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## Report of Meeting

# The Twenty-fifth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities Kościelisko-Zakopane, Poland, February 3–6, 2026

The Twenty-fifth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities was held in Hotel Rewita in Kościelisko, Poland, from February 3 to February 6, 2026. The meeting was organized by the Institute of Mathematics of the University of Silesia and was dedicated to the memory of Professor Roman Ger.

Ten participants came from the University of Debrecen (Hungary), five from the University of Silesia in Katowice (Poland), three from each of the following universities: University of the National Education Commission, Kraków (Poland), University of Rzeszów (Poland), University of Zielona Góra (Poland), one from each of the following universities: Budapest University of Technology and Economics (Hungary), Kazimierz Wielki University (Poland), Jan Kochanowski University of Kielce (Poland), and one with a dual affiliation Corvinus University of Budapest and HUN-REN Alfréd Rényi Institute of Mathematics (Hungary).

Professor Maciej Sablik opened the Seminar and welcomed the participants.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, equations on abstract algebraic structures, functional inequalities, regularity properties of the solutions of certain functional equations, orthogonality equations, inequalities involving stochastic orderings, characterization of premiums for insurances with the use of functional equations, functional equations and inequalities involving mean values, iteration theory, spectral synthesis and spectral analysis.

Interesting discussions were generated by the talks.

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There were also a Problems and remarks sessions and a festive dinner.

The closing address was given by Professor Zsolt Páles. His invitation to the Twenty Sixth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in 2027 in Hungary was gratefully accepted.

Summaries of the talks in alphabetical order of the authors follow in Section 1, problems and remarks in Section 2, and the list of participants in the final section.

## 1. Abstracts of talks

Karol Baron: *On some linear functional equations with continuous and bounded solutions*

The research of Witold Jarczyk, initiated in [3] and continued in [2] and [4], showed that sets of first order non-homogeneous linear functional equations having a continuous solution are small from the topological point of view. We return to these qualitative considerations with higher orders equations in mind.

In the talk we will briefly describe our approach to the equation

$$\varphi(x) = \int_\Omega \varphi\big(f(x,\omega)\big)\mathbb{P}(d\omega) + F(x)$$

assuming that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, $(X, \rho)$ is a complete and separable metric space, $f\colon X \times \Omega \to X$ is continuous with respect to the first variable and measurable for $\mathcal{A}$ with respect to the second variable, and

$$\int_\Omega \rho\big(f(x,\omega), f(z,\omega)\big)\mathbb{P}(d\omega) \le \beta\big(\rho(x,z)\big) \quad \text{for } x, z \in X$$

with a concave $\beta\colon [0,\infty) \to [0,\infty)$ such that $\beta(t) < t$ for $t \in (0,\infty)$.

Based on the ideas from [1] and slightly strengthening the assumption made there about the function $f$, we now consider solutions defined on complete and separable metric spaces, and not, as in [1], on compact spaces.

References


[1] K. Baron, *On some linear functional equations with continuous solutions*, Aequationes Math. **99** (2025), no. 6, 2689–2698.

[2] W. Jarczyk, *A category theorem for linear functional equations in the indeterminate case*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **29** (1981), no. 7–8, 371–372.

[3] W. Jarczyk, *On a set of functional equations having continuous solutions*, Glasnik Mat. Ser. III **17(37)** (1982), no. 1, 59–64.

[4] W. Jarczyk, *On linear functional equations in the determinate case*, Glasnik Mat. Ser. III **18(38)** (1983), no. 1, 91–102.

Zoltán Boros: *Pairs of functions fulfilling an alternative or a cancellative equation on a big set* (Joint work with Rayene Menzer)

In our investigations on the alternative equation

$$f(x)g(y) = 0\,, \tag{1}$$

we have established the following result. Let $k\,,\ m \in \mathbb{N}$. Suppose that $f\colon \mathbb{R}^k \to \mathbb{C}$ and $g\colon \mathbb{R}^m \to \mathbb{C}$ satisfy (1) for all $(x,y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ has a positive $k+m$ dimensional Lebesgue measure or $D$ is a second category Baire set. Then $f$ or $g$ vanishes on a set of positive ($k$ or $m$ dimensional) Lebesgue measure or on a second category Baire set, respectively.

Motivated by a question of Peter Eliaš, investigations were extended to the functional equation

$$F(f(x), g(y)) = 0\,, \tag{2}$$

where $F$ is a given operation. Assuming that $S \subseteq \mathbb{C} \times \mathbb{C}$ and $F\colon S \to \mathbb{C}$ is locally cancellative in the sense that, for all $z \in \mathbb{C}$ and $w \in \mathbb{C}$, the equation $F(z,w) = 0$ has at most one solution in its first (resp., second) variable if the second (resp., first) variable is fixed, we can establish the following theorem: If $f\colon \mathbb{R}^k \to \mathbb{C}$ and $g\colon \mathbb{R}^m \to \mathbb{C}$ satisfy (2) for all $(x,y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ has a positive $k+m$ dimensional Lebesgue measure or $D$ is a second category Baire set, then $f$ and $g$ are constants on sets of positive ($k$ and $m$ dimensional) Lebesgue measure or on second category Baire sets, respectively.

Pál Burai: *A negative result concerning an ordering generated by matrix groups* (Joint work with Hanna Koller)

Let $G \subset GL_n(\mathbb{R})$ be a matrix group, where $GL_n(\mathbb{R})$ denotes the general linear group of degree $n$.

An $x \in \mathbb{R}^n$ is majorized by a $y \in \mathbb{R}^n$ with respect to the group $G$, if

$$x \in conv(O_G(y)),$$

where $O_G(y)$ is the orbit of $y$ under $G$ and *conv* denotes the convex hull operator.

In this talk we present a negative result concerning the characterization of ordering generated by cyclic, doubly stochastic matrices. Possible further research directions are also mentioned.

Jacek Chmieliński: *On approximate preservation of Birkhoff–James orthogonality* (Joint work with Rafał Stypka)

We consider additive operators that approximately preserve (or reverse) the Birkhoff–James orthogonality relation. In particular, we show that such operators must be linear.

Reference

[1] J. Chmieliński and R. Stypka, *Additive operators approximately preserving Birkhoff–James orthogonality*, Aequationes Math. **99** (2025), no. 6, 2847–2854.

Jacek Chudziak: *On comparability of buying and selling prices*

The buying price is a real number that expresses the maximal amount at which the investor is willing to buy a risky asset $X$. If the investor has already received $X$, then the selling price expresses the minimal price at which he is willing to sell $X$. In the expected utility model, where the risks are represented by essentially bounded random variables on a given probability space, the buying price $B_u(w, X)$ for $X$ at a wealth level $w$ is defined implicitly through the equation

$$E[u(w + X - B_u(w, X))] = u(w),$$

where $u : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function. The selling price $S_u(w, X)$ for $X$ at a wealth level $w$ is given by

$$E[u(w + X)] = u(w + S_u(w, X)).$$

Several properties of the buying and selling prices under the expected utility model have been investigated in [1]. In the talk, we introduce the buying and selling prices under rank-dependent utility and we present some results concerning the comparability problem for such prices.

Reference

[1] M. Lewandowski, *Buying and selling price for risky lotteries and expected utility theory with gambling wealth*, J. Risk Uncertain. **48** (2014), no. 3, 253–283.

Borbála Fazekas: *Convergence of sequences of ordered selections* (Joint work with István Fazekas)

Our results are generalizations of well-known theorems on convergence of permutation sequences to permutons. We introduce a convergence notion for ordered selections, which is based on subpermutation densities and convergences of the marginal distributions. We also introduce a family of probability measures called generalized permutons and we embed the set of ordered selections to the set of generalized permutons. We prove that any convergent sequence of ordered selections has a limit which is a generalized permuton. Moreover, any generalized permuton is the limit of a sequence of ordered selections.

Reference

[1] B. Fazekas and I. Fazekas, *Convergence of sequences of ordered selections*, preprint.

Attila Gilányi: *On conditional monomial and polynomial functional equations* (Dedicated to the memory of Roman Ger)

Problems related to extensions and conditionally fulfilled properties often occur in the field of functional equations and their applications (cf., e.g., the 'Program' [1] announced by János Aczél). Related to this topic, some results for conditional monomial and polynomial functional equations are presented.

Reference

[1] J. Aczél, *5. Remark*, in: *Report of meeting. The Forty-second International Symposium on Functional Equations, June 20–27, 2004, Opava, Czech Republic*, Aequationes Math. **69** (2005), no. 1–2, p. 183.

Dorota Głazowska: *Complementary means for generalized classical weighted means with respect to classical weighted means* (Joint work with Janusz Matkowski)

Let $\mathcal{A}_\lambda$, $\mathcal{G}_\lambda$ and $\mathcal{H}_\lambda$, denote, respectively, the bivariable classical weighted arithmetic, geometric and harmonic means with weight $\lambda \in (0,1)$.

Under some simple conditions on the real functions $f$, $g$ and $h$, defined on an interval $I \subset \mathbb{R}$ or $I \subset (0,\infty)$, the bivariable functions $A_f$, $G_g$ and $H_h$, given respectively by

$$A_f(x,y) = f(x) + y - f(y), \quad G_g(x,y) = \frac{g(x)}{g(y)}y,$$
$$H_h(x,y) = \frac{xy}{x - h(x) + h(y)},$$

are means on the interval $I$. These means are natural generalization, respectively, of the classical weighted arithmetic, geometric and harmonic means.

Fixing arbitrarily $\lambda \in (0,1)$ and choosing for $K\colon I^2 \to I$ one of these three classical weighted means, for arbitrary mean $M\colon I^2 \to I$, we examine when the function $N\colon I^2 \to \mathbb{R}$ satisfying the equality

$$K \circ (M,N) = K$$

is a mean, that is when the mean $K$ is invariant with respect to the mean type-mapping $(M,N) : I^2 \to I^2$, where $I \subset \mathbb{R}$ or $I \subset (0,+\infty)$ is an interval.

Recall that if $M$, $N$ are continuous and strict means, then there exists a unique $(M,N)$-invariant mean. In the case when $K$ is the unique $(M,N)$-invariant mean, we say that $N$ is a complementary mean to $M$ with respect to $K$.

The obtained results are applied to determine complementary means for generalized classical weighted means with respect to classical weighted means.

Eszter Gselmann: *Perturbations of Cauchy differences* (Joint work with Tomasz Małolepszy and Janusz Matkowski)

In this talk we investigate functional equations arising from perturbations of Cauchy differences. We study equations of the form

$$f(x+y)-f(x)-f(y)=B(x,y) \quad \text{or} \quad f(xy)-f(x)f(y)=B(x,y),$$

where $B$ is a biadditive mapping, and also more general cases where the inhomogeneity depends on unknown functions

$$\begin{aligned} f(x+y)-f(x)-f(y)&=\alpha xy,\\ f(x+y)-f(x)-f(y)&=\alpha(xy),\\ f(x+y)-f(x)-f(y)&=\alpha(x)(y). \end{aligned}$$

Our results extend previous works on the bilinearity of the Cauchy exponential difference by Horst Alzer and Janusz Matkowski. We characterize solutions under various structural and regularity assumptions, including additive and exponential Cauchy differences, and show that solutions often reduce to additive functions, exponential polynomials, or combinations thereof. For Levi-Civita type equations, we provide explicit representations of solutions in terms of exponential polynomials. Furthermore, we determine conditions under which real-valued solutions exist and describe their forms. The talk concludes with open problems concerning equations of similar type that cannot be solved by the methods presented here, suggesting directions for future research.

Wojciech Jabłoński: *Weak topologies in rings of formal power series*

The operation of substitution (composition) for formal power series is easily defined in a maximal ideal of the ring of formal power series. A definition of the substitution in whole ring of formal power series was given without any explanation in [1, 2, 3, 4]. In order to show the sense of this definition we introduce topologies which are essentially weaker than the strong topology defined in purely algebraic way in the ring of formal power series. As applications we obtain properties of the Hasse derivative and an elementary proof of the existence of composition in a ring of formal power series of several variables over a topological ring.

References


[1] D. Bugajewski, A. Galimberti, and P. Maćkowiak, *On composition and Right Distributive Law for formal power series of multiple variables*, Aequationes Math. **99** (2025), no. 1, 21–35.

[2] X.-X. Gan, *A generalized chain rule for formal power series*, Commun. Math. Anal. **2** (2007), no. 1, 37–44.

[3] X.-X. Gan and D. Bugajewski, *A note on formal power series*, Comment. Math. Univ. Carolin. **51** (2010), no. 4, 595–604.

[4] X.-X. Gan and N. Knox, *On composition of formal power series*, Int. J. Math. Math. Sci. **30** (2002), no. 12, 761–770.

Justyna Jarczyk: *Iterates, fractional iterates and semiflows of Möbius transformations* (Joint work with Steven Finch and Witold Jarczyk)

Let $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be any Möbius transformation given by

$$f(z) = \begin{cases} \frac{az+b}{cz+d}, & \text{when} \quad z \in \mathbb{C} \setminus \left\{-\frac{d}{c}\right\}, \\ \infty, & \text{when} \quad z = -\frac{d}{c}, \\ \frac{a}{c}, & \text{when} \quad z = \infty, \end{cases}$$

where $a, b, c, d \in \mathbb{C}$ are such that $c \neq 0$ and $ad - bc \neq 0$. Using a recurrent method we find a simple explicit form of iterates of $f$. Then, in a neighbourhood of the attractive fixed point $\xi$ of $f$, we determine effective formulas for solutions $\sigma$ of the Schröder equation

$$\sigma(f(z)) = f'(\xi)\sigma(z)$$

which are differentiable at $\xi$. Using this we can find the form of fractional iterates (i.e. iterative roots) of $f$ and, in some cases, embed $f$ into a flow.

Witold Jarczyk: *When are local properties of solutions of iterative functional equations actually global?* (Joint work with Paweł Pasteczka)

The extendability of properties of solutions of simultaneous iterative equations is considered. It turns out that, under some additional assumptions, if the solution of

$$\varphi(f_t(x)) = g(x, \varphi(x)), \quad t \in T,$$

exists, then it is not only unique but its local properties are actually global. More precisely, in the case when the indicated function solves the simultaneous equations, each its local property can be propagated to the entire domain.

Gergely Kiss: *Questions on the regularity of bisymmetric functions*

In recent years, there has been renewed interest in the role of regularity assumptions in Aczél's characterization theorem for (weighted) quasi-arithmetic means, with particular emphasis on how (and to what extent) continuity can be eliminated. Closely related work investigates the regularity of monotone bisymmetric functions defined on real intervals, aiming to understand which structural conclusions can be derived from purely algebraic identities together with monotonicity.

These developments also leave a number of questions open. In this talk, I will briefly survey several recent results on the topic and then turn to questions that, in my view, merit broader attention. Starting from the two-variable setting, I will formulate and discuss concrete questions. I will also present multivariable analogues and pose further open problems.

Most of the results to be discussed are joint work with Pál Burai and Patrícia Szokol.

References


[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia Math. Appl., 31, Cambridge University Press, Cambridge, 1989.

[2] P. Burai, G. Kiss, and P. Szokol, *Characterization of quasi-arithmetic means without regularity condition*, Acta Math. Hungar. **165** (2021), no. 2, 474–485.

[3] P. Burai, G. Kiss, and P. Szokol, *A dichotomy result for strictly increasing bisymmetric maps*, J. Math. Anal. Appl. **526** (2023), no. 2, Paper No. 127269, 9 pp.

[4] G. Kiss, *On noncontinuous bisymmetric strictly monotone operations*, submitted, 2026. Available at arXiv:2601.16247v2.


Tibor Kiss: *Quasi graph-additive functions – on a conjecture of Janusz Matkowski*

On September 18, 2025, the fourth day of the *21st International Conference on Functional Equations and Inequalities*, D. Głasowska discussed in her talk a conjecture originally formulated by J. Matkowski [5], concerning the continuous solutions of the functional equation

$$f\big(f(-x)+x\big) = f\big(-f(x)\big) + f(x), \quad x \in \mathbb{R}.$$

Motivated by W. Jarczyk's related results [2, 3], Matkowski conjectured that, loosely speaking, the continuous solutions of the above equation are precisely those that are positively homogeneous on the non-positive and on the non-negative half-line, respectively. Although the conjecture was well-founded, it turned out that the family of continuous solutions is much richer.

The talk will focus on continuous solutions that are positively homogeneous on the non-positive half-line. Furthermore, we provide a sufficient condition under which the statement appearing in Matkowski's conjecture holds.

References


[1] D. Głazowska and J. Matkowski, *Weakly associative functions*, Aequationes Math. **99** (2025), no. 4, 1827–1841.

[2] W. Jarczyk, *On continuous functions which are additive on their graphs*, Ber. Math.-Statist. Sekt. Forschungsgesellsch. Joanneum, 292, Graz, 1988.

[3] W. Jarczyk, *A recurrent method of solving iterative functional equations*, Prace Nauk. Uniw. Śląsk. Katowic., 1206, Uniwersytet Śląski, Katowice, 1991.

[4] T. Kiss, *A counterexample to Matkowski's conjecture for quasi graph-additive functions*, Aequationes Math. **100** (2026), no. 2, Paper No. 27, 8 pp.

[5] J. Matkowski, *Weakly associative functions and means - new examples and open questions*, Aequationes Math. **99** (2025), no. 6, 2581–2597.


Radosław Łukasik: *Generalized orthogonality equations in normed spaces* (Joint work with Abayomi Epebinu)

Let $X, Y$ be real normed spaces and let $\rho_+^{'}, \rho_-^{'}$ be norm derivatives. In this talk, we solve a system of functional equations

$$\begin{cases} \rho_+^{'}(f(x), f(y)) = g(x)\rho_+^{'}(x,y), \\ \rho_-^{'}(f(x), f(y)) = g(x)\rho_-^{'}(x,y), \end{cases}$$

where the functions $f\colon X \to Y$ and $g\colon X \to \mathbb{R}$ are unknown.

Reference

[1] A. Epebinu and R. Łukasik, *Generalized orthogonality equations in normed spaces*, Aequationes Math. **100** (2026), no. 1, Paper No. 5, 10 pp.

Gergő Nagy: *Resolving sets in spaces of self-adjoint operators*

Concerning certain investigations related to metric spaces, in particular the study of special isometries, it is useful to find such sets which generate these spaces in a kind of metric sense. The resolving sets are particular examples of collections with this property, they are those subsets of a metric space $X$ for which any element of $X$ can be uniquely identified by its distances to the points of such a subset.

This talk is devoted to results on resolving sets in particular metric spaces of self-adjoint operators. These spaces include that of the density operators on a complex Hilbert space $H$, i.e., the positive trace class operators on $H$ with trace 1. Further, the metrics considered come from the Schatten norms. We present results stating that the extreme points of the convex subset of all density operators on $H$ form a resolving set of the latter collection. These statements tell us that the set of extreme points of a certain convex subset $C$ in a normed space generates $C$ in a kind of metric sense. This result is a counterpart of a corollary of the famous Krein-Milman theorem stating that the set $E$ of extreme points of a convex compact set $K$ in a locally convex topological vector space generates $K$ in a certain linear algebraic-topological sense, namely the closed convex hull of $E$ is $K$.

Andrzej Olbryś: *Remarks on Sincov's difference*

Answering the question given by Ludwig Reich we characterize a three place map which can be expressed as the Sincov difference i.e. a map of the form

$$F(x, z) - F(x, y) - F(y, z),$$

as well as, its pexiderized version

$$F(x, z) - G(x, y) - H(y, z).$$

Reference

[1] A.R. Baias, D. Otrocol, and M. Rus, *Report of Meeting. The 61st International Symposium on Functional Equations, Cluj-Napoca (Romania), June 15-21, 2025*, Aequationes Math. **99** (2025), no. 5, 2457–2479. DOI: 10.1007/s00010-025-01236-8.

Zsolt Páles: *On systems of higher-order homogeneous linear differential-algebraic equations* (Joint work with Eszter Gselmann)

In the talk, we consider the following system of homogeneous linear differential-algebraic equations

$$A_k y^{(k)}(t) + \cdots + A_1 y'(t) + A_0 y(t) = 0, \quad t \in I,$$

where $k, n, m \in \mathbb{N}$, $A_k, \dots, A_1, A_0 \in \mathbb{C}^{m\times n}$, $I$ is a nonempty open subinterval of $\mathbb{R}$, and $y\colon I \to \mathbb{C}^n$ denotes a sufficiently smooth unknown function.

The above sytem is called *nondegenerate* if, for some number $\lambda \in \mathbb{C}$, the rank of the $m \times n$ matrix

$$\lambda^k A_k + \cdots + \lambda A_1 + A_0$$

equals $n$. Our main result shows that the solution space of the above system of differential-algebraic equations is finite-dimensional if and only if this nondegeneracy condition holds. Furthermore, in this case, the solution space is a finite-dimensional subspace of the space of $\mathbb{C}^n$-valued exponential polynomials.

Paweł Pasteczka: *On the invariance equation for means of generalized power growth*

We generalize the result of (Witkowski, 2014) which binds orders of homogeneous, symmetric means $M, N, K\colon \mathbb{R}_+^2 \to \mathbb{R}$ of power growth that satisfy the invariance equation

$$K(M(x,y), N(x,y)) = K(x,y)$$

to the broader class of means. Moreover, we define the lower- and the upper-order which gives us insight into the order of the solution of this equation in the case when means do not belong to this class.

References

[1] P. Pasteczka, *On the invariance equation for means of generalized power growth*, Math. Inequal. Appl. **27** (2024), no. 3, 691–702.
[2] A. Witkowski, *On invariance equation for means of power growth*, Math. Inequal. Appl. **17** (2014), no. 3, 1091–1094.

Patryk Rela: *The robust Orlicz premium principle under uncertainty* (Joint work with Jacek Chudziak)

Assume that $(\Omega, \mathcal{F})$ is a measurable space and $\mu_0\colon \mathcal{F} \to [0,1]$ is a capacity, that is a monotone set function satisfying $\mu_0(\emptyset) = 0$ and $\mu_0(\Omega) = 1$. For a capacity $\mu_0$, by $S_{\mu_0}$ we denote a family of all capacities $\mu\colon \mathcal{F} \to [0,1]$ satisfying, for every $A \in \mathcal{F}$, the following condition

$$\mu_0(A) = 0 \implies \mu(A) = 0.$$

Let $\mathcal{X}_{\mu_0}$ be a family of all non-negative $\mathcal{F}$-measurable functions $X\colon \Omega \to [0,\infty)$ such that $\mu_0(\{X > t\}) = 0$ for some $t \in \mathbb{R}$. The robust Orlicz premium principle under uncertainty for the risk $X \in \mathcal{X}_{\mu_0}$ is defined through the equation

$$H_{\alpha,S,\Phi}(X) := \inf\left\{k > 0 : \sup_{\mu\in S} E_\mu\left[\Phi\left(\frac{X}{k}\right)\right] \le 1-\alpha\right\}, \tag{1}$$

where $\alpha \in [0,1)$ is a given parameter, $S \subset S_{\mu_0}$ and the function $\Phi\colon [0,\infty) \to [0,\infty)$ is a normalized Young function, that is a strictly increasing and convex function, satisfying $\Phi(0) = 0$ and $\Phi(1) = 1$. Moreover

$$E_\mu[X] = \int_0^\infty \mu(\{X > x\})dx \quad \text{for } X \in \mathcal{X}_{\mu_0},$$

is the Choquet integral with respect to the capacity $\mu$.

The aim of this talk is to prove the existence of the robust Orlicz premium defined by (1) and to characterize its several important properties.

Reference

[1] F. Bellini, R.J.A. Laeven, and E. Rosazza Gianin, *Robust return risk measures*, Math. Financ. Econ. **12** (2018), no. 1, 5–32.

Maciej Sablik: *Roman, my friend and master*

These are few words (and some pictures) illustrating my half a century long friendship with the late Roman Ger.

Rafał Stypka: *Approximate additivity of operators approximately preserving Birkhoff–James orthogonality*

We consider approximately additive operators between real normed spaces that approximately preserve Birkhoff–James orthogonality. Let $X$ be a normed space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with $\dim X \geq 2$, and let $\varepsilon \in [0,1)$. We introduce the following notions.

— *Birkhoff–James orthogonality*:

$$x \perp_B y \iff \|x + \lambda y\| \geq \|x\| \quad \text{for every } \lambda \in \mathbb{K}.$$

— *Approximate Birkhoff–James orthogonality*:

$$x \underset{\varepsilon}{\perp}_B y \iff \|x + \lambda y\| \geq (1-\varepsilon)\|x\| \quad \text{for every } \lambda \in \mathbb{K}.$$

An operator $T\colon X \to Y$ between normed spaces is said to be an *approximately Birkhoff–James orthogonality preserving operator* if

$$x \perp_B y \quad \Rightarrow \quad T(x) \underset{\varepsilon}{\perp}_B T(y) \quad \text{for all } x, y \in X.$$

The main result of the talk in the setting of real normed spaces establishes that, in a neighborhood of an approximately additive operator, there exists a linear, bounded, and continuous operator which approximately preserves Birkhoff–James orthogonality. Our approach relies on the geometric structure of normed spaces and on inequalities derived from the definition of approximate Birkhoff–James orthogonality.

The obtained result constitutes a generalization of the theorems proved in [1], where it was shown that every additive operator that approximately preserves Birkhoff–James orthogonality is necessarily linear.

Reference

[1] J. Chmieliński and R. Stypka, *Additive operators approximately preserving Birkhoff-James orthogonality*, Aequationes Math. **99** (2025), no. 6, 2847–2854.

László Székelyhidi: *Does spectral analysis imply spectral synthesis?*

Given a system of convolution type functional equations spectral analysis means that every nontrivial extension of the system has exponential solutions. The system is called trivial, if the only solution is the zero function. In terms of linear algebra this means that every nonzero invariant subspace of the solution space includes eigenvectors. A seemingly stronger property of a system is spectral synthesis: it means that in every nontrivial extension of the system the exponential monomial solutions span a dense subspace. In the finite dimensional case this means that the generalized eigenvectors form a basis in every invariant subspace of the solution space. In this talk we show that, in fact, the spectral analysis property implies the spectral synthesizability unless the underlying group has infinitely many linearly independent additive functions.

Patricia Szokol: *On some examples and counterexamples about weighted Lagrange interpolation with Exponential- and Hermite weights* (Joint work with Szilárd Révész)

The famous Bernstein conjecture about optimal node systems of classical polynomial Lagrange interpolation, standing unresolved for over half a century, was solved by T. Kilgore in 1978 [2]. Immediately following him, also the additional conjecture of Erdős was solved by deBoor and Pinkus [1]. These breakthrough achievements were built on a fundamental auxiliary result on nonsingularity of derivative (Jacobian) matrices of certain interval maxima in function of the nodes. After the above breakthrough, a considerable effort was made to extend the results to the case of at least certain restricted classes of functions and Chebyshev-Haar subspaces.

Our aim is to analyze whether this key nonsingularity property holds for exponentially weighted interpolation on a half-line, as well as under Hermite weights on the entire real line. Our case studies also present counterexamples that demonstrate the singularity of the derivative matrices, thereby calling into question the validity of the Bernstein and Erdős conjectures.

References

[1] C. de Boor and A. Pinkus, *Proof of the conjectures of Bernstein and Erdős concerning the optimal nodes for polynomial interpolation*, J. Approx. Theory **24** (1978), no. 4, 289–303.

[2] T.A. Kilgore, *A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm*, J. Approx. Theory **24** (1978), no. 4, 273–288.

Péter Tóth: *On the equality problem of Bajraktarević means*

We discuss the equality problem of two variable generalized Bajraktarević means. That is, we investigate the functional equation

$$f^{(-1)}\left(\frac{p_1(x)f(x)+p_2(y)f(y)}{p_1(x)+p_2(y)}\right)=g^{(-1)}\left(\frac{q_1(x)g(x)+q_2(y)g(y)}{q_1(x)+q_2(y)}\right)\qquad (x,y\in I)$$

where $I$ is an open interval, $f,g\colon I\to\mathbb{R}$ are strictly monotone functions, $f^{(-1)}$ and $g^{(-1)}$ are their generalized left inverses, respectively, while $p_1$, $p_2$, $q_1$, $q_2\colon I\to\mathbb{R}_+$ are positive weight functions. The strongest results about the symmetric version (i.e. the case when $p_1=p_2$ and $q_1=q_2$) and non-symmetric version of this problem are contained in the papers [1] and [2], respectively.

In those works continuous differentiability is assumed for the generator functions $f,g$ and one of the weight functions $p_i$ as well. In our talk we eliminate some of these regularity assumptions, and rely only on the differentiability of some weight functions.

References

[1] Zs. Páles and A. Zakaria, *On the equality problem of two-variable Bajraktarević means under first-order differentiability assumptions*, Aequationes Math. **97** (2023), no. 2, 279–294.

[2] Zs. Páles and A. Zakaria, *On the equality of generalized Bajraktarević means under first-order differentiability assumptions*, Aequationes Math. **99** (2025), no. 6, 2485–2503.

Paweł Wójcik: *Norm derivatives and Cauchy-Schwarz Inequality* (Joint work with S.M. Enderami, M. Abtahi and A. Zamani)

Let $X$ be a complex normed space. Based on the norm derivative $\rho_+\colon X\times X\to\mathbb{R}$, we define a mapping $\rho_\infty\colon X\times X\to\mathbb{C}$ by the following formula:

$$\rho_\infty(x,y):=\frac{1}{\pi}\int_0^{2\pi} e^{i\theta}\rho_+(x,e^{i\theta}y)d\theta.$$

The mapping $\rho_\infty$ has a good response to some geometrical properties of $X$. Some problem connected with the Cauchy-Schwarz Inequality will be discussed.

Reference

[1] S.M. Enderami, M. Abtahi, A. Zamani, and P. Wójcik, *An orthogonality relation in complex normed spaces based on norm derivatives*, Linear Multilinear Algebra **72** (2024), no. 4, 687–705.

## 2. Problems and remarks

1. Remark about extending continuity from the subinterval.

Given a proper interval $I$ (i.e., $I$ is nonempty and not a singleton), let $F : I^n \to I$ be a function satisfying some algebraic identities $\mathcal{A}_1, \dots, \mathcal{A}_m$. Suppose there exists a proper subinterval $J \subsetneq I$ such that $F|_{J^n}$ is continuous.

Questions:

(1) Are there natural assumptions under which this implies that $F$ is continuous on all of $I$?

(2) What changes if we additionally assume that $F$ is partially strictly monotone (or increasing) on $I$ and/or satisfies some algebraic identities on the subinterval $J$ ?

Examples related to (1):

— If $F : I^2 \to I$ satisfies only reflexivity ($F(x, x) = x$ for all $x \in I$), this does not imply continuity.

— If $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies the Cauchy equation ($F(x + y) = F(x) + F(y)$) and is continuous on a subinterval, then $F$ is continuous everywhere.

Examples related to (2):

— If $F : I^2 \to I$ is reflexive, bisymmetric, partially strictly monotone (or increasing) on $I$, and symmetric on a subinterval $J$, then $F$ is continuous on $J$. Moreover, it follows that $F$ is symmetric and continuous everywhere.

— It is unknown what happens if in the previous case the symmetry assumption is removed entirely.

Gergely Kiss

2. Remarks to the talk of László Székelyhidi.

1) Due to the Structure Theorem of LCA groups we are dealing with groups having the general form $\mathrm{G} = \mathbb{R}^a \times \mathbb{Z}^b \times D$, where $D$ is a discrete abelian group. The main theorem I presented in my talk characterizes varieties having spectral synthesis on discrete abelien groups. To extend this theorem for general LCA groups we need to have conditions for the variety so that its "D-part" satisfies the property I presented in the discrete case. For this we have to find out some appropriate meaning of the phrase "D-part of a variety".

2) The question of Prof. Eszter Gselmann to the talk was not completely replied. She asked how to decide in particular cases if the necessary and sufficient condition presented in the talk for spectral synthesis on an LCA group is satisfied. Here is a more detailed answer. In fact, if a system of convolution-type functional equations is given, first we should check its generalized exponential polynomial solutions. Obviously, the first step is to check if there are exponential solutions – if there is none, then we are done: spectral analysis does not hold, hence spectral synthesis fails as well. In fact, we should do

this with every nonzero subvariety. On the other hand, if m is an exponential solution, then we continue with substituting into the system a general m-exponential monomial of the form

$$x \mapsto (A_n(x) + A_{n-1}(x) + A_2(x) + A_1(x) + A_0(x))m(x)$$

where the $A$'s are diagonalizations of unknown symmetric multi-additive functions. Clearly, if there is a solution of this type with nonzero $A_2$, then there is a solution of the type

$$x \mapsto B(x,x)m(x)$$

where $B \neq 0$ is a biadditive symmetric function. If $B$ here is not a quadratic function of finitely many additive functions, then we have a generalized exponential polynomial solution which is not an exponential polynomial solution - spectral synthesis fails. However, if every generalized exponential polynomial solution is an exponential polynomial, then we should check the local Noetherian property given in the theorem. My conjecture is that, in fact, it is equivalent to the property that there are no pathological solutions. In other words, I guess that the local Noetherian property can be replaced simply by the assumption that the system has no $x \mapsto B(x,x)m(x)$ solution, where $B$ is biadditive and symmetric, but it is not a quadratic function of finitely many additive functions.

László Székelyhidi

3. Remark about fractional integrals.

Let $f\colon [x,y] \to \mathbb{R}$ be an integrable function. The Riemann–Liouville integrals $J^{\alpha}_{x+}f(s)$ and $J^{\alpha}_{y-}f(s)$ of order $\alpha > 0$ with $x \geq 0$ are defined by

$$J^{\alpha}_{x+}f(s) = \frac{1}{\Gamma(\alpha)} \int_x^s (s-t)^{\alpha-1} f(t)dt \tag{1}$$

and

$$J^{\alpha}_{y-}f(s) = \frac{1}{\Gamma(\alpha)} \int_s^y (t-s)^{\alpha-1} f(t)dt, \tag{2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{x+}f(s) = J^{0}_{y-}f(s) = f(s)$.

It is known that for all convex functions $f$ the following Hermite-Hadamard type inequality holds

$$f\left(\frac{x+y}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}[J^{\alpha}_{x+}f(y) + J^{\alpha}_{y-}f(x)] \leq \frac{f(x)+f(y)}{2}, \tag{3}$$

see for example [3]. In the paper written jointly with A. Epebinu [1] it was observed that

$$\frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}[J^\alpha_{x+}f(y)+J^\alpha_{y-}f(x)]=\int_x^y f(t)dF_\alpha(t), \tag{4}$$

where

$$F_\alpha(t)=\begin{cases}0, & t<x,\\ \frac{(t-x)^\alpha-(y-t)^\alpha}{2(y-x)^\alpha}+\frac{1}{2}, & t\in[x,y),\\ 1, & t\geq y,\end{cases} \tag{5}$$

and then, based on this observation, some new results were obtained. We may generalize $F_\alpha$ to

$$F_\phi(t)=\frac{\phi(t-x)-\phi(y-t)}{2\phi(y-x)}+\frac{1}{2},$$

where $\phi:[0,\infty)\to[0,\infty)$ is a function, $x,y\in\mathbb{R}$ satisfy $x<y$ and $t\in[0,y-x)$. If we ask for which $\phi$

$$\int_x^y fdF_\phi=\frac{1}{y-x}\int_x^y f(t)dt,$$

then we get the equation

$$\frac{\phi(t-x)-\phi(y-t)}{2\phi(y-x)}+\frac{1}{2}=\frac{t}{y-x}-\frac{x}{y-x}$$

taking here $u=t-x$ and $v=y-t$ we get

$$\phi(u)-\phi(v)=\phi(u+v)\left(\frac{2u}{u+v}-1\right)$$

and

$$\phi(u)(u+v)-\phi(v)(u+v)=\phi(u+v)(u-v)$$

which may be solved using the version of Sablik's lemma from [2] and some other simple calculations. The solutions are $\phi(x)=ax^2+bx$, with any $a,b\in\mathbb{R}$. This means that these are the only functions for which the integral $\int_x^y fdF_\phi$ coincides with the usual integral.

## References

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[2] A Lisak, M Sablik, *Trapezoidal rule revisited*, Bull. Inst. Math. Acad. Sin. (NS) **6** (2011), no. 3, 347–360.

[3] M.Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling **57** (2013), no. 9–10, 2403–2407.

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