

## A NEW PROOF AND CONSEQUENCES OF THE FIXED POINT THEOREM OF MATKOWSKI

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**Abstract.** In this work it was proved Matkowski's fixed point theorem. The consequences of this theorem are also presented.

### 1. Introduction

The presented work concerns Matkowski's fixed point theorem and the conclusions from this theorem. These results were used to study the limit behaviors of quotients  $\frac{F_{n+1}}{F_n}$  of the Fibonacci type numbers. This work thematically refers to works [2] and [3]. For these studies Edelstein's fixed point theorem was used in [2], while in [3], the fixed point theorem was proved and used for the „ $d(f(x), f(y)) \leq \phi(d(x, y))$ ” type mappings of the interval  $\langle a, b \rangle$ , where the function  $\phi$  is right continuous and fulfills additional conditions.

In the presented work there is a new and easy proof of Matkowski's fixed point theorem. In this theorem the function  $\phi$  is not assumed to be continuous. There are also proven conclusions from this theorem. The obtained results concern the mentioned type of mappings of complete spaces. Their application is illustrated by the approximation of the golden number  $\varphi = \frac{1+\sqrt{5}}{2}$ .

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In this work, based on Matkowski's theorem, we present and demonstrate a certain extension of the Hutchinson theorem about the fixed point of the mapping determined by the so-called hyperbolic iterated functional system marked with symbol IFS. In proof of this theorem from 1981, Hutchinson applied Banach Contraction Principle. Banach's principle is a conclusion from Matkowski's theorem. It is worth adding that the basic tool enabling the construction of the so-called self-similar sets, important in fractal theory, is the Hutchinson theorem.

## 2. Fixed point theorems of the Matkowski type generalized contractions and their applications

DEFINITION 1. A map  $f: (X, d) \rightarrow (Y, g)$  of metric spaces that satisfies the inequality  $d(f(x), f(x')) \leq Ld(x, x')$  for some fixed constant  $L$  and all  $x, x' \in X$  is called *Lipschitzian*; the smallest such  $L$  is called the *Lipschitz constant*  $\lambda$  of  $f$ . If  $\lambda < 1$ , the map  $f$  is called the *contraction* (with contraction constant  $\lambda$ ).

DEFINITION 2. Let  $(X, d)$  be a metric space. A map  $f: (X, d) \rightarrow (X, d)$  is called a *Banach contraction*, if there exists constant  $\lambda < 1$  satisfying the inequality  $d(f(x), f(x')) \leq \lambda d(x, x')$  for all  $x, x' \in X$ .

DEFINITION 3. Let  $(X, d)$  be a metric space. For a given map  $\phi: \langle 0, \infty \rangle \rightarrow \langle 0, \infty \rangle$  satisfying the condition

$$\phi(t) < t \quad \text{for all } t > 0,$$

we say that,  $f: X \rightarrow X$  is  $\phi$ -*contraction*, if

$$d(f(x), f(x')) \leq \phi(d(x, x')) \quad \text{for all } x, x' \in X.$$

DEFINITION 4. Let  $(X, d)$  be a metric space. A map  $f: X \rightarrow X$  is called a *Browder contraction*, if  $f$  is  $\phi$ -contraction for some function  $\phi$  which is non-decreasing and right continuous.

DEFINITION 5. Let  $(X, d)$  be a metric space. We say, that  $f: X \rightarrow X$  is a *contraction of Matkowski*, if  $f$  is  $\phi$ -contraction for some function  $\phi$  which is nondecreasing and  $\lim_{n \rightarrow \infty} \phi^n(t) = 0$  for any  $t > 0$ .

DEFINITION 6. *Fibonacci sequence* is a sequence defined recursively as follows:

$$f_1 = f_2 = 1, \quad f_{n+1} = f_{n-1} + f_n, \quad n \geq 2$$

(sometimes formally accepted  $f_0 = 0$  and then the recursive formula is valid for  $n \geq 1$ ).

DEFINITION 7. *Fibonacci numbers* are called consecutive terms of the sequence  $(f_n)$ .

DEFINITION 8. A sequence  $(F_n)$  of the form  $F_{n+1} = F_n + F_{n-1}, n \geq 2$ , where  $F_1$  and  $F_2$  are given positive integers we call a *Fibonacci type sequence*.

For example, this sequence is the so-called *Lucas sequence*  $(l_n)$ :

$$1, 3, 4, 7, 11, 18, 29, \dots$$

These numbers can be described by a formula

$$l_1 = 1, \quad l_2 = 3, \quad l_{n+1} = l_n + l_{n-1}, \quad n \geq 2.$$

DEFINITION 9. A *generalized Fibonacci sequence* is a sequence  $(G_n)$  defined recursively as follows:  $G_{n+1} = G_n + G_{n-1}, n \geq 2$ , with  $G_1 = a$  and  $G_2 = b$ ,  $a, b > 0$ .

Below we present proof of Matkowski's fixed point theorem, which is one of the more general extensions of Banach Contraction Principle. In this proof we will use Cantor's intersection theorem. Before the theorem and its proof, let us note that the last two conditions of Matkowski's contraction imply the condition  $\phi(t) < t$  for all  $t > 0$  (see [1]).

THEOREM 1 ([4, Theorem 3.2, 12 p.], [7]). *Let  $(X, d)$  be a complete metric space. If  $f: X \rightarrow X$  is the contraction of Matkowski, then  $f$  has a unique fixed point  $u$ , and  $f^n(x) \rightarrow u$  for each  $x \in X$ .*

PROOF. Given  $\varepsilon > 0$ , let's choose  $x \in X$ , for  $\delta = \varepsilon - \phi(\varepsilon)$  such that  $d(x, f(x)) \leq \delta$ . We show that  $f$  maps the closed ball  $D = \{y \in X : d(y, x) \leq \varepsilon\}$  into itself: for if  $z \in D$ , then

$$d(f(z), x) \leq d(f(z), f(x)) + d(f(x), x) \leq \phi(d(z, x)) + \delta \leq \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) = \varepsilon,$$

so  $f(z) \in D$ . Let us consider a sequence of sets  $D_n = \overline{f^n(D)}$ . First we have  $D_2 \subset D_1$  from  $f^2(D) \subset f(D) \subset D$ . Now suppose that  $f^k(D) \subset f^{k-1}(D)$ ,

$k \geq 2$ , then  $f^{k+1}(D) = f(f^k(D)) \subset f^k(D)$ , hence  $D_{k+1} = \overline{f^{k+1}(D)} \subset \overline{f^k(D)} = D_k$ . Therefore we have a descending sequence of closed sets  $D \supset D_1 \supset D_2 \supset \dots$ .

We shall show that  $\text{diam}(D_n) \rightarrow 0$ . For this purpose, observe first that

$$\text{diam}(D_1) = \text{diam}(\overline{f(D)}) = \text{diam}(f(D)) \leq \phi(\text{diam}(D)) = \phi(2\varepsilon)$$

and, by induction  $\text{diam}(D_n) \leq \phi^n(2\varepsilon)$  for each  $n \in \mathbb{N}$ .

Because  $\phi^n(2\varepsilon) \rightarrow 0$  ( $n \rightarrow \infty$ ), so  $\text{diam}(D_n) \rightarrow 0$ . Also

$$f\left(\bigcap_{n \geq 1} D_n\right) = f\left(\bigcap_{n \geq 1} \overline{f^n(D_n)}\right) \subset \bigcap_{n \geq 1} f\left(\overline{f^n(D)}\right) \subset \bigcap_{n \geq 1} \overline{f(D_n)} \subset \bigcap_{n \geq 1} D_n.$$

Consequently, using Cantor's Theorem we deduce that  $\bigcap_{n \geq 1} D_n$  consists of a unique point  $u = f(u)$ . Because from this equality we have  $u = f^n(u)$  for every  $n \in \mathbb{N}$ , so

$$(*) \quad d(f^n(y), u) \leq \phi(d(f^{n-1}(y), f^{n-1}(u))) \leq \dots \leq \phi^n(d(y, u))$$

for any  $y \in X$  and for any  $n \in \mathbb{N}$ , and hence  $f^n(y) \rightarrow u$ , when  $n \rightarrow \infty$ .  $\square$

Note that the above theorem can be proved in another way by considering, instead of Matkowski's contraction  $f$ , its second iteration  $f^2 = f \circ f$ , which is the Browder contraction (see [6]). Based on Browder's fixed point theorem (see [4, Theorem 6.10, p. 18])  $f^2$  has a unique fixed point  $u$ , so  $u$  is the only fixed point for  $f$ . Indeed, since  $u = f^2(u)$ , then from the equality  $f(u) = f^2(f(u))$  we get the fixed point  $f(u)$ , so  $f(u) = u$ . It is easy to show that  $u$  is the only fixed point of  $f$  (by  $f^2(v) = v$  for another point  $v = f(v)$ ).

Theorem 1 has a useful local version:

**COROLLARY 1.** *Let  $(X, d)$  be a complete metric space and  $D = D(x_0, r)$  be the set  $\{x \in X : d(x, x_0) \leq r\}$ . If  $D \rightarrow X$  is the contraction of Matkowski such that*

$$(**) \quad d(x_0, f(x_0)) \leq r - \phi(r),$$

*then  $f$  has a unique fixed point  $u$ , and  $f^n(x) \rightarrow u$  for each  $x \in D$ .*

PROOF. For any  $x \in D$  we have

$$\begin{aligned} d(f(x), x_0) &\leq d(f(x), f(x_0)) + d(f(x_0), x_0) \leq \phi(d(x, x_0)) + r - \phi(r) \\ &\leq \phi(r) + r - \phi(r) = r. \end{aligned}$$

Therefore  $f: D \rightarrow D$ . Since  $D$  is complete, the conclusion follows from Theorem 1. □

REMARK 1. Let  $\phi(t) = \lambda t$ ,  $t \in (0, \infty)$ ,  $\lambda < 1$ . On this assumption  $f: X \rightarrow X$  in Theorem 1 is the Banach contraction, and Theorem 1 is the Banach Contraction Principle. The assumption  $(**)$  in Corollary 1 takes the form

$$d(x_0, f(x_0)) \leq (1 - \lambda)r.$$

EXAMPLE 1 (Application of Theorem 1 to study the convergence of the quotient of neighboring terms of the Lucas sequence). Let us recall that Lucas numbers are:

$$1, 3, 4, 7, 11, 18, 29, 47, \dots$$

They are terms of the sequence  $(l_n)$  starting with  $l_1 = 1, l_2 = 3$ , whose successive terms satisfy the relationship  $l_n = l_{n-1} + l_{n-2}$  for  $n > 2$ . The mapping  $f: \langle \frac{4}{3}, 3 \rangle \rightarrow \langle \frac{4}{3}, 3 \rangle$ ,  $f(x) = 1 + \frac{1}{x}$  is a contraction with the constant  $\lambda = \frac{9}{16}$ . Indeed for  $x, x' \in \langle \frac{4}{3}, 3 \rangle$  we have

$$|f(x) - f(x')| = \frac{|x - x'|}{xx'} \leq \left(\frac{3}{4}\right)^2 |x - x'|.$$

Therefore, based on Banach Contraction Principle the sequence  $(x_n), x_n = f(x_{n-1}), n \geq 1, x_0 = \frac{4}{3} = \frac{l_3}{l_2}$  converges to the fixed point  $u = \varphi = \frac{1+\sqrt{5}}{2}$ , which is the solution of the equation  $x = 1 + \frac{1}{x}$  in  $\langle \frac{4}{3}, 3 \rangle$ . Also for  $x_0 = 3 = \frac{l_2}{l_1}$  we have  $x_n = f^n(x_0) \rightarrow \varphi$ . Therefore we finally have

$$\lim_{n \rightarrow \infty} f^n \left( \frac{l_2}{l_1} \right) = \lim_{n \rightarrow \infty} f^n \left( \frac{l_3}{l_2} \right) = \varphi.$$

EXAMPLE 2 (Application of Corollary 1 to study the convergence of the quotient of neighboring terms of the Fibonacci sequence). Let  $D = D(\varphi, \frac{1}{2})$ , thus  $D = \langle \varphi - \frac{1}{2}, \varphi + \frac{1}{2} \rangle$ . Note that the function  $f: D \rightarrow \mathbb{R}$  given by the

formula  $f(x) = 1 + \frac{1}{x}$  is a contraction with the constant  $\lambda = \frac{4}{5}$ , because from equality  $\varphi - \frac{1}{2} = \frac{\sqrt{5}}{2}$  we have

$$|f(x) - f(x')| = \frac{|x - x'|}{xx'} \leq \frac{4}{5}|x - x'| \quad \text{for } x, x' \in D.$$

Hence we have

$$|\varphi - f(\varphi)| = 0 \leq \left(1 - \frac{4}{5}\right) \cdot \frac{1}{2}.$$

Based on Corollary 1  $f^n(d) \rightarrow \varphi$  for any  $d \in D$ .

Let  $d = \frac{f_3}{f_2} = 2$ , then  $d \in D = \langle \varphi - \frac{1}{2}, \varphi + \frac{1}{2} \rangle$ . Therefore  $f^n\left(\frac{f_3}{f_2}\right) \rightarrow \varphi$ .

Now taking  $d = \frac{f_4}{f_3} = \frac{3}{2} \in D$  we get  $f^n\left(\frac{f_4}{f_3}\right) \rightarrow \varphi$ . We finally have

$$\lim_{n \rightarrow \infty} f^n\left(\frac{f_3}{f_2}\right) = \varphi = \lim_{n \rightarrow \infty} f^n\left(\frac{f_4}{f_3}\right).$$

**THEOREM 2** (compare [4], [5]). *Let  $(X, d)$  be a complete metric space and let  $f: X \rightarrow X$ . Suppose that there is a natural number  $N > 1$  such that  $f^N$  is the contraction of Matkowski. Then  $f$  has a unique fixed point  $u$  and the sequence of iterates  $f^N(x) \rightarrow u$  for each  $x \in X$ .*

**PROOF.** Based on Theorem 1  $f^N$  has a unique fixed point  $u = f^N(u)$ . However  $f^N(f(u)) = f(f^N(u)) = f(u)$ , therefore  $f(u)$  is also a fixed point of  $f^N$ . Because the fixed point of  $f^N$  is only one, so  $f(u) = u$ . If for another point  $v = f(v)$ , then from  $f^N(v) = v, n \in \mathbb{N}$ , we have  $f^N(v) = v$ , so  $v = u$ .

Proof of the second part of the thesis is analogous to the last part of the proof of Theorem 1 (comp. (\*)).  $\square$

**EXAMPLE 3** (Application of Theorem 2 to study the convergence of the quotients  $\frac{F_{n+1}}{F_n}$  of the Fibonacci type sequence  $(F_n)$ ). We will justify that successive quotients  $\frac{F_{n+1}}{F_n}$  of terms of the Fibonacci type sequence  $(F_n)$  approach the value of  $\varphi$ . We will assume that the initial terms  $F_1$  and  $F_2$  of this sequence, which are natural numbers, satisfy the inequality  $F_1 \leq F_2$ . Since for  $f(x) = 1 + \frac{1}{x}$ :  $f(1) \leq 2, f(2) \geq 1$  and  $f$  is decreasing we have  $f(\langle 1, 2 \rangle) \subset \langle 1, 2 \rangle$ . So  $f^2(\langle 1, 2 \rangle) \subset \langle 1, 2 \rangle$ . Because  $f$  is not a contraction on the interval  $\langle 1, 2 \rangle$  (as  $|f(x) - f(x')| = \frac{|x - x'|}{xx'} \leq |x - x'|$  for  $x, x' \in \langle 1, 2 \rangle$ ), we will

examine whether  $f^2$  is a contraction. We have  $f^2(x) = f(1 + \frac{1}{x}) = 1 + \frac{x}{x+1}$  for each  $x \in \langle 1, 2 \rangle$ , hence

$$|f^2(x) - f^2(x')| = \frac{|x - x'|}{xx' + x + x' + 1} \leq \frac{1}{4}|x - x'| \quad \text{for all } x, x' \in \langle 1, 2 \rangle.$$

Therefore  $f^2$  is the contraction with constant  $\lambda = \frac{1}{4}$ . We can now apply Theorem 2 assuming  $\phi(t) = \frac{1}{4}t, t \geq 0$ . By Theorem 2  $f$  has in  $\langle 1, 2 \rangle$  a unique fixed point  $u$ , and the sequence of iterates  $f^n(y_0) \rightarrow u$  for each  $y_0 \in \langle 1, 2 \rangle$ . Let  $x_0 = \frac{F_2}{F_1}$ , then  $y_0 = f(x_0) = 1 + \frac{F_1}{F_2}$  and  $y_0 \in \langle 1, 2 \rangle$ . Because  $y_0 = 1 + \frac{F_1}{F_2} = \frac{F_2 + F_1}{F_2} = \frac{F_3}{F_2}$ , so  $f^n\left(\frac{F_3}{F_2}\right) \rightarrow u = \varphi$  ( $u = \varphi$  because  $u = 1 + \frac{1}{u}$ ).

REMARK 2. It is worth adding that, using Banach Contraction Principle as a conclusion from the fixed point theorem of Matkowski, we can study the limit behavior of the quotients  $\frac{G_{n+1}}{G_n}$  of the corresponding terms of the generalized Fibonacci sequence  $(G_n)$  (see [3]).

Let  $K(X)$  be a family of non-empty and compact subset of the metric space  $(X, d)$ . In the set  $K(X)$  we define the metric using the definition: an epsilon extension of the set  $A$  we call the set

$$A_\varepsilon = \{x \in X; d(a, x) \leq \varepsilon \text{ for some } a \in A\}.$$

$A_\varepsilon$  is also called the  $\varepsilon$ -envelope of the set  $A$ .

It can be shown that the function  $d_H: K(X) \times K(X) \rightarrow \langle 0, \infty \rangle$  given by the formula

$$d_H(A, B) = \inf \{\varepsilon \geq 0; A \subset B_\varepsilon \wedge B \subset A_\varepsilon\}$$

is a metric. We call it the Hausdorff metric on the set  $K(X)$ .  $(K(X), d_H)$  is a complete metric space, if  $(X, d)$  is a complete metric space. Let the mapping  $F: K(X) \rightarrow K(X)$  be given by the formula  $F(A) = f_1(A) \cup \dots \cup f_k(A)$  for  $A \in K(X)$ , where  $f_i: X \rightarrow X, i = 1, \dots, k$  are functions.

THEOREM 3. *If all functions  $f_i: X \rightarrow X, i \in \{1, \dots, k\}$  are Matkowski contractions for the same non-decreasing function  $\phi: \langle 0, \infty \rangle \rightarrow \langle 0, \infty \rangle$ , then the mapping  $F: K(X) \rightarrow K(X)$  is the Matkowski contraction with the function  $\phi$  (also the same).*

PROOF. Since every function  $f_i$  is the Matkowski contraction with  $\phi$ , so for any  $p, q \in X$  and  $i = 1, \dots, k$  we have  $d(f_i(p), f_i(q)) \leq \phi(d(p, q))$ . Let  $A, B \in K(X)$  and let  $\delta = d_H(A, B)$ . Then for every  $p \in A$  there exists such  $q \in B$  that  $d(p, q) \leq \delta$ . Therefore for each  $i$  we have  $d(f_i(p), f_i(q)) \leq \phi(\delta)$ . It follows that  $f_i(A)$  is a set contained in the epsilon extension  $f_i(B)$  for  $\varepsilon = \phi(\delta)$ .

So we have  $F(A) = \bigcup_{i=1}^k f_i(A) \subset \bigcup_{i=1}^k (f_i(B))_\varepsilon = (F(B))_\varepsilon$ . Similarly we prove that  $F(B) \subset (F(A))_\varepsilon$ . Therefore

$$d_H(F(A), F(B)) \leq \varepsilon = \phi(\delta) = \phi(d_H(A, B)). \quad \square$$

We will now present one of the extensions of Hutchinson's theorem on the fixed point of mapping  $F$  which concerned the Banach contraction system  $\{f_1, \dots, f_k\}$ .

THEOREM 4. *If the space  $(X, d)$  is complete and the mapping  $F: K(X) \rightarrow K(X)$  is defined by the formula  $F(A) = f_1(A) \cup \dots \cup f_k(A)$  for  $A \in K(X)$ , where each function  $f_i$  ( $i = 1, \dots, k$ ) is the Matkowski contraction with the same non-decreasing function  $\phi: (0, \infty) \rightarrow (0, \infty)$ , then there exists exactly one set  $A_* \in K(X)$  such that*

$$(***) \quad A_* = F(A_*) = f_1(A_*) \cup \dots \cup f_k(A_*).$$

Moreover, for any  $K_0 \in K(X)$  the iteration sequence  $(F^n(K_0))$  converges to  $A_*$  relative to the Hausdorff metric.

Sets  $A_* \in K(X)$  satisfying the condition (\*\*\*) are called self-similar (relative to  $f_1, \dots, f_k$ ) or fractals.

PROOF. It is enough to recall that:

- (i)  $(K(X), d_H)$  is a complete space,
- (ii)  $F: K(X) \rightarrow K(X)$  is the Matkowski contraction

and refer to Theorem 1. □

REMARK 3. If  $X$  is the Euclidean space  $(\mathbb{R}^n, d)$  and  $F$  is the Matkowski contraction with the function  $\phi$  of the form  $\phi(t) = \lambda t, \lambda < 1, t \geq 0$ , then we can obtain, among others, the Cantor set. Namely let  $S$  be the family of all closed nonempty subsets of the unit interval  $\langle 0, 1 \rangle$ . Let  $f: S \rightarrow S$  be a transformation that assigns to each set  $A \in S$  the set  $F(A) = \frac{1}{3}A \cup (\frac{2}{3} + \frac{1}{3}A)$ . Let's put  $D_0 = \langle 0, 1 \rangle$ . Finding successive iterations of the transformation  $F$  of set  $D_0$  we get:

$$D_1 = F(D_0) = \left\langle 0, \frac{1}{3} \right\rangle \cup \left\langle \frac{2}{3}, 1 \right\rangle,$$



$$D_2 = F^2(D_0) = F(D_1) = \left\langle 0, \frac{1}{9} \right\rangle \cup \left\langle \frac{2}{5}, \frac{1}{3} \right\rangle \cup \left\langle \frac{2}{3}, \frac{7}{9} \right\rangle \cup \left\langle \frac{8}{9}, 1 \right\rangle.$$

In the same way we construct the next sets  $D_3, D_4, \dots$ . The set  $C = \bigcap_{n \geq 0} D_n$  which is a unique fixed point of the transformation  $F$  is a self-similar set and is known as the Cantor set.

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