose $n = 2^\alpha \cdot 3^\beta \cdot q^\gamma \cdots$ with $q \geq 5$. Let $q \geq 5$ be the smallest prime divisor of n greater than 3. Then every prime divisor r of $q-1$ satisfies $r < q$.

Since $r \mid n$, the minimality of q implies $r \in \{2, 3\}$. Hence $q - 1 = 2^a 3^b$ with $a \geq 1$, since q is odd.

The valuation argument then gives

$$v_2(\phi(n)) \geq (\alpha - 1) + 1 + a = \alpha + a,$$

whereas

$$v_2(\kappa_m(n)) = \min\{m - 1, \alpha\}.$$

The divisibility condition would therefore imply

$$\alpha + a \leq \min\{m - 1, \alpha\},$$

which is impossible because $a \geq 1$.

We conclude from the study that it is impossible to have three or more distinct prime factors. The total number of solutions for the equation stated in the theorem is equal to $1 + m + (m - 1)m = 1 + m + m^2 - m = m^2 + 1$. This completes the proof of the theorem. □

The last result involves the Diophantine equation $\kappa_m(n) = d$.

THEOREM 3.3. *Let $m \geq 2$ be a fixed integer and let $d \in \mathbb{N}^*$. The equation*

$$\kappa_m(n) = d$$

admits a positive integer solution n if and only if d is m-power-free. In this case, the set of all solutions is

$$S_d = \left\{ n = d \cdot \prod_{\substack{p \mid d \\ v_p(d) = m-1}} p^{\gamma_p} : \gamma_p \geq 0 \right\}.$$

Equivalently, n verifies one of the following conditions:
- *For every prime $p \mid d$ with $v_p(d) < m - 1$, we have $v_p(n) = v_p(d)$;*
- *For every prime $p \mid d$ with $v_p(d) = m - 1$, we get $v_p(n) \geq m - 1$;*
- *For every prime $p \nmid d$, we obtain $v_p(n) = 0$.*

PROOF. By Lemma 2.2, $\kappa_m(n)$ is the largest m-power-free divisor of n. In particular, $\kappa_m(n)$ itself is always m-power-free. Therefore, if $\kappa_m(n) = d$, then d must be m-power-free. This establishes the necessary condition on d.

Conversely, assume d is m-power-free. We characterize all n such that $\kappa_m(n) = d$. Let $n = \prod_p p^{\alpha_p}$ and $d = \prod_p p^{\delta_p}$ be their prime factorizations. The condition $\kappa_m(n) = d$ is equivalent to $\min\{m - 1, \alpha_p\} = \delta_p$ for all primes p. Therefore, we can observe the following cases:
- If $p \nmid d$ (i.e., $\delta_p = 0$), then $\min\{m - 1, \alpha_p\} = 0$, which forces $\alpha_p = 0$. Thus, we see that $p \nmid n$.

– If $p \mid d$ with $\delta_p < m-1$, then $\min\{m-1, \alpha_p\} = \delta_p < m-1$, which implies that $\alpha_p = \delta_p$. Thus, we obtain $v_p(n) = v_p(d)$.
– If $p \mid d$ with $\delta_p = m-1$, then $\min\{m-1, \alpha_p\} = m-1$, which is satisfied for any $\alpha_p \geq m-1$. Thus, we deduce that $v_p(n) \geq m-1 = v_p(d)$.

Equivalently, we can write $v_p(n) = v_p(d) + \gamma_p$, where $\gamma_p \geq 0$ is arbitrary. Combining all cases, n must be of the form

$$n = \Bigg(\prod_{\substack{p \mid d \\ v_p(d) < m-1}} p^{v_p(d)} \Bigg) \cdot \Bigg(\prod_{\substack{p \mid d \\ v_p(d) = m-1}} p^{\gamma_p} \Bigg).$$

This is equivalent to

$$n = d \cdot \prod_{\substack{p \mid d \\ v_p(d) = m-1}} p^{\gamma_p}, \quad \gamma_p \geq 0.$$

This completes the proof of the theorem. □

Remark 3.4. The results established in this paper resonate with the celebrated conjecture of Carmichael [3], which asserts that the equation $\phi(x) = n$ cannot have a unique solution for any positive integer n. While Carmichael's conjecture remains open despite significant progress by K. Ford [4], our investigation reveals a complementary perspective through the lens of the truncated power-residue function $\kappa_m(n)$.

Specifically, Theorem 3.1 demonstrates that the equation $\phi(n) = \kappa_m(n)$ admits exactly $m+1$ solutions for each fixed $m \geq 2$, namely $n \in \{1, 2^m\} \cup \{2^\alpha \cdot 3^m : 1 \leq \alpha \leq m-1\}$. This explicit classification shows that when the codomain is restricted to values of the form $\kappa_m(n)$, the preimage under ϕ is never a singleton, a phenomenon consistent with Carmichael's assertion for the classical equation.

Moreover, the structural characterization of $\kappa_m(n)$ as the largest m-power-free divisor of n (Lemma 2.2) provides a novel tool for analyzing the multiplicity structure of ϕ-preimages. An interesting open question, and a possible direction for future research, is to investigate whether for any m-power-free integer d in the range of κ_m, the equation $\phi(x) = d$ admits at least two solutions whenever it admits one. This suggests a potential refinement of Carmichael's conjecture restricted to m-power-free codomains, which could be more tractable than the general case.

Future work could investigate whether the techniques developed here, in particular the p-adic valuation analysis and the maximal divisor characterization, extend to broader classes of arithmetic functions and yield new insights into the long-standing problem of solution multiplicity for $\phi(x) = n$.

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