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MHD EQUATIONS IN A BOUNDED DOMAIN

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Abstract. We consider the MHD system in a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, with Dirichlet boundary conditions. Using Dan Henry's semigroup approach and Giga–Miyakawa estimates we construct global in time, unique solutions to fractional approximations of the MHD system in the base space $(L^2(\Omega))^N \times (L^2(\Omega))^N$. Solutions to MHD system are obtained next as a limits of that fractional approximations.

1. Introduction

We consider the Dirichlet boundary value problem for the incompressible magnetohydrodynamical (MHD) system

$$
\begin{aligned}
& u_t - \nu\Delta u + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0,\\
& b_t - \eta\Delta b + u\cdot\nabla b = b\cdot\nabla u, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0,\\
& \operatorname{div} u = \operatorname{div} b = 0, \qquad\qquad (1.1)\\
& u = 0, \quad b = 0 \quad \text{on} \quad \partial\Omega,\\
& u(0,x) = u_0(x), \quad b(0,x) = b_0(x), \quad x\in\Omega,
\end{aligned}
$$

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in a bounded domain $\Omega \subset \mathbb{R}^N$ with C^2 boundary, where $N = 2, 3$. Here u is the velocity of the fluid flow and b is the magnetic field. These functions are the vector-valued functions of $x \in \Omega$ and $t \geq 0$ ($u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$, $b(t,x) = (b_1(t,x), \ldots, b_N(t,x))$). The total pressure $p = p(t,x)$ is real-valued function of $x \in \Omega$ and $t \geq 0$. The constant $\nu > 0$ is the viscosity of the fluid and $\eta > 0$ is the magnetic diffusivity. Here $u \cdot \nabla b$ denotes the matrix product $u(\nabla b)^T$, i.e.

$$u \cdot \nabla b = u(\nabla b)^T = \left(u_1 \frac{\partial b_1}{x_1} + \ldots + u_N \frac{\partial b_1}{x_N}, \ldots, u_1 \frac{\partial b_N}{x_1} + \ldots + u_N \frac{\partial b_N}{x_N} \right).$$

The MHD system is called *ideal MHD system* if $\nu = 0$ and $\eta > 0$ and *non-resistive MHD system* if $\nu > 0$ and $\eta = 0$ (see [20]). In the paper [23], Miao and Yuan called MHD system with $\nu = \eta = 0$ also ideal. If the magnetic field $b(t,x)$ identically equals zero, MHD system reduces to the incompressible Navier–Stokes equation. It is impossible to recall even the most important results devoted to N-S problem, since corresponding literature is too large, see anyway [3, 4, 10, 13, 17, 21, 27] together with the references cited there.

MHD equations in the whole of $\mathbb{R}^N$ studies the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas, liquid metals, and salt water or electrolytes. Besides their physical applications, the MHD equations remain also an outstanding mathematical problem. Fundamental mathematical issues such as well-posedness and regularity of their solutions have generated extensive research and many interesting results have been already obtained. The most of them concern MHD equations in unbounded domain $\mathbb{R}^N$, $N = 2, 3$. We would like to mention some of them. Recently, the Cauchy's problem of MHD system in $\mathbb{R}^3$ was considered in the papers [24, 29, 14, 20]. For small data, the existence of a global mild solution in BMO^{-1} as well as a locall mild solution in bmo^{-1} was established in [24]. Obtained solutions are unique in the spaces $C([0,\infty); BMO^{-1})$ and $C([0,T); bmo^{-1})$, respectively. The global existence of the mild solutions was obtained in the paper [29] when the norms of the initial data (in the spaces χ^{-1}) are bounded exactly by the minimal value of the viscosity coefficients. He et al. ([14]) got the global smooth solution under the assumption that the difference between the magnetic field and the velocity is small initially and $\frac{|\nu-\eta|}{\nu+\eta} < 1$. In paper [20], authors established the global existence of smooth solutions for a class of large initial data. In [30], Wu studied Generalized MHD equations (GMHD), where the viscosity terms $-\Delta u$ and $-\Delta b$, were replaced by $(-\Delta)^\alpha u$ and $(-\Delta)^\beta b$ with $\alpha, \beta > 0$. He showed when $\nu, \eta > 0$ the GMHD equations with any $\alpha, \beta > 0$ possess a global week solution corresponding to any L^2 initial data. Moreover, week solutions associated with $\alpha \geq \frac{1}{2} + \frac{N}{4}$ and $\beta \geq \frac{1}{2} + \frac{N}{4}$ are global classical solutions, when their initial date are sufficiently smooth.

In bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ the MHD system with the additional term $f(t, x)$ on the right hand side in first equation and the boundary condition

$$u = 0, \quad b \cdot n = 0, \quad \operatorname{curl} b \times n = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

where n is the unit outward normal vector to $\partial\Omega$ was studied by Duvaut and Lions long time ago. They constructed in [8] a class of weak and strong solutions to it. Next, Sermange and Temam recalled and completed their results in [26]. According to [26, Theorem 3.1] under the assumptions $f \in L^2(0, T, V_1^{'})$ and $(u_0, b_0) \in H$ there exists week solution for MHD system satisfying $(u, b) \in L^2(0, T, V) \cap L^\infty(0, T, H)$, where

$$V = \{v \in H_0^1(\Omega) : \operatorname{div} v = 0\} \times \{C \in H^1(\Omega) : \operatorname{div} C = 0 \text{ and } C \cdot n|_{\partial\Omega} = 0\},$$

$$V_1^{'} \text{ denotes dual space of } V_1 = \{v \in H_0^1(\Omega) : \operatorname{div} v = 0\},$$

$$H_1 = \{v \in L^2(\Omega) : \operatorname{div} v = 0 \text{ and } v \cdot n|_{\partial\Omega} = 0\}, \quad \text{and} \quad H = H_1 \times H_1.$$

The characterisation of spaces V_1, $V_1^{'}$ and H_1 is discussed in [27, section 1.4, chapter 3] . Moreover, when f and initial data are more regular, that is $f \in L^\infty(0, T, H_1)$ and $(u_0, b_0) \in V$, then the unique local solution (global when the dimension is $N = 2$) is strong one (see [26, Theorem 3.2]). In [19] Kozono showed the existence and uniqueness of a global weak solution and a local classical solution for the ideal MHD system ($\nu = 0$, $\eta = 1$) considered in a bounded domain $\Omega \subset \mathbb{R}^2$ with the boundary condition

$$u \cdot n = 0, \quad b \cdot n = 0, \quad \operatorname{rot} b = 0 \quad \text{on} \quad \partial\Omega \times (0, T).$$

Recently, MHD system was considered in [28] with the boundary condition

$$u \cdot n = 0, \quad b \cdot n = 0, \quad \operatorname{curl} u \times n = 0, \quad \operatorname{curl} b \times n = 0,$$

where n is the unit outward normal vector to $\partial\Omega$ and Ω is bounded in $\mathbb{R}^3$. It was shown there that under the specified relation between u_0 and b_0 the MHD system admits a unique global smooth solution.

This paper is devoted to the global in time solvability and properties of solutions to problem (1.1) in a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. In section 3 we will obtain solutions of 2-D MHD equations as limits of solutions to sub-critical approximations (2.4) (where the operators $-P\Delta u$ and $-P\Delta b$ are replace with their fractional powers $(-P\Delta)^\alpha u$ and $(-P\Delta)^\beta b$, respectively, $\alpha, \beta > 1$) when $\alpha, \beta \to 1^+$. The main result in 2-D case is the following.

Theorem 1.1. *Let* $(u^{\alpha_n}_{\beta_n}, b^{\alpha_n}_{\beta_n})$ *be the solution of the subcritical problem* (2.4) *(constructed in Theorem* 2.4 *in the base space* $\mathbf{L}^2(\Omega)$*) corresponding to the initial condition* $u_0, b_0 \in D(A^{\frac{1+\epsilon}{2}}) \subset \mathcal{H}^{1+\epsilon}(\Omega)$ *and the fractional exponents* $\alpha_n, \beta_n \in (1, \frac{5}{4}]$*. Then passing, over a subsequence (denoted the same), with* α_n, β_n *to* 1 *in system* (2.4) *we get a weak solution* (u, b) *(not necessarily unique) to the critical problem (*$\alpha = \beta = 1$*) satisfying, for each test function* $\phi \in D(A) \subset \mathcal{H}^2(\Omega)$*, the equalities* (3.8) *and* (3.9)*.*

In 3-D, the higher order diffusion terms like $\nu_3(-P\Delta)^\alpha u$ and $\eta_3(-P\Delta)^\beta b$, where $\alpha, \beta > \frac{5}{4}$ and $\nu_3, \eta_3 > 0$, are added to the system (1.1). Instead of equations (1.1) we study their regularization (2.5). Next we will obtain solutions of 3-D MHD equations as limits of solutions of (2.5) when $\nu_3, \eta_3 \to 0^+$ (section 4). The main result in 3-D case is as follows.

Theorem 1.2. *Let the fractional exponents* $\alpha, \beta > \frac{5}{4}$ *be fixed (but close to* $\frac{5}{4}$*), the parameters* $\nu_n, \eta_n \in (0, 1]$ *and* $(u^{\nu_n}_{\eta_n}, b^{\nu_n}_{\eta_n})$ *be the solution of the regularization problem* (2.5) *(constructed in Theorem* 2.4 *in the base space* $\mathbf{L}^2(\Omega)$*) corresponding to the initial condition* $u_0, b_0 \in D(A^{\frac{3+2\epsilon}{4}}) \subset \mathcal{H}^{\frac{3}{2}+\epsilon}(\Omega)$ *and the parameters* ν_n, η_n*. Then passing, over a subsequence (denoted the same), with* ν_n, η_n *to 0 in problem* (2.5) *we get a weak solution* (U, B) *(not necessarily unique) to the MHD problem* (1.1) *satisfying, for each test function* $\psi \in D(A^{\frac{\max\{\alpha,\beta\}}{2}})$*, the equalities* (4.9) *and* (4.10)*.*

We consider approximation problems (2.4) and (2.5) in the framework of semilinear parabolic equations with a sectorial positive operator (see [15, 2]). This offers a simple but formalized proof of local solvability as well as the regularity of solutions. There are different possible choices of the phase spaces for these problems. We choose $(L^2(\Omega))^N \times (L^2(\Omega))^N$ as the *base spaces* (in which the equations are fulfilled). Section 2 of the paper is devoted to the global in time solvability of the problem (2.4) and (2.5). We obtained there also the global in time solvability of (1.1) in phase space $\left(D(A^{\frac{N+2\epsilon}{4}})\right)^2$ for small data. In this study we use a technique proposed in our recent publications [5, 6, 7, 16] and in recent papers devoted to the Navier–Stokes equation [3, 4].

Notation. Standard notation for Sobolev spaces is used. When necessary, for clarity of the presentation, we indicate the dependence of solution (u, b) of (2.6) on α, β in 2-D and ν_3, η_3 in 3-D, calling it $(u^\alpha_\beta, b^\alpha_\beta)$, $(u^{\nu_3}_{\eta_3}, b^{\nu_3}_{\eta_3})$, respectively. Let $r - \epsilon$, $\epsilon > 0$, denotes a number strictly less than r but arbitrarily close to it. Moreover, we set $P := P_2$ and $A := A_2$ (see (2.1) for definitions A_2 and P_2).

2. Solvability of fractional approximations of MHD system

We start with recalling some results on the Stokes operator (see [11, 12, 13]). Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary. For simplicity of the notation let us introduce the following list of vector spaces:

$$\mathcal{L}^r(\Omega) := (L^r(\Omega))^N, \quad \mathbf{L}^r(\Omega) := \mathcal{L}^r(\Omega) \times \mathcal{L}^r(\Omega),$$

$$\mathcal{W}^{2,r}(\Omega) := (W^{2,r}(\Omega))^N, \quad X_r := cl_{\mathcal{L}^r(\Omega)}\{\phi \in (C_0^\infty(\Omega))^N : \operatorname{div}\phi = 0\},$$

for $1 < r < \infty$. We define the Stokes operator A_r in X_r

$$A_r = -P_r \begin{bmatrix} \Delta & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \Delta \end{bmatrix}_{N\times N} \tag{2.1}$$

with domain $D(A_r) = X_r \cap D(-\Delta) = X_r \cap \{\phi \in \mathcal{W}^{2,r}(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$, where P_r denotes the continuous projection from $\mathcal{L}^r(\Omega)$ to X_r given by the decomposition of $\mathcal{L}^r(\Omega)$ onto the space of divergence-free vector fields and scalar function gradient. The operator $-A_r$ generates on X_r a bounded holomorphic semigroup $\{e^{-tA_r}\}$ of class C_0 (see [11, Theorem 2]). To simplify notation we will set $P := P_2$ and $A := A_2$. In our work we use the Balakrishnan's definition of the fractional powers operators (see [18, 22]).

Definition 2.1. Let B be a closed linear densely defined operator in a Banach space X, such that interval $(-\infty, 0)$ is included in the resolvent set $\rho(B)$ and

$$M = \sup_{\lambda>0}\left\{\|\lambda(\lambda + B)^{-1}\|\right\} < \infty.$$

Then, for $\eta \in (0,1)$,

$$B^\eta\phi = \frac{\sin(\pi\eta)}{\pi}\int_0^\infty s^{\eta-1}B(s+B)^{-1}\phi ds.$$

This definition can be used to study problem (2.6) both in the case of a bounded and unbounded domain. For complete of presentation we recall from paper [7] a lemma justifying convergence of negative powers:

Lemma 2.2. *Let B be a positive operator in a Banach space X. For arbitrary $\phi \in X$, and $\epsilon > 0$ there exists $L > 0$ such that for $\beta \in (0, \frac{1}{2})$:*

$$\|(I - B^{-\beta})\phi\|_X \leq \frac{\sin(\pi\beta)}{\pi}\big(2L(1+M) + ML^{-1})\|\phi\|_X + \epsilon.$$

Consequently, the left side tends to zero as $0 < \beta \to 0^+$.

and important results ([13, p. 270]), which are crucial in the proof of local and global solvability of (2.6).

Lemma 2.3. *Let $0 \leq \delta < \frac{1}{2} + \frac{N}{2}(1 - \frac{1}{r})$. Then*

$$\|A_r^{-\delta} P_r(u \cdot \nabla v)\|_{\mathcal{L}^r(\Omega)} \leq M\|A_r^{\theta} u\|_{\mathcal{L}^r(\Omega)}\|A_r^{\rho} v\|_{\mathcal{L}^r(\Omega)} \tag{2.2}$$

with a constant $M = M(\delta, \theta, \rho, r)$, provided that

$$\delta + \theta + \rho \geq \frac{1}{2}(\frac{N}{r} + 1), \quad \theta, \rho > 0, \quad \rho + \delta > \frac{1}{2}.$$

Moreover, if $\delta \geq \frac{1}{2}$ then

$$\|A_r^{-\delta} P_r(u \cdot \nabla v)\|_{\mathcal{L}^r(\Omega)} \leq C\||u| \cdot |v|\|_{\mathcal{L}^s(\Omega)} \tag{2.3}$$

with $\frac{1}{s} = \frac{1}{r} + \frac{2\epsilon}{N}$ and $\epsilon = \delta - \frac{1}{2}$.

It is also a familiar fact, that if B_i, $i = 1, \ldots, m$, with domains $D(B_i)$ respectively, are sectorial positive operators on the Banach space Z_i, then the product operator $\mathbf{B} = (B_1, \ldots, B_m)$, considered with the domain $D(B_1) \times \cdots \times D(B_m)$, will be sectorial positive (product) operator on the space $Z_1 \times \cdots \times Z_m$ (see e.g. [2, Example 1.3.2, p. 37]). Consequently, the operator $\mathbf{A}_N^{\alpha,\beta} : D(\mathbf{A}_N^{\alpha,\beta}) \subset \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ defined by the formula

$$\mathbf{A}_N^{\alpha,\beta}(\phi, \psi) = \begin{bmatrix} \nu_N A^\alpha & 0 \\ 0 & \eta_N A^\beta \end{bmatrix} (\phi, \psi) = \left(\nu_N A^\alpha \phi, \eta_N A^\beta \psi\right), \quad N = 2,$$

and

$$\begin{aligned} \mathbf{A}_N^{\alpha,\beta}(\phi, \psi) &= \begin{bmatrix} \nu A + \nu_N A^\alpha & 0 \\ 0 & \eta A + \eta_N A^\beta \end{bmatrix} (\phi, \psi) \\ &= \left((\nu A + \nu_N A^\alpha)\phi, (\eta A + \eta_N A^\beta)\psi\right), \quad N = 3, \end{aligned}$$

where $\nu, \eta \geq 0$ and $\eta_N, \nu_N > 0$ for $N = 2, 3$, is sectorial positive on the space $\mathbf{L}^2(\Omega)$.

2.1. Formulation of the problem and its local solvability

In 2-D MHD, we improve the viscosity terms $Au = -P\Delta u$ and $Ab = -P\Delta b$ so that the resulting problems are sub-critical. The way for obtaining such effect is to replace the classical viscosity terms through a bit higher fractional diffusion terms $A^\alpha u = (-P\Delta)^\alpha u$ and $A^\beta b = (-P\Delta)^\beta b$ with $\alpha, \beta > 1$. So improved viscosity together with $\mathbf{H}^1(\Omega)$ a priori estimate allow to control to the nonlinear term. Using Dan Henry's *semigroup approach* we construct regular solutions to such approximations. Next we will study the process of letting with $\alpha, \beta \to 1^+$. Solution in critical case $\alpha = \beta = 1$ is obtained as a limit of such regular solutions of the approximations.

Precisely speaking, in 2-D case instead (1.1) we consider the family of sub-critical problems with $\alpha, \beta > 1$:

$$\begin{aligned}
&u_t + \nu_2(-P\Delta)^\alpha u + P(u\cdot\nabla u) = P(b\cdot\nabla b), \quad x\in\Omega\subset\mathbb{R}^2,\ t>0,\\
&b_t + \eta_2(-P\Delta)^\beta b + P(u\cdot\nabla b) = P(b\cdot\nabla u), \quad x\in\Omega\subset\mathbb{R}^2,\ t>0,\\
&u=0, \quad b=0 \quad \text{on} \quad \partial\Omega,\\
&u(0,x)=u_0(x), \quad b(0,x)=b_0(x), \quad x\in\Omega.
\end{aligned} \tag{2.4}$$

In 3-D case, we consider the approximation/regularization of (1.1) having the form:

$$\begin{aligned}
&u_t + \left[\nu(-P\Delta) + \nu_3(-P\Delta)^\alpha\right] u + P(u\cdot\nabla u) = P(b\cdot\nabla b), \quad x\in\Omega\subset\mathbb{R}^3,\ t>0,\\
&b_t + \left[\eta(-P\Delta) + \eta_3(-P\Delta)^\beta\right] b + P(u\cdot\nabla b) = P(b\cdot\nabla u), \quad x\in\Omega\subset\mathbb{R}^3,\ t>0,\\
&u=0, \quad b=0 \quad \text{on} \quad \partial\Omega,\\
&u(0,x)=u_0(x), \quad b(0,x)=b_0(x), \quad x\in\Omega,
\end{aligned} \tag{2.5}$$

with the parameters $\alpha, \beta > \frac{5}{4}$ (to be chosen) and $\nu, \nu_3, \eta, \eta_3 > 0$.

Our first task is the local in time solvability of the problems (2.4) and (2.5), when the equations are treated in the *base space* $\mathbf{L}^2(\Omega)$. We will use the standard approach proposed by Dan Henry ([15]) for semilinear 'parabolic' equations. Working with the *sectorial positive operator* $\mathbf{A}_N^{\alpha,\beta}$ we rewrite problems (2.4) and (2.5) in an abstract form:

$$\begin{aligned}
&(u_t, b_t) + \mathbf{A}_N^{\alpha,\beta}(u,b) = \mathbf{F}(u,b), \quad t>0,\\
&(u(0,x), b(0,x)) = (u_0(x), b_0(x)),
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}(2.7)\qquad \mathbf{F}(u,b) &= (\mathbf{F}_1(u,b),\mathbf{F}_2(u,b))\\ &= (P(b\cdot\nabla b)-P(u\cdot\nabla u),\ P(b\cdot\nabla u)-P(u\cdot\nabla b))\end{aligned}$$

is the Nemytskii operator corresponding to a nonlinear term.

The following local existence result holds:

THEOREM 2.4. *Let $\alpha,\beta > \frac{N}{4}$, $N = 2,3$, the parameters $\eta_N,\nu_N > 0$, $\eta,\nu \geq 0$ and $\frac{N+2\epsilon}{4}$ be a number strictly greater than $\frac{N}{4}$ but arbitrarily close to it. For each $u_0,b_0 \in D(A^{\frac{N+2\epsilon}{4}}) \subset \mathcal{H}^{\frac{N}{2}+\epsilon}(\Omega)$, there exists a unique local in time mild solution $(u(t),b(t))$ to the sub-critical problem (2.6) in the base space $\mathbf{L}^2(\Omega)$ (see [15]). Moreover,*

$$(u,b)\in C\left((0,\tau_{u_0});D(A^\alpha)\times D(A^\beta)\right)\cap C\left([0,\tau_{u_0});\left(D\left(A^{\frac{N+2\epsilon}{4}}\right)\right)^2\right),$$

$$(u_t,b_t)\in C\left((0,\tau_{u_0});D(A^\gamma)\times D(A^\delta)\right),$$

with arbitrary $\gamma<\alpha$ and $\delta<\beta$. Here τ_{u_0,b_0} is the 'life time' of this local in time solution. Furthermore, the Cauchy formula is satisfied:

$$(u(t),b(t)) = e^{-\mathbf{A}_N^{\alpha,\beta}t}(u_0,b_0)+\int_0^t e^{-\mathbf{A}_N^{\alpha,\beta}(t-s)}\mathbf{F}(u(s),b(s))ds,\ t\in[0,\tau_{u_0}),$$

where $e^{-\mathbf{A}_N^{\alpha,\beta}t}$ denotes the linear semigroup corresponding to the operator $\mathbf{A}_N^{\alpha,\beta}$ in $\mathbf{L}^2(\Omega)$.

PROOF. To guarantee the local solvability (see e.g. [2, p. 55]) we need to check if the nonlinearity (2.7) is Lipschitz continuous on bounded sets as a map from $(D(A^{\frac{N+2\epsilon}{4}}))^2$ into $\mathbf{L}^2(\Omega)$, that is for (v,w), $(\phi,\psi)\in B$ (B bounded in $(D(A^{\frac{N+2\epsilon}{4}}))^2$),

$$\|\mathbf{F}(v,w)-\mathbf{F}(\phi,\psi)\|_{\mathbf{L}^2(\Omega)} \leq L(B)\|(v-\phi,w-\psi)\|_{(D(A^{\frac{N+2\epsilon}{4}}))^2}.$$

Since the form $P(\xi\cdot\nabla\zeta)$ is bilinear in ξ,ζ for $(v,w),(\phi,\psi)$ varying in a bounded set $B\subset(D(A^{\frac{N+2\epsilon}{4}}))^2$ we obtain

$$\begin{aligned}&\|\mathbf{F}(v,w)-\mathbf{F}(\phi,\psi)\|_{\mathbf{L}^2(\Omega)}\\ &= \|P\phi\cdot\nabla(\phi-v)+P(\phi-v)\cdot\nabla v+P\psi\cdot\nabla(\psi-w)+P(\psi-w)\cdot\nabla w\|_{\mathcal{L}^2(\Omega)}\\ &+\|P\phi\cdot\nabla(\psi-w)+P(\phi-v)\cdot\nabla w+Pw\cdot\nabla(v-\phi)+P(\psi-w)\cdot\nabla\phi\|_{\mathcal{L}^2(\Omega)}.\end{aligned}$$

We present the calculations only for the component $P(w \cdot \nabla(v - \phi))$. The way of handling another components is similar. Using the estimate (2.2) with $r = 2$, $\delta = 0$, $\theta = \frac{1}{2}$ and $\rho = \frac{N+2\epsilon}{4}$, $\epsilon > 0$, we get

$$\begin{aligned} \|P(w \cdot \nabla(v - \phi))\|_{\mathcal{L}^2(\Omega)} &\leq M\|A^{\frac{1}{2}}w\|_{\mathcal{L}^2(\Omega)}\|A^{\frac{N+2\epsilon}{4}}(v - \phi)\|_{\mathcal{L}^2(\Omega)} \\ &\leq L(B)\|A^{\frac{N+2\epsilon}{4}}(v - \phi)\|_{\mathcal{L}^2(\Omega)}. \end{aligned}$$ □

2.2. Global solvability

Having obtained the local in time solution of (2.6), to guarantee its global extensibility we need suitable *a priori estimates* and the subordination condition (see [2, Theorem 3.1.1]). First we will discuss useful properties of divergence-free functions:

REMARK 2.5. Note that for divergence-free functions $f = (f_1, \ldots, f_N), g = (g_1, \ldots, g_N)$, $h = (h_1, \ldots, h_N) \colon \bar{\Omega} \to \mathbb{R}^N$ such that $f, g, h \in C^1(\bar{\Omega})$ and $f = g = h = 0$ on $\partial\Omega$ we have:
(a) $\int_\Omega (f \cdot \nabla g) \cdot g \, dx = -\frac{1}{2}\int_\Omega \operatorname{div} f \sum_{i=1}^N g_i^2 \, dx = 0$,
(b) $\int_\Omega [(f \cdot \nabla g) \cdot h + (f \cdot \nabla h) \cdot g] \, dx = -\int_\Omega \operatorname{div} f \sum_{i=1}^N g_i h_i \, dx = 0$.

LEMMA 2.6. *For a sufficiently regular solution of* (2.6)*, the following estimate holds:*

$$\|u\|^2_{\mathcal{L}^2(\Omega)} + \|b\|^2_{\mathcal{L}^2(\Omega)} \leq \|u_0\|^2_{\mathcal{L}^2(\Omega)} + \|b_0\|^2_{\mathcal{L}^2(\Omega)}. \tag{2.8}$$

PROOF. Multiplying the first equation in (2.6) by u, the second by b and adding the results, we get in 2-D

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|^2_{\mathcal{L}^2(\Omega)} + \|b\|^2_{\mathcal{L}^2(\Omega)}) + \nu_2\|A^{\frac{\alpha}{2}}u\|^2_{\mathcal{L}^2(\Omega)} + \eta_2\|A^{\frac{\beta}{2}}b\|_{\mathcal{L}^2(\Omega)} \\ &\qquad = \int_\Omega [P(b \cdot \nabla b) - P(u \cdot \nabla u)] \cdot u \, dx + \int_\Omega [P(b \cdot \nabla u) - P(u \cdot \nabla b)] \cdot b \, dx \end{aligned}$$

and in 3-D

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|^2_{\mathcal{L}^2(\Omega)} + \|b\|^2_{\mathcal{L}^2(\Omega)}) + \nu\|A^{\frac{1}{2}}u\|^2_{\mathcal{L}^2(\Omega)} + \eta\|A^{\frac{1}{2}}b\|_{\mathcal{L}^2(\Omega)} \\ &\qquad + \nu_3\|A^{\frac{\alpha}{2}}u\|^2_{\mathcal{L}^2(\Omega)} + \eta_3\|A^{\frac{\beta}{2}}b\|_{\mathcal{L}^2(\Omega)} \\ &\qquad = \int_\Omega [P(b \cdot \nabla b) - P(u \cdot \nabla u)] \cdot u \, dx + \int_\Omega [P(b \cdot \nabla u) - P(u \cdot \nabla b)] \cdot b \, dx. \end{aligned} \tag{2.9}$$

Since for divergence-free functions the nonlinear term vanishes (see Remark 2.5) we have

$$(2.10)\quad \frac{d}{dt}(\|u\|^2_{\mathcal{L}^2(\Omega)}+\|b\|^2_{\mathcal{L}^2(\Omega)})+2\nu_N\|A^{\frac{\alpha}{2}}u\|^2_{\mathcal{L}^2(\Omega)}+2\eta_N\|A^{\frac{\beta}{2}}b\|_{\mathcal{L}^2(\Omega)}\le 0. \qquad \square$$

REMARK 2.7. In 2-D case, integrating (2.10) over $(0,T)$, due to (2.8), we get a uniform in $\alpha,\beta\in(1,\frac{5}{4}]$ estimate of u, b in $L^2(0,T;D(A^{\frac{1}{2}}))$, where $T>0$ is fixed but arbitrarily large.

REMARK 2.8. In 3-D case, from (2.9), since for regular solution the nonlinear term vanishes (see Remark 2.5), we obtain a differential inequality of the form

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_{\mathcal{L}^2(\Omega)}+\|b\|^2_{\mathcal{L}^2(\Omega)})+\nu\|A^{\frac{1}{2}}u\|^2_{\mathcal{L}^2(\Omega)}+\eta\|A^{\frac{1}{2}}b\|^2_{\mathcal{L}^2(\Omega)} + \nu_3\|A^{\frac{\alpha}{2}}u\|^2_{\mathcal{L}^2(\Omega)}+\eta_3\|A^{\frac{\beta}{2}}b\|_{\mathcal{L}^2(\Omega)}\le 0.$$

Integrating the above inequality over $(0,T)$ we get

$$(2.11)\qquad \int_0^T(\nu\|A^{\frac{1}{2}}u\|^2_{\mathcal{L}^2(\Omega)}+\eta\|A^{\frac{1}{2}}b\|^2_{\mathcal{L}^2(\Omega))})\,ds\le c(\|(u_0,b_0)\|_{\mathbf{L}^2(\Omega)}),$$

and

$$(2.12)\qquad \int_0^T(\nu_3\|A^{\frac{\alpha}{2}}u\|^2_{\mathcal{L}^2(\Omega)}+\eta_3\|A^{\frac{\beta}{2}}b\|^2_{\mathcal{L}^2(\Omega))})\,ds\le c(\|(u_0,b_0)\|_{\mathbf{L}^2(\Omega)}),$$

with a positive constant c independent of ν_3 and η_3. The inequality (2.11) implies a uniform in $\nu_3,\eta_3>0$ estimate of u, b in $L^2(0,T;D(A^{\frac{1}{2}}))$, where $T>0$ is fixed but arbitrarily large. From estimate (2.12) follows that

$$(2.13)\quad \sqrt{\nu_3}\|A^{\frac{\alpha}{2}}u\|_{L^2(0,T;\mathcal{L}^2(\Omega))}+\sqrt{\eta_3}\|A^{\frac{\beta}{2}}b\|_{L^2(0,T;\mathcal{L}^2(\Omega))}\le const$$

with the const independent of ν_3 and η_3.

The $\mathbf{L}^2(\Omega)$ a priori estimate obtained in Lemma 2.6 is unfortunately too weak to guarantee the global in time solvability of (2.6) in phase space $\Big(D(A^{\frac{N+2\epsilon}{4}})\Big)^2$. For this purpose, we need to estimate higher Sobolev norm of the solutions to (2.6).

LEMMA 2.9. *Let* $\alpha, \beta > \frac{1}{2} + \frac{N}{4}$, $N = 2, 3$, *the parameters* $\nu_N, \eta_N > 0$ *and* $\nu, \eta \geq 0$. *For a sufficiently regular solution of* (2.6), *the following estimate holds:*

$$\|A^{\frac{1}{2}} u\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}} b\|^2_{\mathcal{L}^2(\Omega)} \leq c(\alpha, \beta, \nu_N, \eta_N, \|(u_0, b_0)\|_{\mathbf{L}^2(\Omega)}).$$

PROOF. Multiplying the first equation in (2.6) by Au and the second by Ab, we get

$$(2.14)\quad \frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}} u\|^2_{\mathcal{L}^2(\Omega)} + \nu_N \|A^{\frac{1+\alpha}{2}} u\|^2_{\mathcal{L}^2(\Omega)} \leq \int_\Omega P(b \cdot \nabla b) \cdot Au\, dx - \int_\Omega P(u \cdot \nabla u) \cdot Au\; dx$$

and

$$(2.15)\quad \frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}} b\|^2_{\mathcal{L}^2(\Omega)} + \eta_N \|A^{\frac{1+\beta}{2}} b\|^2_{\mathcal{L}^2(\Omega)} \leq \int_\Omega P(b \cdot \nabla u) \cdot Ab\, dx - \int_\Omega P(u \cdot \nabla b) \cdot Ab\; dx.$$

Using the Schwarz inequality and the estimate (2.2) (with $r = 2$, $\delta = 0$, $\theta = \frac{N}{8}$ and $\rho = \frac{1}{2} + \frac{N}{8}$), we get

$$(2.16)\quad \int_\Omega P(u \cdot \nabla b) \cdot Ab\; dx \leq \|P(u \cdot \nabla b)\|_{\mathcal{L}^2(\Omega)} \|Ab\|_{\mathcal{L}^2(\Omega)} \leq \|A^{\frac{N}{8}} u\|_{\mathcal{L}^2(\Omega)} \|A^{\frac{1}{2}+\frac{N}{8}} b\|_{\mathcal{L}^2(\Omega)} \|Ab\|_{\mathcal{L}^2(\Omega)}.$$

Next, from the theory of interpolation (see [25, Lemma 3.27]), we have

$$(2.17)\quad \|Ab\|_{\mathcal{L}^2(\Omega)} \leq c(\beta) \|b\|^{\frac{\beta-1}{\beta+1}}_{\mathcal{L}^2(\Omega)} \|A^{\frac{1+\beta}{2}} b\|^{\frac{2}{\beta+1}}_{\mathcal{L}^2(\Omega)},$$

$$(2.18)\quad \|A^{\frac{1}{2}+\frac{N}{8}} b\|_{\mathcal{L}^2(\Omega)} \leq c(\beta) \|b\|^{\frac{4\beta-N}{4(\beta+1)}}_{\mathcal{L}^2(\Omega)} \|A^{\frac{1+\beta}{2}} b\|^{\frac{4+N}{4(\beta+1)}}_{\mathcal{L}^2(\Omega)},$$

and

$$(2.19)\quad \|A^{\frac{N}{8}} u\|_{\mathcal{L}^2(\Omega)} \leq c(\alpha) \|u\|^{\frac{4\alpha+4-N}{4(\alpha+1)}}_{\mathcal{L}^2(\Omega)} \|A^{\frac{1+\alpha}{2}} u\|^{\frac{N}{4(\alpha+1)}}_{\mathcal{L}^2(\Omega)}.$$

Consequently, collecting the above estimates, thanks to the Young inequality and (2.8), we obtain

$$|\int_\Omega P(u\cdot\nabla b)\cdot Ab\,dx| \le \frac{\nu_N}{4}\|A^{\frac{1+\alpha}{2}}u\|^2_{\mathcal{L}^2(\Omega)} + \frac{\eta_N}{3}\|A^{\frac{1+\beta}{2}}b\|^2_{\mathcal{L}^2(\Omega)} + c^{\alpha,\beta}_{\nu_N,\eta_N}(\|(u_0,b_0)\|_{\mathbf{L}^2(\Omega)}).$$

Finally, estimating another components in (2.14) and (2.15) in a similar way, we get

$$\frac{d}{dt}\left(\|A^{\frac12}u\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac12}b\|^2_{\mathcal{L}^2(\Omega)}\right) + \nu_N\|A^{\frac{1+\alpha}{2}}b\|^2_{\mathcal{L}^2(\Omega)} + \eta_N\|A^{\frac{1+\beta}{2}}u\|^2_{\mathcal{L}^2(\Omega)} \le c(\alpha,\beta,\nu_N,\eta_N,\|(u_0,b_0)\|_{\mathbf{L}^2(\Omega)}). \qquad \square$$

REMARK 2.10. For (2.6) with the fractional exponents $\alpha=\beta=\frac{2+N}{4}$, $N=2,3$, thanks to the estimates (2.17) - (2.19) (with $\alpha=\beta=\frac{2+N}{4}$), we obtain

$$\frac{d}{dt}\left(\|A^{\frac12}u\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac12}b\|^2_{\mathcal{L}^2(\Omega)}\right) + 2\nu_N\|A^{\frac{6+N}{8}}u\|^2_{\mathcal{L}^2(\Omega)} + 2\eta_N\|A^{\frac{6+N}{8}}b\|^2_{\mathcal{L}^2(\Omega)}$$
$$\le c_1(\|(u_0,b_0)\|_{\mathbf{L}^2(\Omega)})\|A^{\frac{6+N}{8}}u\|^2_{\mathcal{L}^2(\Omega)} + c_2(\|(u_0,b_0)\|_{\mathbf{L}^2(\Omega)})\|A^{\frac{6+N}{8}}b\|^2_{\mathcal{L}^2(\Omega)}.$$

Consequently, when the data are small:

$$\frac12 c_1(\|(u_0,b_0)\|_{\mathbf{L}^2(\Omega)}) < \nu_N \quad \text{and} \quad \frac12 c_2(\|(u_0,b_0)\|_{\mathbf{L}^2(\Omega)}) < \eta_N \tag{2.20}$$

we get the estimates

$$\|u\|_{L^\infty\left(0,T;D\left(A^{\frac12}\right)\right)} + \|u\|_{L^2\left(0,T;D\left(A^{\frac{6+N}{8}}\right)\right)} \le const,$$

$$\|b\|_{L^\infty\left(0,T;D\left(A^{\frac12}\right)\right)} + \|b\|_{L^2\left(0,T;D\left(A^{\frac{6+N}{8}}\right)\right)} \le const,$$

where $T>0$ is fixed but arbitrarily large.

Recall, see [2, pp. 72–73], that to be able to extend globally in time the local in time solution constructed above, with bounded orbits of bounded sets, we need to have a subordination condition of the form:

$$\|\mathbf{F}(u(t),b(t))\|_{\mathbf{L}^2(\Omega)}$$
$$\leq const(\|(u(t),b(t))\|_Y)\left(1+\|(u(t),b(t))\|^{\gamma}_{(D(A^{\frac{N+2\epsilon}{4}}))^2}\right), \quad t\in(0,\tau_{u_0}),$$

with a certain auxiliary Banach space $D(\mathbf{A}_N^{\alpha,\beta})\subset Y$ and a certain $\gamma\in[0,1)$. Let $Y:=(D(A^{\frac{1}{2}}))^2$. We have

$$\|\mathbf{F}(u,b)\|_{\mathbf{L}^2(\Omega)}\leq \|P(b\cdot\nabla b)\|_{\mathcal{L}^2(\Omega)}+\|P(u\cdot\nabla u)\|_{\mathcal{L}^2(\Omega)}$$
$$+\|P(b\cdot\nabla u)\|_{\mathcal{L}^2(\Omega)}+\|P(u\cdot\nabla b)\|_{\mathcal{L}^2(\Omega)}.$$

Using the estimate (2.2) with $r=2$, $\delta=0$, $\theta=\frac{1}{2}$, $\rho=\frac{N+\epsilon}{4}$ and the theory of interpolation (see [25, Lemma 3.27]), we get

$$\|P(b\cdot\nabla u)\|_{\mathcal{L}^2(\Omega)}\leq c\|A^{\frac{1}{2}}b\|_{\mathcal{L}^2(\Omega)}\|A^{\frac{N+\epsilon}{4}}u\|_{\mathcal{L}^2(\Omega)}$$
$$\leq c\|A^{\frac{1}{2}}b\|_{\mathcal{L}^2(\Omega)}\|A^{\frac{1}{2}}u\|^{1-\gamma}_{\mathcal{L}^2(\Omega)}\|A^{\frac{N+2\epsilon}{4}}u\|^{\gamma}_{\mathcal{L}^2(\Omega)}$$

with $\gamma=\frac{N+\epsilon-2}{N+2\epsilon-2}<1$, which is a form of a subordination condition. Estimating another components in a similar way, we conclude the last claim as a theorem:

Theorem 2.11. *Let $N=2,3$. When $\alpha,\beta>\frac{1}{2}+\frac{N}{4}$ and $\eta_N,\nu_N>0$, then the local solution of* (2.6) *constructed in Theorem* 2.4 *exists globally in time.*

Theorem 2.12. *Let $\alpha,\beta=\frac{1}{2}+\frac{N}{4}$ and $\eta_N,\nu_N>0$, $N=2,3$. When the data are small that is the condition* (2.20) *holds, then the local solution of* (2.6) *constructed in Theorem* 2.4 *exists globally in time.*

2.3. Global in time solutions of 3-D MHD for small data

As well know (see [2]) global in time extendibility of the local solution constructed in the Theorem 2.4 is possible if we have sufficiently well a priori estimate. We will show such type estimation for the solution of 3-D MHD equation when the data (u_0,b_0) are small.

Remark 2.13. Let $\nu,\eta>0$. Note that taking $\nu_3=\eta_3=0$ or $\alpha=\beta=1$ in system (2.5) we obtain equivalent problems.

LEMMA 2.14. *Let $\nu_3 = \eta_3 = 0$ and $\nu, \eta > 0$. If the initial date (u_0, b_0) fulfills the smallness restriction* (2.22)*, then for a sufficiently regular solution of* (2.5)*, the $D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}})$ norm of the solution (u,b) is bounded uniformly in time $t \geq 0$. That is the following estimate holds:*

$$\|A^{\frac{1}{2}}u\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}}b\|^2_{\mathcal{L}^2(\Omega)} \leq \|A^{\frac{1}{2}}u_0\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}}b_0\|^2_{\mathcal{L}^2(\Omega)}.$$

PROOF. Multiplying the first equation in (2.5) by Au and the second by Ab and adding the results, we get

$$\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{2}}u\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}}b\|^2_{\mathcal{L}^2(\Omega)}) + \nu\|Au\|^2_{\mathcal{L}^2(\Omega)} + \eta\|Ab\|^2_{\mathcal{L}^2(\Omega)} \tag{2.21}$$
$$= \int_\Omega (P(b\cdot\nabla b) - P(u\cdot\nabla u))\cdot Au\,dx + \int_\Omega (P(b\cdot\nabla u) - P(u\cdot\nabla b))\cdot Ab\,dx.$$

From (2.16), for $N = 3$, using the interpolation inequality (see [25, Lemma 3.27])

$$\|A^{\frac{7}{8}}b\|_{\mathcal{L}^2(\Omega)} \leq c\|A^{\frac{1}{2}}b\|^{\frac{1}{4}}_{\mathcal{L}^2(\Omega)}\|Ab\|^{\frac{3}{4}}_{\mathcal{L}^2(\Omega)}$$

and the Young inequality, we get

$$\int_\Omega P(u\cdot\nabla b)\cdot Ab\ dx \leq \frac{\eta}{6}\|Ab\|^2_{\mathcal{L}^2(\Omega)} + C(\eta)(\|A^{\frac{1}{2}}b\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}}u\|^2_{\mathcal{L}^2(\Omega)})^5.$$

Estimating another components in (2.21) in similar way, we obtain

$$\frac{d}{dt}\Big(\|A^{\frac{1}{2}}u\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}}b\|^2_{\mathcal{L}^2(\Omega)}\Big) + \frac{\min\{\nu,\eta\}}{c_1}(\|A^{\frac{1}{2}}b\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}}u\|^2_{\mathcal{L}^2(\Omega)})$$
$$\leq c_2(\nu,\eta)(\|A^{\frac{1}{2}}b\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}}u\|^2_{\mathcal{L}^2(\Omega)})^5,$$

where c_1 is the square of the embedding constant $(D(A) \subset D(A^{\frac{1}{2}}))$. Denoting $y(t) = \|A^{\frac{1}{2}}u(t)\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}}b(t)\|^2_{\mathcal{L}^2(\Omega)}$ and $c_3 = \frac{\min\{\nu,\eta\}}{c_1}$ we rewrite the above inequality in the following form

$$\frac{d}{dt}y(t) + c_3y(t) \leq c_2y^5(t).$$

We consider the above differential inequality together with the initial date

$$y(0) = \|A^{\frac{1}{2}}u(0)\|^2_{\mathcal{L}^2(\Omega)} + \|A^{\frac{1}{2}}b(0)\|^2_{\mathcal{L}^2(\Omega)}.$$

Finally, when the data are small

$$(2.22)\qquad y(0) \le \left(\frac{\min\{\nu,\eta\}}{c_1c_2}\right)^{\frac14} = \left(\frac{c_3}{c_2}\right)^{\frac14}$$

we obtain the bound

$$y(t) \le y(0)\left(\frac{c_3}{c_2y^4(0) + (c_3 - c_2y^4(0))e^{4c_3t}}\right)^{\frac14} \le y(0). \qquad \square$$

THEOREM 2.15. *Let* $\nu_3 = \eta_3 = 0$. *When the data are small that is the condition* (2.22) *holds, the local solution of* (2.6) *constructed in Theorem* 2.4 *exists globally in time.*

3. 2-D MHD system

Using the Lions-Aubin compactness lemma we will show now that the solutions of sub-critical problems (2.4) converge as $\alpha,\beta \to 1^+$ to the weak solution of the critical problem. In this section we denote the solution of (2.4) as $(u^\alpha_\beta, b^\alpha_\beta)$. We indicate the dependence of solutions on α,β for clarity of presentation. Such solutions, for any $\alpha,\beta \in (1,\frac54]$, vary continuously in $D(A)\times D(A) \subset \mathcal{H}^2(\Omega)\times\mathcal{H}^2(\Omega)$. Furthermore, they fulfill the *uniform in* α,β estimates in $\mathbf{L}^2(\Omega)$ of Lemma 2.6. To be more precise, there exists a *const* > 0 (depending on $\|(u_0,b_0)\|_{\mathbf{L}^2(\Omega)}$) for each $\alpha,\beta \in (1,\frac54]$, such that

$$(3.1)\qquad \|u^\alpha_\beta\|_{L^\infty(0,T;\,\mathcal{L}^2(\Omega))} + \|b^\alpha_\beta\|_{L^\infty(0,T;\,\mathcal{L}^2(\Omega))} \le const,$$

where $T > 0$ is fixed but arbitrarily large. Moreover (see Remark 2.7)

$$(3.2)\qquad \|u^\alpha_\beta\|_{L^2\left(0,T;\,D\left(A^{\frac12}\right)\right)} + \|b^\alpha_\beta\|_{L^2\left(0,T;\,D\left(A^{\frac12}\right)\right)} \le c,$$

with a positive constant c independent on α,β. This is the main information allowing us to let $\alpha,\beta \to 1^+$ in equation (2.4).

Applying the operator $A^{-\frac34}$ to (2.4) we obtain

$$(3.3)\qquad \|A^{-\frac34}(u^\alpha_\beta)_t\|^2_{\mathcal{L}^2(\Omega))} \le c\|A^{\alpha-\frac34}u^\alpha_\beta\|^2_{\mathcal{L}^2(\Omega)} + c\|A^{-\frac34}P(u^\alpha_\beta\cdot\nabla u^\alpha_\beta)\|^2_{\mathcal{L}^2(\Omega)}$$
$$+\, c\|A^{-\frac34}P(b^\alpha_\beta\cdot\nabla b^\alpha_\beta)\|^2_{\mathcal{L}^2(\Omega)}$$

and

$$\|A^{-\frac{3}{4}}(b^\alpha_\beta)_t\|^2_{\mathcal{L}^2(\Omega)} \le c\|A^{\beta-\frac{3}{4}}b^\alpha_\beta\|^2_{\mathcal{L}^2(\Omega)} + c\|A^{-\frac{3}{4}}P(u^\alpha_\beta\cdot\nabla b^\alpha_\beta)\|^2_{\mathcal{L}^2(\Omega)} + c\|A^{-\frac{3}{4}}P(b^\alpha_\beta\cdot\nabla u^\alpha_\beta)\|^2_{\mathcal{L}^2(\Omega)}. \tag{3.4}$$

Thanks to the Poincaré inequality and estimate (3.2), for $\alpha\in(1,\frac{5}{4}]$, we obtain

$$\int_0^T \|A^{\alpha-\frac{3}{4}}u^\alpha_\beta\|^2_{\mathcal{L}^2(\Omega)}\,dt \le \lambda_1^{2\alpha-\frac{5}{2}}\int_0^T \|A^{\frac{1}{2}}u^\alpha_\beta\|^2_{\mathcal{L}^2(\Omega)}\,dt \le const, \tag{3.5}$$

where λ_1 is the Poincaré constant. Using the estimate (2.3) with $\delta=\frac{3}{4}$ and Hölder inequality, due to estimates (3.1) and (3.2), we get

$$\int_0^T \|A^{-\frac{3}{4}}P(u^\alpha_\beta\cdot\nabla b^\alpha_\beta)\|^2_{\mathcal{L}^2(\Omega)}\,dt \le c\int_0^T \||u^\alpha_\beta|\cdot|b^\alpha_\beta|\|^2_{\mathcal{L}^{\frac{4}{3}}(\Omega)}\,dt \tag{3.6}$$

$$\le c\int_0^T \|u^\alpha_\beta\|^2_{\mathcal{L}^2(\Omega)}\|b^\alpha_\beta\|^2_{\mathcal{L}^4(\Omega)}\,dt \le c\|u^\alpha_\beta\|_{L^\infty(0,T;\mathcal{L}^2(\Omega))}\|b^\alpha_\beta\|_{L^2(0,T;D(A^{\frac{1}{2}}))} \le const.$$

Integrating (3.3) and (3.4) over $(0,T)$ and estimating obtained components like in (3.5) and (3.6), we have

$$\int_0^T \|A^{-\frac{3}{4}}(u^\alpha_\beta)_t\|^2_{\mathcal{L}^2(\Omega)}\,dt + \int_0^T \|A^{-\frac{3}{4}}(b^\alpha_\beta)_t\|^2_{\mathcal{L}^2(\Omega)}\,dt \le c \tag{3.7}$$

with a positive constant c independent on α,β. This implies the uniform in $\alpha,\beta\in(1,\frac{5}{4}]$ estimate of $((u^\alpha_\beta)_t,(b^\alpha_\beta)_t)$ in $L^2(0,T;D(A^{-\frac{3}{4}})\times D(A^{-\frac{3}{4}}))$, where $T>0$ is fixed but arbitrarily large. Thanks to (3.2) and (3.7) we know that the family

$$\left\{(u^\alpha_\beta,b^\alpha_\beta);\quad \alpha,\beta\in\left(1,\frac{5}{4}\right]\right\}$$

is bounded in the space

$$W_2=\Big\{\psi:\psi\in L^2(0,T;\,D(A^{\frac{1}{2}})\times D(A^{\frac{1}{2}})),\quad \psi_t\in L^2\left(0,T;\,D(A^{-\frac{3}{4}})\times D(A^{-\frac{3}{4}})\right)\Big\}.$$

Consequently, using the Lions-Aubin compactness lemma we claim that this family is precompact in the space $L^2(0,T;D(A^{\frac{1-\epsilon}{2}})\times D(A^{\frac{1-\epsilon}{2}}))$, where $\frac{1-\epsilon}{2}$

is a number strictly less than $\frac{1}{2}$ but arbitrarily close to it. Therefore, any sequence $\{(u^{\alpha_n}_{\beta_n}, b^{\alpha_n}_{\beta_n})\}$, $\alpha_n, \beta_n \to 1^+$ has a subsequence (denoted the same) convergent in $L^2(0,T; D(A^{\frac{1-\epsilon}{2}}) \times D(A^{\frac{1-\epsilon}{2}}))$ to some (u,b).

We look at (2.4) as the system in $\mathbf{L}^2(\Omega)$. Multiplying the first equation by a 'test function' $A^{-\alpha_n}\phi$ and the second by $A^{-\beta_n}\phi$, where $\phi \in D(A) \subset \mathcal{H}^2(\Omega)$, we obtain

$$\int_\Omega \Big((u^{\alpha_n}_{\beta_n})_t + P(u^{\alpha_n}_{\beta_n} \cdot \nabla u^{\alpha_n}_{\beta_n}) - P(b^{\alpha_n}_{\beta_n} \cdot \nabla b^{\alpha_n}_{\beta_n})\Big) \cdot A^{-\alpha_n}\phi \, dx$$

$$= -\nu \int_\Omega A^{\alpha_n} u^{\alpha_n}_{\beta_n} \cdot A^{-\alpha_n}\phi \, dx$$

and

$$\int_\Omega \big((b^{\alpha_n}_{\beta_n})_t + P(u^{\alpha_n}_{\beta_n} \cdot \nabla b^{\alpha_n}_{\beta_n}) - P(b^{\alpha_n}_{\beta_n} \cdot \nabla u^{\alpha_n}_{\beta_n})\big) \cdot A^{-\beta_n}\phi \, dx$$

$$= -\eta \int_\Omega A^{\beta_n} b^{\alpha_n}_{\beta_n} \cdot A^{-\beta_n}\phi \, dx.$$

We will discuss now the convergence of components in the above equalities one by one. We have

$$\Big|\int_0^T \int_\Omega A^{-\frac{\epsilon+1}{2}} P(u^{\alpha_n}_{\beta_n} \cdot \nabla b^{\alpha_n}_{\beta_n}) \cdot A^{-\beta_n + \frac{\epsilon+1}{2}}\phi - A^{-\frac{\epsilon+1}{2}} P(u \cdot \nabla b) \cdot A^{\frac{\epsilon-1}{2}}\phi \, dx \, dt\Big|$$

$$\leq \int_0^T \int_\Omega \Big|A^{-\frac{\epsilon+1}{2}} P(u^{\alpha_n}_{\beta_n} \cdot \nabla b^{\alpha_n}_{\beta_n}) \cdot (A^{-\beta_n + \frac{\epsilon+1}{2}}\phi - A^{\frac{\epsilon-1}{2}}\phi)\Big| \, dx \, dt$$

$$+ \int_0^T \int_\Omega \Big|A^{-\frac{\epsilon+1}{2}} P(u^{\alpha_n}_{\beta_n} \cdot \nabla b^{\alpha_n}_{\beta_n}) \cdot A^{\frac{\epsilon-1}{2}}\phi - A^{-\frac{\epsilon+1}{2}} P(u \cdot \nabla b) \cdot A^{\frac{\epsilon-1}{2}}\phi\Big| \, dx \, dt.$$

We estimate the first component on the right hand side. The second component is estimated in a similar way. Using the estimate (2.2) with $\delta = \frac{\epsilon+1}{2}$, $\theta = \rho = \frac{1-\epsilon}{2}$ and the Hölder inequality, we get

$$\int_0^T \int_\Omega \Big|A^{-\frac{\epsilon+1}{2}} P(u^{\alpha_n}_{\beta_n} \cdot \nabla b^{\alpha_n}_{\beta_n}) \cdot (A^{-\beta_n + \frac{\epsilon+1}{2}}\phi - A^{\frac{\epsilon-1}{2}}\phi)\Big| \, dx \, dt$$

$$\leq \int_0^T \|A^{-\frac{\epsilon+1}{2}} P(u^{\alpha_n}_{\beta_n} \cdot \nabla b^{\alpha_n}_{\beta_n})\|_{\mathcal{L}^2(\Omega)} \|A^{-\beta_n + \frac{\epsilon+1}{2}}\phi - A^{\frac{\epsilon-1}{2}}\phi\|_{\mathcal{L}^2(\Omega)} \, dt$$

$$\leq c\int_0^T \|A^{\frac{1-\epsilon}{2}}u^{\alpha_n}_{\beta_n}\|_{\mathcal{L}^2(\Omega)}\|A^{\frac{1-\epsilon}{2}}b^{\alpha_n}_{\beta_n}\|_{\mathcal{L}^2(\Omega)}\|A^{-\beta_n+\frac{\epsilon+1}{2}}\phi - A^{\frac{\epsilon-1}{2}}\phi\|_{\mathcal{L}^2(\Omega)}\,dt$$

$$\leq c\|A^{\frac{1-\epsilon}{2}}u^{\alpha_n}_{\beta_n}\|_{L^2(0,T;\mathcal{L}^2(\Omega))}\|A^{\frac{1-\epsilon}{2}}b^{\alpha_n}_{\beta_n}\|_{L^2(0,T;\mathcal{L}^2(\Omega))}\|A^{-\beta_n+\frac{\epsilon+1}{2}}\phi - A^{\frac{\epsilon-1}{2}}\phi\|_{\mathcal{L}^2(\Omega)}.$$

Consequently,

$$\Big|\int_0^T\int_\Omega A^{-\frac{\epsilon+1}{2}}P(u^{\alpha_n}_{\beta_n}\cdot\nabla b^{\alpha_n}_{\beta_n})\cdot A^{-\beta_n+\frac{\epsilon+1}{2}}\phi - A^{-\frac{\epsilon+1}{2}}P(u\cdot\nabla b)\cdot A^{\frac{\epsilon-1}{2}}\phi\,dx\,dt\Big|$$

$$\leq c\|A^{-\beta_n+\frac{\epsilon+1}{2}}\phi - A^{\frac{\epsilon-1}{2}}\phi\|_{\mathcal{L}^2(\Omega)}\|A^{\frac{1-\epsilon}{2}}u^{\alpha_n}_{\beta_n}\|_{L^2(0,T;\mathcal{L}^2(\Omega))}\|A^{\frac{1-\epsilon}{2}}b^{\alpha_n}_{\beta_n}\|_{L^2(0,T;\mathcal{L}^2(\Omega))}$$

$$+ c\|A^{\frac{\epsilon-1}{2}}\phi\|_{\mathcal{L}^2(\Omega)}\Big[\|A^{\frac{1-\epsilon}{2}}u^{\alpha_n}_{\beta_n}\|_{L^2(0,T;\mathcal{L}^2(\Omega))}\|A^{\frac{1-\epsilon}{2}}(b^{\alpha_n}_{\beta_n}-b)\|_{L^2(0,T;\mathcal{L}^2(\Omega))}$$

$$+\|A^{\frac{1-\epsilon}{2}}(u^{\alpha_n}_{\beta_n}-u)\|_{L^2(0,T;\mathcal{L}^2(\Omega))}\|A^{\frac{1-\epsilon}{2}}b\|_{L^2(0,T;\mathcal{L}^2(\Omega))}\Big].$$

By Lemma 2.2, for arbitrary $A^{\frac{\epsilon-1}{2}}\phi\in\mathcal{L}^2(\Omega)$, we know that $\|A^{-\beta_n+1+\frac{\epsilon-1}{2}}\phi - A^{\frac{\epsilon-1}{2}}\phi\|_{\mathcal{L}^2(\Omega)}$ tends to 0 as $\beta_n\to 1^+$. Moreover, thanks to precompactness of the family $\left\{(u^\alpha_\beta, b^\alpha_\beta);\ \alpha,\beta\in\left(1,\frac{5}{4}\right]\right\}$ in the space $L^2(0,T;(D(A^{\frac{1-\epsilon}{2}}))^2)$, we get the convergences

$$\Big(\int_0^T\|A^{\frac{1-\epsilon}{2}}(b^{\alpha_n}_{\beta_n}-b)\|^2_{\mathcal{L}^2(\Omega)}dt\Big)^{\frac{1}{2}}\to 0 \quad\text{and}\quad \Big(\int_0^T\|A^{\frac{1-\epsilon}{2}}(u^{\alpha_n}_{\beta_n}-u)\|^2_{\mathcal{L}^2(\Omega)}dt\Big)^{\frac{1}{2}}\to 0$$

as $\alpha_n,\beta_n\to 1^+$ over a subsequence (denoted the same). Finally, the estimate (3.2) and the above convergences implies

$$\int_0^T\int_\Omega P(u^{\alpha_n}_{\beta_n}\cdot\nabla b^{\alpha_n}_{\beta_n})\cdot A^{-\beta_n}\phi\,dx\,dt \to \int_0^T\int_\Omega A^{-\frac{\epsilon+1}{2}}P(u\cdot\nabla b)\cdot A^{\frac{\epsilon-1}{2}}\phi\,dx\,dt$$

as $\alpha_n,\beta_n\to 1^+$ over a subsequence. Moreover

$$\int_0^T\int_\Omega\Big|A^{\frac{1-\epsilon}{2}}(u^{\alpha_n}_{\beta_n}-u)\cdot A^{\frac{\epsilon-1}{2}}\phi\Big|\,dx\,dt$$

$$\leq\sqrt{T}\|u^{\alpha_n}_{\beta_n}-u\|_{L^2(0,T;D(A^{\frac{1-\epsilon}{2}}))}\|A^{\frac{\epsilon-1}{2}}\phi\|_{\mathcal{L}^2(\Omega)}\to 0.$$

In the term containing the time derivative $(u^{\alpha_n}_{\beta_n})_t$ we can pass to the limit in the sense of 'scalar distributions'. Indeed, by [27, Lemma 1.1, Chapt.III], for all $\phi\in\mathcal{H}^2(\Omega)$

$$<(u^{\alpha_n}_{\beta_n})_t,\phi> = \frac{d}{dt}<u^{\alpha_n}_{\beta_n},\phi>\to\frac{d}{dt}<u,\phi>,$$

the derivative $\frac{d}{dt}$ and the convergence are in $\mathcal{D}'(0,T)$ (space of the 'scalar distributions'). Consequently,

$$\int_\Omega (u^{\alpha_n}_{\beta_n})_t \cdot A^{-\alpha_n}\phi dx = \frac{d}{dt} < u^{\alpha_n}_{\beta_n}, A^{-\alpha_n}\phi >_{\mathcal{L}^2(\Omega)} \to \frac{d}{dt} < u, A^{-1}\phi >_{\mathcal{L}^2(\Omega)}$$

in $\mathcal{D}'(0,T)$ as $\alpha_n, \beta_n \searrow 1^+$.

By collecting all the limits together, we find the form of the limiting *critical system*:

$$\text{(3.8)}\quad \frac{d}{dt} < u, A^{-1}\phi > = \int_\Omega (A^{-\frac{\epsilon+1}{2}} P(b\cdot\nabla b) - A^{-\frac{\epsilon+1}{2}} P(u\cdot\nabla u)) \cdot A^{\frac{\epsilon-1}{2}}\phi\, dx - \nu \int_\Omega A^{\frac{1-\epsilon}{2}} u \cdot A^{\frac{\epsilon-1}{2}}\phi\, dx,$$

$$\text{(3.9)}\quad \frac{d}{dt} < b, A^{-1}\phi > = \int_\Omega (A^{-\frac{\epsilon+1}{2}} P(b\cdot\nabla u) - A^{-\frac{\epsilon+1}{2}} P(u\cdot\nabla b)) \cdot A^{\frac{\epsilon-1}{2}}\phi\, dx - \eta \int_\Omega A^{\frac{1-\epsilon}{2}} b \cdot A^{\frac{\epsilon-1}{2}}\phi\, dx.$$

REMARK 3.1. Note that the uniform in α, β estimate (3.1) implies that any sequence $\{(u^{\alpha_n}_{\beta_n}, b^{\alpha_n}_{\beta_n})\}$ is bounded in $\mathbf{L}^2(\Omega)$. Consequently, [1, Theorem 3.18] implies that there exists a subsequence (denoted the same) and some $(\bar{u}, \bar{b}) \in \mathbf{L}^2(\Omega)$ such that $(u^{\alpha_n}_{\beta_n}, b^{\alpha_n}_{\beta_n}) \to (\bar{u}, \bar{b})$ weakly in $\mathbf{L}^2(\Omega)$ as $\alpha_n, \beta_n \to 1^+$. In particular

$$\int_\Omega u^{\alpha_n}_{\beta_n} \cdot \phi\, dx \to \int_\Omega \bar{u}\cdot\phi\, dx \quad \text{and} \quad \int_\Omega b^{\alpha_n}_{\beta_n} \cdot \phi\, dx \to \int_\Omega \bar{b}\cdot\phi\, dx.$$

Moreover, we have (see [1, Proposition 3.5])

$$\|(\bar{u},\bar{b})\|_{\mathbf{L}^2(\Omega)} \le \liminf \|(u^{\alpha_n}_{\beta_n}, b^{\alpha_n}_{\beta_n})\|_{\mathbf{L}^2(\Omega)}.$$

4. 3-D MHD system

We will describe now shortly the process of passing to the limit, as $\nu_3, \eta_3 \to 0^+$ in equations (2.5) where the parameters $\alpha, \beta > \frac{5}{4}$ are fixed (but close to $\frac{5}{4}$). The idea of passing to the limit follows the considerations in [21]. For clarity of presentation we denote solution of (2.5) by $(u^{\nu_3}_{\eta_3}, b^{\nu_3}_{\eta_3})$. Note that without loss of generality we can assume that $\nu_3, \eta_3 \in (0,1]$. Applying the operator $A^{-\alpha+\frac{1}{2}}$ to the first equation in (2.5) and $A^{-\beta+\frac{1}{2}}$ to the second, we obtain

$$(4.1)\quad \|A^{-\alpha+\frac{1}{2}}(u^{\nu_3}_{\eta_3})_t\|^2_{\mathcal{L}^2(\Omega))} \le c\left(\|A^{-\alpha+\frac{3}{2}}u^{\nu_3}_{\eta_3}\|^2_{\mathcal{L}^2(\Omega)} + \nu_3^2\|A^{\frac{1}{2}}u^{\nu_3}_{\eta_3}\|^2_{\mathcal{L}^2(\Omega)}\right) + c\left(\|A^{-\alpha+\frac{1}{2}}P(u^{\nu_3}_{\eta_3}\cdot\nabla u^{\nu_3}_{\eta_3})\|^2_{\mathcal{L}^2(\Omega)} + \|A^{-\alpha+\frac{1}{2}}P(b^{\nu_3}_{\eta_3}\cdot\nabla b^{\nu_3}_{\eta_3})\|^2_{\mathcal{L}^2(\Omega)}\right)$$

and

$$(4.2)\quad \|A^{-\beta+\frac{1}{2}}(b^{\nu_3}_{\eta_3})_t\|^2_{\mathcal{L}^2(\Omega)} \le c\left(\|A^{-\beta+\frac{3}{2}}b^{\nu_3}_{\eta_3}\|^2_{\mathcal{L}^2(\Omega)} + \eta_3^2\|A^{\frac{1}{2}}b^{\nu_3}_{\eta_3}\|^2_{\mathcal{L}^2(\Omega)}\right) + c\left(\|A^{-\beta+\frac{1}{2}}P(u^{\nu_3}_{\eta_3}\cdot\nabla b^{\nu_3}_{\eta_3})\|^2_{\mathcal{L}^2(\Omega)} + \|A^{-\beta+\frac{1}{2}}P(b^{\nu_3}_{\eta_3}\cdot\nabla u^{\nu_3}_{\eta_3})\|^2_{\mathcal{L}^2(\Omega)}\right).$$

Since $-\beta+\frac{3}{2}\le\frac{1}{2}$ and $\eta_3\in(0,1]$, due to estimate (2.11), we have

$$(4.3)\quad \int_0^T (\|A^{-\beta+\frac{3}{2}}b^{\nu_3}_{\eta_3}\|^2_{\mathcal{L}^2(\Omega)} + \eta_3^2\|A^{\frac{1}{2}}b^{\nu_3}_{\eta_3}\|^2_{\mathcal{L}^2(\Omega)})\,dt \le c\int_0^T \|A^{\frac{1}{2}}b^{\nu_3}_{\eta_3}\|^2_{\mathcal{L}^2(\Omega)}\,dt \le c_1.$$

Using the estimate (2.3) with $\delta=\beta-\frac{1}{2}$, Hölder inequality and the Sobolev embeddings $D(A^{\frac{1}{2}})\subset\mathcal{H}^1(\Omega)\subset\mathcal{L}^{\frac{3}{2(\beta-1)}}(\Omega)$ for $\beta>\frac{5}{4}$, due to estimates (2.8) and (2.11), we get

$$(4.4)\quad \int_0^T \|A^{-\beta+\frac{1}{2}}P(u^{\nu_3}_{\eta_3}\cdot\nabla b^{\nu_3}_{\eta_3})\|^2_{\mathcal{L}^2(\Omega)}\,dt \le c\int_0^T \||u^{\nu_3}_{\eta_3}|\cdot|b^{\nu_3}_{\eta_3}|\|^2_{\mathcal{L}^{\frac{6}{4\beta-1}}(\Omega)}\,dt \le c\|u^{\nu_3}_{\eta_3}\|^2_{L^\infty(0,T;\mathcal{L}^2(\Omega))}\|b^{\nu_3}_{\eta_3}\|^2_{L^2(0,T;D(A^{\frac{1}{2}}))} \le c_2.$$

Integrating (4.1) and (4.2) over (0,T) and estimating obtained components like in (4.3) and (4.4), we have

$$\int_0^T \|A^{-\alpha+\frac{1}{2}}(u^{\nu_3}_{\eta_3})_t\|^2_{\mathcal{L}^2(\Omega)}\,dt + \int_0^T \|A^{-\beta+\frac{1}{2}}(b^{\nu_3}_{\eta_3})_t\|^2_{\mathcal{L}^2(\Omega)}\,dt \le c$$

with a positive constant c independent on $\eta_3,\nu_3\in(0,1]$ (since the constants c_1 and c_2 are independent on $\eta_3,\nu_3\in(0,1]$). This implies the uniform in $\eta_3,\nu_3\in(0,1]$ estimate of $(u^{\nu_3}_{\eta_3})_t$ in $L^2(0,T;D(A^{-\alpha+\frac{1}{2}}))$ and $(b^{\nu_3}_{\eta_3})_t$ in

$L^2(0,T;D(A^{-\beta+\frac{1}{2}}))$, where $T>0$ is fixed but arbitrarily large. Consequently, due to (2.11), the family $\{(u^{\nu_3}_{\eta_3}, b^{\nu_3}_{\eta_3});\ \nu_3,\eta_3\in(0,1]\}$ is bounded in the space

$$W_3=\Big\{\psi:\psi\in L^2(0,T;\ D(A^{\frac{1}{2}})\times D(A^{\frac{1}{2}}))\qquad \psi_t\in L^2\left(0,T;\ D(A^{-\alpha+\frac{1}{2}}))\times D(A^{-\beta+\frac{1}{2}})\right)\Big\}.$$

Thus, thanks to the Lions-Aubin compactness lemma we conclude that any sequence $\{(u^{\nu_n}_{\eta_n}, b^{\nu_n}_{\eta_n})\}$, where $\nu_n,\eta_n\to 0^+$, has a subsequence (denoted the same) convergent in $L^2(0,T;D(A^{\frac{1-\epsilon}{2}})\times D(A^{\frac{1-\epsilon}{2}}))$ to some (U,B), where $\frac{1-\epsilon}{2}$ is a number strictly less than $\frac{1}{2}$ but arbitrarily close to it. This information allows us to pass to the limit in the nonlinear term. Multiplying (2.5) by a 'test function' $\psi\in D(A^{\frac{\max\{\alpha,\beta\}}{2}})$, we obtain

$$(4.5)\quad \int_\Omega (u^{\nu_n}_{\eta_n})_t\cdot\psi\,dx=-\nu\int_\Omega A^{\frac{1-\epsilon}{2}}u^{\nu_n}_{\eta_n}\cdot A^{\frac{1+\epsilon}{2}}\psi\,dx-\nu_n\int_\Omega A^{\frac{\alpha}{2}}u^{\nu_n}_{\eta_n}\cdot A^{\frac{\alpha}{2}}\psi\,dx + \int_\Omega\left(P(b^{\nu_n}_{\eta_n}\cdot\nabla b^{\nu_n}_{\eta_n})-P(u^{\nu_n}_{\eta_n}\cdot\nabla u^{\nu_n}_{\eta_n})\right)\cdot\psi\,dx$$

and

$$(4.6)\quad \int_\Omega (b^{\nu_n}_{\eta_n})_t\cdot\psi\,dx=\int_\Omega\left(P(b^{\nu_n}_{\eta_n}\cdot\nabla u^{\nu_n}_{\eta_n})-P(u^{\nu_n}_{\eta_n}\cdot\nabla b^{\nu_n}_{\eta_n})\right)\cdot\psi\,dx -\eta\int_\Omega A^{\frac{1-\epsilon}{2}}b^{\nu_n}_{\eta_n}\cdot A^{\frac{1+\epsilon}{2}}\psi\,dx-\eta_n\int_\Omega A^{\frac{\beta}{2}}b^{\nu_n}_{\eta_n}\cdot A^{\frac{\beta}{2}}\psi\,dx.$$

We will discuss now the convergence of components in the above equalities one by one.

Using Hölder inequality, due to (2.13), we obtain that the right hand side of below inequality

$$(4.7)\quad \nu_n\int_0^T\int_\Omega\left|A^{\frac{\alpha}{2}}u^{\nu_n}_{\eta_n}\cdot A^{\frac{\alpha}{2}}\psi\right|\,dx\,dt \le\nu_n\left(\int_0^T\|A^{\frac{\alpha}{2}}u^{\nu_n}_{\eta_n}\|^2_{\mathcal{L}^2(\Omega)}\,dt\right)^{\frac{1}{2}}\left(\int_0^T\|A^{\frac{\alpha}{2}}\psi\|^2_{\mathcal{L}^2(\Omega)}\,dt\right)^{\frac{1}{2}} \le\sqrt{T}\nu_n\|A^{\frac{\alpha}{2}}u^{\nu_n}_{\eta_n}\|_{L^2(0,T;\mathcal{L}^2(\Omega))}\|A^{\frac{\alpha}{2}}\psi\|_{\mathcal{L}^2(\Omega)}\le c\sqrt{T\nu_n}$$

vanishes when $\nu_n \to 0^+$. Since any sequence $\{u_{\eta_n}^{\nu_n}\}$ has a subsequence (denoted the same) convergent in $L^2(0,T;D(A^{\frac{1-\epsilon}{2}}))$ we have

$$\int_0^T \int_\Omega \left| A^{\frac{1-\epsilon}{2}}(u_{\eta_n}^{\nu_n} - U) \cdot A^{\frac{1+\epsilon}{2}}\psi \right| dx\, dt$$
$$\le \sqrt{T} \|u_{\eta_n}^{\nu_n} - U\|_{L^2(0,T;D(A^{\frac{1-\epsilon}{2}}))} \|A^{\frac{1+\epsilon}{2}}\psi\|_{\mathcal{L}^2(\Omega)} \to 0.$$

For the nonlinear component, using the estimate (2.2) with $\delta = \frac{1+\epsilon}{2}$, $\theta = \rho = \frac{1-\epsilon}{2}$ and the Hölder inequality, we get

$$\int_0^T \int_\Omega |A^{-\frac{1+\epsilon}{2}} P[(u_{\eta_n}^{\nu_n} \cdot \nabla b_{\eta_n}^{\nu_n}) - (U \cdot \nabla B)] \cdot A^{\frac{1+\epsilon}{2}}\psi|\, dx\, dt$$
$$\le \|A^{\frac{1+\epsilon}{2}}\psi\|_{\mathcal{L}^2(\Omega)} \|u_{\eta_n}^{\nu_n}\|_{L^2(0,T;D(A^{\frac{1-\epsilon}{2}}))} \|b_{\eta_n}^{\nu_n} - B\|_{L^2(0,T;D(A^{\frac{1-\epsilon}{2}}))}$$
$$+ \|A^{\frac{1+\epsilon}{2}}\psi\|_{\mathcal{L}^2(\Omega)} \|u_{\eta_n}^{\nu_n} - U\|_{L^2(0,T;D(A^{\frac{1-\epsilon}{2}}))} \|B\|_{L^2(0,T;D(A^{\frac{1-\epsilon}{2}}))} \to 0.$$

Thanks to [27, Lemma 1.1, Chapt.III] and the convergence $u_{\eta_n}^{\nu_n} \to U$ in $L^2(0,T;D(A^{\frac{1-\epsilon}{2}}))$, we have

$$< (u_{\eta_n}^{\nu_n})_t, \psi > = \frac{d}{dt} < u_{\eta_n}^{\nu_n}, \psi > \to \frac{d}{dt} < U, \psi >, \tag{4.8}$$

in the sense of 'scalar distributions' (the derivative $\frac{d}{dt}$ is in the sense of distributions). Consequently, passing to the limit in another components in (4.5) and (4.6) like in (4.7)–(4.8), we obtain

$$\frac{d}{dt} < U, \psi >_{\mathcal{L}^2(\Omega)} = -\nu < A^{\frac{1-\epsilon}{2}} U, A^{\frac{1+\epsilon}{2}}\psi >_{\mathcal{L}^2(\Omega)} + < A^{-\frac{1+\epsilon}{2}} \mathbf{F}_1(U,B), A^{\frac{1+\epsilon}{2}}\psi >_{\mathcal{L}^2(\Omega)} \tag{4.9}$$

and

$$\frac{d}{dt} < B, \psi >_{\mathcal{L}^2(\Omega)} = -\eta < A^{\frac{1-\epsilon}{2}} B, A^{\frac{1+\epsilon}{2}}\psi >_{\mathcal{L}^2(\Omega)} + < A^{-\frac{1+\epsilon}{2}} \mathbf{F}_2(U,B), A^{\frac{1+\epsilon}{2}}\psi >_{\mathcal{L}^2(\Omega)} . \tag{4.10}$$

5. Properties of the weak solutions to the MHD system

We will start from collecting the properties inherited by the solution (u, b) of the (2.4) ((U,B) of the (2.5)) in the process of passing to the limit. We have:

$$u, b, U, B \in L^2(0,T; D(A^{\frac{1}{2}-\epsilon})) \cap L^\infty(0,T; L^2(\Omega)),$$

$$u_t, b_t, U_t, B_t \in L^2(0,T; D(A^{-\frac{1}{2}})).$$

We will show now that the local solutions of the MHD system varying in space $D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}})$, are unique.

Lemma 5.1. *The solution of the MHD equations* (1.1) *satisfying:*

$$u, b \in L^\infty([0,\tau); D(A^{\frac{1}{2}})) \quad \textit{with} \quad u_t, b_t \in L^2(0,\tau; D(A^{-\frac{1}{2}}))$$

or

$$u, b \in L^2(0,\tau; D(A^{\frac{1}{2}})) \cap L^{\frac{4}{4-N}}(0,\tau; D(A^{\frac{1}{2}})) \quad \textit{with} \quad u_t, b_t \in L^2(0,\tau; D(A^{-\frac{1}{2}}))$$

is unique.

Proof. Let $(\bar{U}, \bar{B}) = (u - v, b - w)$, where (u, b) and (v, w) are the local in time solutions of the problem (1.1) (in the above class) corresponding to the same initial condition (u_0, b_0). Then $(\bar{U}, \bar{B})$ satisfies

$$\bar{U}_t - \nu\Delta\bar{U} + u \cdot \nabla\bar{U} + \bar{U} \cdot \nabla v = -\nabla P\bar{U} + b \cdot \nabla\bar{B} + \bar{B} \cdot \nabla w, \ \ x \in \Omega \subset \mathbb{R}^N, t > 0,$$

$$B_t - \eta\Delta\bar{B} + u \cdot \nabla\bar{B} + \bar{U} \cdot \nabla w = b \cdot \nabla\bar{U} + B \cdot \nabla v, \ \ x \in \Omega \subset \mathbb{R}^N, t > 0,$$

$$(\bar{U}, \bar{B}) = (0, 0) \ \text{ on } \ \partial\Omega,$$

$$(\bar{U}, \bar{B})(0, x) = (0, 0).$$

Multiplying the first equation in $\mathcal{L}^2(\Omega)$ by $\bar{U}$, the second by $\bar{B}$, adding the results, thanks to integration by parts and Remark 2.5, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\bar{U}\|^2_{\mathcal{L}^2(\Omega)} + \|B\|^2_{\mathcal{L}^2(\Omega)}) + \nu\|\nabla\bar{U}\|^2_{\mathcal{L}^2(\Omega)} + \eta\|\nabla B\|^2_{\mathcal{L}^2(\Omega)}$$
$$= \int_\Omega (\bar{B} \cdot \nabla w - \bar{U} \cdot \nabla v) \cdot \bar{U}\, dx + \int_\Omega (\bar{B} \cdot \nabla v - \bar{U} \cdot \nabla w) \cdot \bar{B}\, dx.$$

From the Hölder and the Cauchy inequality, thanks to the Nirenberg–Gagliardo inequality

$$\|\bar{U}\|_{\mathcal{L}^4(\Omega)} \le c\|\bar{U}\|_{\mathcal{L}^2(\Omega)}^{\frac{4-N}{4}} \|\nabla\bar{U}\|_{\mathcal{L}^2(\Omega)}^{\frac{N}{4}},$$

the first component in the nonlinear term can be transformed as follows

$$\begin{aligned}
&\int_\Omega (\bar{B}\cdot\nabla w - \bar{U}\cdot\nabla v)\cdot\bar{U}\,dx \\
&\qquad\le \big(\|\bar{B}\|_{\mathcal{L}^4(\Omega)}\|\nabla w\|_{\mathcal{L}^2(\Omega)} + \|\bar{U}\|_{\mathcal{L}^4(\Omega)}\|\nabla v\|_{\mathcal{L}^2(\Omega)}\big)\,\|\bar{U}\|_{\mathcal{L}^4(\Omega)} \\
&\qquad\le \frac{\nu}{4}\|\nabla\bar{U}\|_{\mathcal{L}^2(\Omega)}^2 + \frac{\eta}{4}\|\nabla\bar{B}\|_{\mathcal{L}^2(\Omega)}^2 \\
&\qquad\quad + c(\|\nabla v\|_{\mathcal{L}^2(\Omega)}^{\frac{4}{4-N}} + \|\nabla w\|_{\mathcal{L}^2(\Omega)}^{\frac{4}{4-N}})(\|\bar{U}\|_{\mathcal{L}^2(\Omega)}^2 + \|\bar{B}\|_{\mathcal{L}^2(\Omega)}^2).
\end{aligned}$$

Estimating the last component in a similar way, we get a differential inequality for the $\mathcal{L}^2(\Omega)$ norm

$$\begin{aligned}
&\frac{d}{dt}(\|\bar{U}(t)\|_{\mathcal{L}^2(\Omega)}^2 + \|\bar{B}(t)\|_{\mathcal{L}^2(\Omega)}^2) \\
&\qquad\le c(\|\nabla v(t)\|_{\mathcal{L}^2(\Omega)}^{\frac{4}{4-N}} + \|\nabla w(t)\|_{\mathcal{L}^2(\Omega)}^{\frac{4}{4-N}})(\|\bar{U}(t)\|_{\mathcal{L}^2(\Omega)}^2 + \|\bar{B}(t)\|_{\mathcal{L}^2(\Omega)}^2), \\
&\qquad(\bar{U},\bar{B})(0,x) = (0,0).
\end{aligned}$$

Since the mapping $t \mapsto \|u\|_{\mathcal{L}^2(\Omega)}^2 + \|b\|_{\mathcal{L}^2(\Omega)}^2$ is absolutely continuous (see [9, Theorem 3, p. 287]), thanks to the Gronwall inequality (see [9, p. 624]), we obtain $\|\bar{U}(t)\|_{\mathcal{L}^2(\Omega)}^2 + \|\bar{B}(t)\|_{\mathcal{L}^2(\Omega)}^2 = 0$ for all $t \in [0,\tau]$. □

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