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A NOTE ON TWO FUNDAMENTAL RECURSIVE SEQUENCES

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Abstract. In this note, we establish some general results for two fundamental recursive sequences that are the basis of many well-known recursive sequences, as the Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, etc. We establish some general limit formulas, where the product of the first n terms of these sequences appears. Furthermore, we prove some general limits that connect these sequences to the number $e(\approx 2.71828...)$.

1. Introduction

In mathematics, a recursive sequence is a sequence in which terms are defined using one or more previous terms which are given. There are many recursive sequences and the most well-known of them are the following two fundamental and primordial sequences:

$$U_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ mU_{n-1} + U_{n-2} & \text{otherwise,} \end{cases} \qquad m \in \mathbb{R}, \tag{1.1}$$

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and

$$V_n = \begin{cases} 2 & \text{if } n = 0, \\ m & \text{if } n = 1, \\ mV_{n-1} + V_{n-2} & \text{otherwise,} \end{cases} \qquad m \in \mathbb{R}. \tag{1.2}$$

Furthermore, sequences (1.1) and (1.2) can be defined by the following closed-form solutions (known as *Binet style formulas*), respectively:

$$U_n = \frac{\alpha^n - \beta^n}{\Delta}, \tag{1.3}$$

and

$$V_n = \alpha^n + \beta^n, \tag{1.4}$$

where $\Delta = \sqrt{m^2+4}$, $\alpha = \frac{m+\Delta}{2}$, and $\beta = \frac{m-\Delta}{2}$.

Two typical examples of sequences in forms (1.1) and (1.2) (corresponding to $m = 1$) are the Fibonacci and Lucas sequences, respectively. Thus, if $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ denote the Fibonacci and Lucas sequences, respectively, then

$$F_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1} + F_{n-2} & \text{otherwise,} \end{cases} \qquad L_n = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ L_{n-1} + L_{n-2} & \text{otherwise.} \end{cases}$$

The Binet forms for F_n and L_n are

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}, \qquad L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Further examples of sequences in recursive forms (1.1) and (1.2) are the Pell and Pell–Lucas sequences that the first is of the recursive form (1.1) and the second is of the recursive form (1.2). These sequences are defined as follows (for $n \geq 0$), respectively:

$$P_{n+2} = 2P_{n+1} + P_n, \quad P_0 = 0,\ P_1 = 1,$$

$$Q_{n+2} = 2Q_{n+1} + Q_n, \quad Q_0 = 2,\ Q_1 = 2.$$

Thus, the following closed-form solutions (according to (1.3) and (1.4)) exist for the Pell and Pell–Lucas numbers, respectively:

$$P_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}},$$

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

For more information about the above sequences see [1]–[6].

The aim of this note is to establish some general results for fundamental recursive sequences (1.1) and (1.2). We prove some general limit formulas, where the product of the first n terms of sequences (1.1) and (1.2) appears. We also prove some limit formulas that connect sequences (1.1) and (1.2) to the number $e(\approx 2.71828...)$.

2. Preliminaries

In this section we present some preliminary results. Note that throughout this paper, the symbol $\sim$ means asymptotic equivalence.

Lemma 2.1. *If $\{U_n\}_{n\geq 0}$ is a sequence in recursive form (1.1) corresponding to $m>0$, then*

$$U_n \sim \frac{\alpha^n}{\Delta} \qquad (n\to\infty), \tag{2.1}$$

and if $\{V_n\}_{n\geq 0}$ is a sequence in recursive form (1.2), then

$$V_n \sim \alpha^n \qquad (n\to\infty).$$

Proof. Use identities (1.3), (1.4), and also consider the fact that if $m>0$, then $\alpha = \frac{m+\sqrt{m^2+4}}{2} > 1$. The lemma is proved. □

Lemma 2.2. *For $\alpha > 1$ and $s \geq 1$, the series*

$$\sum_{i=1}^{\infty} \log\left(1+\frac{(-1)^{i^s+1}}{\alpha^{2i^s}}\right)$$

is absolutely convergent.

Proof. By simple calculations we have the following inequalities:

$$\log(1+x) < x \qquad (x > 0), \tag{2.2}$$

$$-\log(1-x) < \frac{x}{1-x} \qquad (0 < x < 1). \tag{2.3}$$

We have also the following well-known geometric power series:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \qquad |x| < 1. \tag{2.4}$$

Inequality (2.2) and identity (2.4) give

$$\sum_{i\ odd} \log\left(1 + \frac{1}{\alpha^{2i^s}}\right) < \sum_{i\ odd} \frac{1}{\alpha^{2i^s}} < \sum_{i=0}^{\infty} \frac{1}{\alpha^i} = \sum_{i=0}^{\infty} \left(\frac{1}{\alpha}\right)^i = \frac{1}{1-\frac{1}{\alpha}}.$$

Therefore the series of positive terms $\sum_{i\ odd} \log\left(1+\frac{1}{\alpha^{2i^s}}\right)$, where the sum runs over all odd numbers i, converges. Inequality (2.3) gives

$$\sum_{i\ even} -\log\left(1 - \frac{1}{\alpha^{2i^s}}\right) < \sum_{i=1}^{\infty} -\log\left(1-\frac{1}{\alpha^i}\right) < \sum_{i=1}^{\infty} \frac{\frac{1}{\alpha^i}}{1-\frac{1}{\alpha^i}} = \sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1},$$

where the series $\sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1}$ converges, since if i is sufficiently large we have

$$\frac{1}{\alpha^i - 1} < \frac{1}{\alpha^i - \frac{\alpha^i}{2}} = 2\left(\frac{1}{\alpha}\right)^i.$$

Therefore the series of positive terms $\sum_{i\ even} -\log\left(1-\frac{1}{\alpha^{2i^s}}\right)$, where the sum runs over all even numbers i, also converges. Now, we have

$$\begin{aligned}\sum_{i=1}^{\infty} \left|\log\left(1 + \frac{(-1)^{i^s+1}}{\alpha^{2i^s}}\right)\right| \\ = \sum_{i\ even} -\log\left(1-\frac{1}{\alpha^{2i^s}}\right) + \sum_{i\ odd} \log\left(1+\frac{1}{\alpha^{2i^s}}\right),\end{aligned} \tag{2.5}$$

where the two series in the right hand side of (2.5) we have proved are convergent. Therefore the series $\sum_{i=1}^{\infty} \log\left(1+\frac{(-1)^{i^s+1}}{\alpha^{2i^s}}\right)$ is absolutely convergent. The lemma is proved. □

3. Main results

We start this section with the following theorem.

Theorem 3.1. *Let $\{U_n\}_{n\geq 0}$ be a sequence in recursive form (1.1) corresponding to $m \geq 1$. Then for any $s \in \mathbb{N}$ the following limit holds:*

$$\lim_{n\to\infty} \sqrt[n^{s+1}]{U_{1^s}U_{2^s}\dots U_{n^s}} = \sqrt[s+1]{\alpha}. \tag{3.1}$$

Proof. For a sequence in recursive form (1.1), we have (see (1.3))

$$U_i = \frac{\alpha^i - \beta^i}{\Delta} = \frac{\left(\frac{m+\sqrt{m^2+4}}{2}\right)^i - \left(\frac{m-\sqrt{m^2+4}}{2}\right)^i}{\sqrt{m^2+4}}.$$

A simple computation shows that

$$\frac{m-\sqrt{m^2+4}}{2} = \frac{-1}{\frac{m+\sqrt{m^2+4}}{2}}, \qquad \forall m \in \mathbb{R}.$$

Therefore,

$$U_i = \frac{\alpha^i - \left(-\frac{1}{\alpha}\right)^i}{\Delta} = \frac{1}{\Delta}\left(\alpha^i + \frac{(-1)^{i+1}}{\alpha^i}\right) = \frac{\alpha^i}{\Delta}\left(1 + \frac{(-1)^{i+1}}{\alpha^{2i}}\right).$$

Hence,

$$\log U_i = i\log\alpha + \log\Delta^{-1} + \log\left(1 + \frac{(-1)^{i+1}}{\alpha^{2i}}\right). \tag{3.2}$$

Consequently, for any $s \in \mathbb{N}$, we have

$$\log U_{i^s} = i^s\log\alpha + \log\Delta^{-1} + \log\left(1 + \frac{(-1)^{i^s+1}}{\alpha^{2i^s}}\right). \tag{3.3}$$

We know that if $m > 0$, then $\alpha = \frac{m+\sqrt{m^2+4}}{2} > 1$. Since $x^s\log\alpha$ (with $s \in \mathbb{N}$ and $\alpha > 1$) is strictly increasing and positive in the interval $[1,\infty)$, we find that

$$\begin{aligned}\sum_{i=1}^{n} i^s\log\alpha &= \int_1^n x^s\log\alpha\, dx + O(n^s\log\alpha) \\ &= \frac{(n^{s+1}-1)}{s+1}\log\alpha + O(n^s\log\alpha) = \frac{n^{s+1}}{s+1}\log\alpha + o(n^{s+1}).\end{aligned} \tag{3.4}$$

Hence, (3.3), (3.4), and Lemma 2.2 give

$$\log \sqrt[n^{s+1}]{U_{1^s}U_{2^s}\dots U_{n^s}} = \frac{1}{n^{s+1}}\Big(\log U_{1^s} + \log U_{2^s} + \dots + \log U_{n^s}\Big)$$
$$= \frac{1}{n^{s+1}}\left(\frac{n^{s+1}}{s+1}\log\alpha + n\log\Delta^{-1} + \sum_{i=1}^{n}\log\left(1+\frac{(-1)^{i^s+1}}{\alpha^{2i^s}}\right) + o(n^{s+1})\right)$$
$$= \frac{1}{s+1}\log\alpha + o(1).$$

This completes the proof. □

Remark 3.2. By a proof similar to the proof of Theorem 3.1, it can be shown that relation (3.1) holds for a sequence $\{V_n\}_{n\geq 0}$ in recursive form (1.2) corresponding to $m \geq 1$.

For the sequence $\{F_n\}_{n\geq 0}$ of Fibonacci numbers we have (by (1.3)) $\alpha = \frac{1+\sqrt{5}}{2}$. The algebraic number $\frac{1+\sqrt{5}}{2}$ is called the *golden ratio* and is usually denoted by the Greek letter φ (phi). It is well-known that the ratio of two consecutive Fibonacci numbers tends to the golden ratio, i.e.,

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \varphi.$$

The golden ratio φ, can also be expressed exactly by the following infinite series of continued fractions and that of nested square roots (see, for example, [8]):

$$\varphi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}},$$

and

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}.$$

Moreover, if $\wp(x)$ denotes the counting function of the Fibonacci numbers, i.e., the number of F_n not exceeding x, then (see [3])

$$\lim_{n\to\infty}\sqrt[\wp(n)]{\wp(1)+\wp(2)+\dots+\wp(n)} = \varphi.$$

Here, the following example gives a new expansion of the golden ratio φ.

Example 3.3. The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is of the recursive form (1.1) corresponding to $m = 1$. Hence, by Theorem 3.1 the following limit holds:

$$\lim_{n\to\infty} \sqrt[n^{s+1}]{F_{1^s} F_{2^s} \ldots F_{n^s}} = \sqrt[s+1]{\varphi}, \qquad \forall s \in \mathbb{N}.$$

In particular, if $s = 1, 2$, and 3, we have respectively,

$$\lim_{n\to\infty} \sqrt[n^2]{F_1 F_2 \ldots F_n} = \sqrt{\varphi},$$

$$\lim_{n\to\infty} \sqrt[n^3]{F_1 F_4 \ldots F_{n^2}} = \sqrt[3]{\varphi},$$

$$\lim_{n\to\infty} \sqrt[n^4]{F_1 F_8 \ldots F_{n^3}} = \sqrt[4]{\varphi}.$$

Theorem 3.4. *If $\{U_n\}_{n\geq 0}$ is a sequence in recursive form* (1.1) *corresponding to $m \geq 1$, then*

$$\lim_{n\to\infty} \frac{\sqrt[n]{U_1 U_2 \ldots U_n}}{\sqrt{U_n}} = \sqrt{\frac{\alpha}{\Delta}}, \tag{3.5}$$

and if $\{V_n\}_{n\geq 0}$ is a sequence in recursive form (1.2) *corresponding to $m \geq 1$, then*

$$\lim_{n\to\infty} \frac{\sqrt[n]{V_1 V_2 \ldots V_n}}{\sqrt{V_n}} = \sqrt{\alpha}. \tag{3.6}$$

Proof. We first prove (3.5). For a sequence in recursive form (1.1) we have (see (3.2))

$$\log U_i = i \log \alpha - \log \Delta + \log\left(1 + \frac{(-1)^{i+1}}{\alpha^{2i}}\right).$$

Hence,

$$\sum_{i=1}^{n} \log U_i = \frac{(n^2+n)}{2} \log \alpha - n \log \Delta + \sum_{i=1}^{n} \log\left(1 + \frac{(-1)^{i+1}}{\alpha^{2i}}\right). \tag{3.7}$$

By subtracting $\frac{n^2}{2}\log\alpha$ from both sides of (3.7), then multiplying by $\frac{1}{n}$, we obtain

$$(3.8)\quad \frac{1}{n}\sum_{i=1}^{n}\log U_i-\frac{n}{2}\log\alpha=\frac{1}{2}\log\alpha-\log\Delta+\frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{(-1)^{i+1}}{\alpha^{2i}}\right).$$

Now, we take the exponential of both sides of (3.8) to obtain

$$(3.9)\qquad \frac{\sqrt[n]{U_1U_2\dots U_n}}{\sqrt{\alpha^n}}=\frac{\sqrt{\alpha}}{\Delta}\left(\prod_{i=1}^{n}\left(1+\frac{(-1)^{i+1}}{\alpha^{2i}}\right)\right)^{\frac{1}{n}}.$$

Taking the limit of (3.9) as $n\to\infty$ we get

$$(3.10)\qquad \lim_{n\to\infty}\frac{\sqrt[n]{U_1U_2\dots U_n}}{\sqrt{\alpha^n}}=\frac{\sqrt{\alpha}}{\Delta}\lim_{n\to\infty}\left(\prod_{i=1}^{n}\left(1+\frac{(-1)^{i+1}}{\alpha^{2i}}\right)\right)^{\frac{1}{n}}.$$

We know that for any recursive sequence of the form (1.1) or (1.2) corresponding to $m>0$ we have $\alpha=\frac{m+\sqrt{m^2+4}}{2}>1$, consequently by Lemma 2.2 the series

$$\sum_{i=1}^{\infty}\log\left(1+\frac{(-1)^{i+1}}{\alpha^{2i}}\right)$$

converges absolutely, hence, $\frac{1}{n}\sum_{i=1}^{\infty}\log\left(1+\frac{(-1)^{i+1}}{\alpha^{2i}}\right)$ tends to zero as $n\to\infty$, consequently

$$(3.11)\quad \left(\prod_{i=1}^{n}\left(1+\frac{(-1)^{i+1}}{\alpha^{2i}}\right)\right)^{\frac{1}{n}} =\exp\left(\frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{(-1)^{i+1}}{\alpha^{2i}}\right)\right)\to 1,\qquad \text{as } n\to\infty.$$

Hence, (3.10) and (3.11) give

$$(3.12)\qquad \lim_{n\to\infty}\frac{\sqrt[n]{U_1U_2\dots U_n}}{\sqrt{\alpha^n}}=\frac{\sqrt{\alpha}}{\Delta}.$$

Combining the property $U_n\sim\frac{\alpha^n}{\Delta}$ (see (2.1)) with (3.12), this completes the proof of (3.5). For a sequence $\{V_n\}_{n\geq 0}$ in recursive form (1.2) the proof of relation (3.6) is the same as above and easier. The theorem is proved. $\square$

Next, we prove some limit formulas that connect sequences in recursive forms (1.1) and (1.2) to the number e.

THEOREM 3.5. *Let $\{U_n\}_{n\geq 0}$ be a sequence in recursive form (1.1) corresponding to $m > 0$. If $v, s \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ are such that $r < s$, then*

$$\lim_{n\to\infty} \left(\frac{\log U_{vn+s}}{\log U_{vn+r}}\right)^n = \sqrt[v]{e^{s-r}}. \tag{3.13}$$

PROOF. We have (see (3.2))

$$\log U_n = n \log \alpha + \log \Delta^{-1} + \log \left(1 + \frac{(-1)^{n+1}}{\alpha^{2n}}\right).$$

Therefore if we put, for sake of simplicity, $c_n = \frac{\log \Delta^{-1} + \log\left(1+\frac{(-1)^{n+1}}{\alpha^{2n}}\right)}{\log \alpha} \to c = \frac{\log \Delta^{-1}}{\log \alpha}$ (since we have $\alpha = \frac{m+\sqrt{m^2+4}}{2} > 1$ for $m > 0$, hence $\log\left(1+\frac{(-1)^{n+1}}{\alpha^{2n}}\right) \to 0$ as $n \to \infty$), then

$$\left(\frac{\log U_{vn+s}}{\log U_{vn+r}}\right)^n = \left(\frac{(vn+s)\log\alpha + \log\Delta^{-1} + \log\left(1+\frac{(-1)^{vn+s+1}}{\alpha^{2(vn+s)}}\right)}{(vn+r)\log\alpha + \log\Delta^{-1} + \log\left(1+\frac{(-1)^{vn+r+1}}{\alpha^{2(vn+r)}}\right)}\right)^n$$

$$= \left(1 + \frac{s-r}{vn+r}\right)^n \frac{\left(1+\frac{c_{vn+s}}{vn+s}\right)^n}{\left(1+\frac{c_{vn+r}}{vn+r}\right)^n} \to e^{\frac{s-r}{v}} \frac{e^{\frac{c}{v}}}{e^{\frac{c}{v}}} = e^{\frac{s-r}{v}} :$$

in fact, it is well-known that if the sequence $a_n \to \infty$, then

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e,$$

hence,

$$\left(1+\frac{c_{vn+s}}{vn+s}\right)^n = \left(\left(1+\frac{1}{\frac{vn+s}{c_{vn+s}}}\right)^{\frac{vn+s}{c_{vn+s}}}\right)^{\frac{n}{vn+s}c_{vn+s}} \to e^{\frac{c}{v}},$$

analogously the other limit. This completes the proof. $\square$

REMARK 3.6. By a proof similar to the proof of Theorem 3.5, it can be shown that relation (3.13) holds for a sequence $\{V_n\}_{n\geq 0}$ in recursive form (1.2) corresponding to $m > 0$.

EXAMPLE 3.7. The Fibonacci sequence $\{F_n\}_{n\geq 0}$ and the Lucas sequence $\{L_n\}_{n\geq 0}$ are of the recursive forms (1.1) and (1.2) corresponding to $m = 1$, respectively. Hence, by Theorem 3.5 and Remark 3.6, if $v = 2$, $s = 3$, and $r = 1$, then

$$\lim_{n\to\infty}\left(\frac{\log F_{2n+3}}{\log F_{2n+1}}\right)^n = \lim_{n\to\infty}\left(\frac{\log L_{2n+3}}{\log L_{2n+1}}\right)^n = e.$$

Here, we shall recall the well-known prime number theorem (PNT), which states that the n-th prime number p_n is asymptotically equivalent to $n\ln n$ (i.e., $p_n \sim n\ln n$). We use the PNT to prove the next theorem.

THEOREM 3.8. *Let $\{U_n\}_{n\geq 0}$ be a sequence in recursive form* (1.1) *corresponding to $m = 1$. Then*

$$\lim_{n\to\infty} \sqrt[\phi_n]{U_1^{U_0}U_2^{U_1}\dots U_n^{U_{n-1}}} = e, \tag{3.14}$$

where $\phi_n = p_{_{U_{n+1}}}$, that is, the U_{n+1}-th prime number.

PROOF. The function $\ln x$ is continuous on the interval $[U_n, U_{n+1}]$ for all $n \in \mathbb{N}$. By the integral mean value theorem, we have $\int_{U_n}^{U_{n+1}} \ln x\ dx = (U_{n+1} - U_n)\ln c = U_{n-1}\ln c$ for some c with $U_n < c < U_{n+1}$. Hence

$$U_{n-1}\ln U_n < \int_{U_n}^{U_{n+1}} \ln x\ dx < U_{n-1}\ln U_{n+1}.$$

Since $\ln U_{n+1} \sim \ln U_n$ (by Lemma 2.1), we have

$$1 < \frac{\int_{U_n}^{U_{n+1}} \ln x\ dx}{U_{n-1}\ln U_n} < \frac{\ln U_{n+1}}{\ln U_n} \to 1,$$

that is,

$$U_{n-1}\ln U_n \sim \int_{U_n}^{U_{n+1}} \ln x\ dx. \tag{3.15}$$

Now, let us recall the well-known proposition (see [7, page 332]) that states for two series of positive terms $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$, if $\sum_{i=1}^{\infty} b_i$ diverges and

$a_i \sim b_i$, then $\sum_{i=1}^{n} a_i \sim \sum_{i=1}^{n} b_i$. Using this fact and by use of (3.15), we have (use also PNT),

$$\sum_{i=1}^{n} U_{i-1} \ln U_i \sim \sum_{i=1}^{n} \int_{U_i}^{U_{i+1}} \ln x \, dx = \int_{U_1}^{U_{n+1}} \ln x \, dx$$

$$\sim U_{n+1} \ln U_{n+1} \sim p_{_{U_{n+1}}}.$$

This gives

$$\frac{\sum_{i=1}^{n} U_{n-1} \ln U_i}{p_{_{U_{n+1}}}} \to 1.$$

This completes the proof. □

Remark 3.9. By a proof similar to the proof of Theorem 3.8, it can be shown that relation (3.14) holds for a sequence $\{V_n\}_{n\geq 0}$ in recursive form (1.2) corresponding to $m = 1$.

Example 3.10. The Lucas sequence $\{L_n\}_{n\geq 0}$ is of the recursive form (1.2) corresponding to $m = 1$. Hence, by Remark 3.9 we have

$$\lim_{n\to\infty} \sqrt[\phi_n]{L_1^{L_0} L_2^{L_1} \ldots L_n^{L_{n-1}}} = e,$$

where ϕ_n is the L_{n+1}-th prime number.

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References

[1] M. Bicknell, *A primer on the Pell sequence and related sequences*, Fibonacci Quart. **13** (1975), 345–349.

[2] G. Bilgici and T.D. Şentürk, *Some addition formulas for Fibonacci, Pell and Jacobsthal numbers*, Ann. Math. Sil. **33** (2019), 55–65.

[3] R. Farhadian and R. Jakimczuk, *Notes on a general sequence*, Ann. Math. Sil. **34** (2020), 193–202.

[4] A.F. Horadam, *Pell identities*, Fibonacci Quart. **9** (1971), 245–252, 263.

[5] A.F. Horadam and J.M. Mahon, *Pell and Pell–Lucas polynomials*, Fibonacci Quart. **23** (1985), 7–20.

[6] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, 2001.

[7] J. Rey Pastor, P. Pi Calleja, and C.A. Trejo, *Análisis Matemático*, Vol. 1, Editorial Kapelusz, Buenos Aires, 1969.
[8] A.P Stakhov, *The golden section in the measurement theory*, Comput. Appl. Math. **17** (1989), 613–638.

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