

THE LAW OF THE ITERATED LOGARITHM FOR RANDOM DYNAMICAL SYSTEM WITH JUMPS AND STATE-DEPENDENT JUMP INTENSITY

JOANNA KUBIENIEC 

Abstract. In this paper our considerations are focused on some Markov chain associated with certain piecewise-deterministic Markov process with a state-dependent jump intensity for which the exponential ergodicity was obtained in [4]. Using the results from [3] we show that the law of iterated logarithm holds for such a model.

1. Introduction

We conduct our considerations for some subclass of piecewise-deterministic Markov processes (PDMP). These processes are governed by deterministic semiflows which are intermittent by jumps. PDMP's have been introduced by Davis in [5] and have found their application, among others, in modeling phenomena in biology, as stochastic model for gene expression ([2, 14, 15]).

Most results on such processes are formulated in situation when the periods between jumps have a Poisson distribution with constant parameter λ . We are interested in the properties of these type of systems but in case when the jump intensity depends on the trajectory of the process. The asymptotic stability and exponential ergodicity for a model in which the intensity of jumps depends on the state of the system was examined in [4, 11]. In this work we focus to

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prove one of the limit theorems, namely the law of the iterated logarithm (LIL), for a such process. Limit theorems for Markov processes have recently been the subject of intense research (see e.g. [1, 6, 9, 10, 16]). The LIL defines a range in which, with probability 1, from a certain point the trajectories of the stochastic process will be found. In other words the LIL examines the greatest deviations from the mean of stochastic process. It can be located between the strong law of large numbers and the central limit theorem. Originally it was formulated by A. Khintchine in [7] and independently by A. Kolmogorov in [8].

The article consists of three parts. In Section 2 we introduce a notation and formulate basic definitions and facts related to Markov operators. Section 3 contains a formal description and main assumption of the considered model. In the last section we formulate and prove the LIL for the process described in Section 3.

2. Preliminaries

2.1. Basic notation and definition

Let (S, d) be a Polish metric space and let \mathcal{B}_S denotes the σ -field of all Borel subsets of S . As usual, by $B(x, r)$ we denote the open ball in (S, d) with center at $x \in S$ and radius $r > 0$. We use symbol $\mathbb{1}_A$ and δ_x for indicator function of set $A \subset S$ and Dirac measure in point $x \in S$, respectively. We use letters \mathbb{R} and \mathbb{N} to denote successively the set of real and natural numbers. Additionally, \mathbb{R}_+ stands for the set of nonnegative real numbers and \mathbb{N}_0 for $\mathbb{N} \cup \{0\}$.

Within the set $B(S)$, which states for all bounded, Borel measurable functions $f: S \rightarrow \mathbb{R}$, we specify two subsets: $C(S)$ and $Lip(S)$ consisting of all continuous functions and Lipschitz-continuous functions, respectively.

Let $\mathcal{M}_s(S)$ be the set of all finite, countably additive functions on \mathcal{B}_S . By $\mathcal{M}(S)$ and $\mathcal{M}_1(S)$ we denote the subsets of $\mathcal{M}_s(S)$ consisting of all non-negative measures and all probability measures, respectively.

We write $\mathcal{M}_{1,k}^{\mathcal{L}}(S)$ for the set of all $\mu \in \mathcal{M}_1(S)$ satisfying

$$\int_S (\mathcal{L}(x))^k \mu(dx) < \infty,$$

where $k > 0$ and $\mathcal{L}: S \rightarrow \mathbb{R}_+$ is a *Lyapunov function* i.e \mathcal{L} is bounded on bounded sets and for some $x_0 \in S$

$$\lim_{d(x,x_0) \rightarrow \infty} \mathcal{L}(x) = \infty,$$

if S is unbounded.

The set $\mathcal{M}_s(S)$ is considered with the *Fortet-Mourier norm* $\|\cdot\|_{FM}$ ([12, 13]), given by

$$\|\mu\|_{FM} = \sup \left\{ \left| \int_S f(x) \mu(dx) \right| : f \in \mathcal{F}_{FM}(S) \right\} \quad \text{for } \mu \in \mathcal{M}_s(S),$$

where

$$\mathcal{F}_{FM}(S) = \{f \in C(S) : |f(x)| \leq 1, |f(x) - f(y)| \leq d(x,y), x, y \in S\}.$$

2.2. Markov operators

An operator $\mathcal{P}: \mathcal{M}(S) \rightarrow \mathcal{M}(S)$ is called a *Markov operator* if

- (i) $\mathcal{P}(a\mu_1 + b\mu_2) = a\mathcal{P}\mu_1 + b\mathcal{P}\mu_2$ for $a, b \in \mathbb{R}_+, \mu_1, \mu_2 \in \mathcal{M}(S)$,
- (ii) $(\mathcal{P}\mu)(S) = \mu(S)$ for $\mu \in \mathcal{M}(S)$.

If for the Markov operator \mathcal{P} there exists a dual operator $U: B(S) \rightarrow B(S)$ i.e

$$(1) \quad \int_S f(x) (\mathcal{P}\mu)(dx) = \int_S (Uf)(x) \mu(dx) \quad \text{for } f \in B(S), \mu \in \mathcal{M}(S),$$

then \mathcal{P} is called *regular*.

A function $\mathcal{K}: S \times \mathcal{B}_S \rightarrow [0, 1]$ is called a *substochastic kernel* if

- (i) $\mathcal{K}(\cdot, A): S \rightarrow [0, 1]$ is measurable for every $A \in \mathcal{B}_S$,
- (ii) $\mathcal{K}(x, \cdot): \mathcal{B}_S \rightarrow [0, 1]$ is a subprobability Borel measure for every $x \in S$.

If \mathcal{K} is substochastic kernel and $\mathcal{K}(x, S) = 1$ for $x \in S$ then \mathcal{K} is called *stochastic kernel*.

For given stochastic kernel \mathcal{K} we can always set two mappings $\mathcal{P}: \mathcal{M}(S) \rightarrow \mathcal{M}(S)$ and $U: B(S) \rightarrow B(S)$ by formulas:

$$(2) \quad (\mathcal{P}\mu)(A) = \int_S \mathcal{K}(x, A) \mu(dx) \quad \text{for } \mu \in \mathcal{M}(S), A \in \mathcal{B}_S,$$

and

$$(3) \quad (Uf)(x) = \int_S f(y) \mathcal{K}(x, dy) \quad \text{for } x \in S, f \in B(S).$$

Then \mathcal{P} is a *Markov operator* and U is its dual operator. Let us notice that using (1) and (3) we obtain

$$(4) \quad \int_S f(x)(\mathcal{P}\mu)(dx) = \int_S \int_S f(y) \mathcal{K}(x, dy) \mu(dx) \quad \text{for } f \in B(S).$$

We want to emphasize that the operator \mathcal{P} can be extended to a linear operator defined on the space of all bounded below Borel functions with keeping the duality property (1).

A regular Markov operator \mathcal{P} is called *Feller* if $Uf \in C(S)$ for any $f \in C(S)$.

We say that the $\mu_* \in \mathcal{M}(S)$ is *invariant* for operator \mathcal{P} if $\mathcal{P}\mu_* = \mu_*$.

The operator \mathcal{P} is said to be *exponentially ergodic* if there exists invariant measure $\mu_* \in \mathcal{M}_1(S)$ and constant $\beta \in [0, 1)$ such that, for every $\mu \in \mathcal{M}_{1,1}^{\mathcal{L}}(S)$ and some constant $C_\mu \in \mathbb{R}$, we have

$$\|\mathcal{P}^n \mu - \mu_*\|_{FM} \leq C_\mu \beta^n \quad \text{for all } n \in \mathbb{N}.$$

2.3. Markov chains

It is well known that if we take a stochastic kernel \mathcal{K} and a measure $\mu \in \mathcal{M}_1(S)$ then we can always define on relevant probability space, say $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$, a homogeneous Markov chain $(\chi_n)_{n \in \mathbb{N}_0}$ for which

$$\mathbb{P}_\mu(\chi_0 \in A) = \mu(A) \quad \text{for } A \in \mathcal{B}_S,$$

$$(5) \quad \mathcal{K}(x, A) = \mathbb{P}_\mu(\chi_{n+1} \in A | \chi_n = x) \quad \text{for } x \in S, A \in \mathcal{B}_S, n \in \mathbb{N}_0.$$

If we consider the Markov operator \mathcal{P} for the kernel (5) according formula (2), then

$$\mu_{n+1} = \mathcal{P}\mu_n \quad \text{for } n \in \mathbb{N}_0$$

where

$$\mu_n(\cdot) := \mathbb{P}_\mu(\chi_n \in \cdot).$$

In our further considerations we will use the symbol \mathbb{E}_μ for the expectation with respect to \mathbb{P}_μ . If $\mu = \delta_x$ for some fixed $x \in S$, we will write \mathbb{E}_x instead of \mathbb{E}_{δ_x} .

We say that a time-homogeneous Markov chain evolving on the space S^2 is a *Markovian coupling* of some stochastic kernel $\mathcal{K}: S \times \mathcal{B}_S \rightarrow [0, 1]$ whenever its stochastic kernel $\mathcal{J}: S^2 \times \mathcal{B}_{S^2} \rightarrow [0, 1]$ satisfies

$$\mathcal{J}(x, y, A \times S) = \mathcal{K}(x, A) \quad \text{and} \quad \mathcal{J}(x, y, S \times A) = \mathcal{K}(y, A)$$

for all $x, y \in S$ and $A \in \mathcal{B}_S$. Let us underline this, that if $\underline{\mathcal{J}}: S^2 \times \mathcal{B}_{S^2} \rightarrow [0, 1]$ is a substochastic kernel satisfying

$$\underline{\mathcal{J}}(x, y, A \times S) \leq \mathcal{K}(x, A) \quad \text{and} \quad \underline{\mathcal{J}}(x, y, S \times A) \leq \mathcal{K}(y, A),$$

for all $x, y \in S$ and $A \in \mathcal{B}_S$, then we are always able to construct a Markovian coupling of \mathcal{K} whose stochastic kernel \mathcal{J} satisfies $\underline{\mathcal{J}} \leq \mathcal{J}$.

2.4. The law of the iterated logarithm for Markov chains

Let $\mu \in \mathcal{M}_1(S)$ be a initial distribution of the Markov chain $(\chi_n)_{n \in \mathbb{N}_0}$. For any $n \in \mathbb{N}$ and $f \in Lip(S)$ we define

$$s_n(f) = \begin{cases} \frac{f(\chi_0) + \dots + f(\chi_{n-1})}{\sqrt{2n \ln(\ln(n))}} & \text{for } n > e, \\ 0 & \text{for } n \leq e. \end{cases}$$

Suppose that there exists the unique invariant measure $\mu_* \in \mathcal{M}_1(S)$ for $(\chi_n)_{n \in \mathbb{N}_0}$. We say that the LIL holds for Markov chain $(f(\chi_n))_{n \in \mathbb{N}_0}$ if for some $\sigma(\tilde{f}) \in (0, \infty)$

$$\mathbb{P}_\mu \left(\limsup_{n \rightarrow \infty} s_n(\tilde{f}) = \sigma(\tilde{f}) \right) = 1 \quad \text{and} \quad \mathbb{P}_\mu \left(\liminf_{n \rightarrow \infty} s_n(\tilde{f}) = -\sigma(\tilde{f}) \right) = 1,$$

where $\tilde{f} = f - \int_S f(x) \mu_*(dx)$.

The following theorem is proved in [3, Theorem 4.7].

THEOREM 1. *Let $(\chi_n)_{n \in \mathbb{N}_0}$ be a time-homogeneous Markov chain with values in S and let $\mathcal{K}: S \times \mathcal{B}_S \rightarrow [0, 1]$ be the stochastic kernel of $(\chi_n)_{n \in \mathbb{N}_0}$ for which the following conditions hold:*

- (B0) *The Markov operator \mathcal{P} corresponding to \mathcal{K} is Feller operator.*
- (B1) *There exist a Lyapunov function $\mathcal{L}: S \rightarrow \mathbb{R}_+$ and constants $a \in (0, 1)$ and $b \in (0, \infty)$ such that*

$$(U\mathcal{L})(x) \leq a\mathcal{L}(x) + b.$$

(B1') For some fixed $\bar{x} \in S$ and some $r \in (0, 2)$ there exist $a^* \in (0, 1)$ and $b^* \in (0, \infty)$ such that for every $\nu \in \mathcal{M}_{1,2+r}^{d(\cdot, \bar{x})}(S)$

$$\left(\int_S \left(d(x, \bar{x}) \right)^{2+r} (\mathcal{P}\nu)(dx) \right)^{\frac{1}{2+r}} \leq a^* \left(\int_S \left(d(x, \bar{x}) \right)^{2+r} \nu(dx) \right)^{\frac{1}{2+r}} + b^*.$$

Moreover, assume that there is a substochastic kernel $\underline{\mathcal{J}}: S^2 \times \mathcal{B}_{S^2} \rightarrow [0, 1]$ that for every $A \in \mathcal{B}_S, x, y \in S,$

$$\underline{\mathcal{J}}(x, y, A \times S) \leq \mathcal{K}(x, A) \quad \text{and} \quad \underline{\mathcal{J}}(x, y, S \times A) \leq \mathcal{K}(y, A).$$

(B2) There exist $F \subset S^2$ and $q \in (0, 1)$ such that $\text{supp } \underline{\mathcal{J}}(x, y, \cdot) \subset F$ and

$$\int_{S^2} d(u, v) \underline{\mathcal{J}}(x, y, du \times dv) \leq qd(x, y) \quad \text{for } (x, y) \in F.$$

(B3) For $U(r) = \{(x, y) \in F : d(x, y) \leq r\}, r > 0,$ we have

$$\inf_{(x,y) \in F} \underline{\mathcal{J}}(x, y, U(qd(x, y))) > 0.$$

(B4) There exist constants $v \in (0, 1]$ and $l > 0$ such that

$$\underline{\mathcal{J}}(x, y, S^2) \geq 1 - ld(x, y)^v \text{ for every } (x, y) \in F.$$

(B5) There is a coupling $((\chi_n^{(1)}, \chi_n^{(2)}))_{n \in \mathbb{N}_0}$ of \mathcal{K} with stochastic kernel $\mathcal{J},$ satisfying $\underline{\mathcal{J}} \leq \mathcal{J},$ such that for some $R > 0$ and

$$K := \{(x, y) \in F : \mathcal{L}(x) + \mathcal{L}(y) < R\}$$

we can find $\zeta \in (0, 1)$ and $\bar{C} > 0$ satisfying

$$\mathbb{E}_{(x,y)}(\zeta^{-\sigma_K}) \leq \bar{C} \quad \text{whenever } \mathcal{L}(x) + \mathcal{L}(y) < 4b(1 - a)^{-1},$$

where

$$\sigma_K = \inf\{n \in \mathbb{N} : (\chi_n^{(1)}, \chi_n^{(2)}) \in K\}.$$

Let $\mu \in \mathcal{M}_{1,2+r}^{d(\cdot, \bar{x})}(S)$ be an initial distribution of $(\chi_n)_{n \in \mathbb{N}_0}.$ If $f \in \text{Lip}(S)$ and f is not constant function, then $(f(\chi_n))_{n \in \mathbb{N}_0}$ satisfies the LIL.

3. Model description and assumptions

Let us fix natural number N , set of indexes $I = \{1, \dots, N\}$ and a Polish metric space (Y, ρ) . We introduce metric space (X, ρ_c) , where

$$X := Y \times I,$$

$$(6) \quad \rho_c((y_1, i), (y_2, j)) = \rho(y_1, y_2) + c\phi(i, j), \quad (y_1, i), (y_2, j) \in X$$

and

$$(7) \quad \phi(i, j) = \begin{cases} 1 & \text{for } i \neq j, \\ 0 & \text{for } i = j, \end{cases}$$

and c is some fixed positive constant. Let Θ be a compact interval.

Assume that for every $i \in I$ the mapping $\Pi_i : \mathbb{R}_+ \times Y \rightarrow Y$ is a semiflow. It means that Π_i is continuous with respect to each variable and

$$\Pi_i(0, y) = y \quad \text{and} \quad \Pi_i(s + t, y) = \Pi_i(s, \Pi_i(t, y)) \quad \text{for } i \in I, y \in Y, s, t \in \mathbb{R}_+.$$

Our considerations are focused on discrete-time dynamical system described in detail in [4] and determined by stochastic process $((Y(t), \xi(t)))_{t \geq 0}$ evolving through random jumps in the space X .

On a time frame $[t_{n-1}, t_n]$ the process $(Y(t))_{t \geq 0}$ is driven accordingly with Π_i , where index i is appointed by $(\xi(t))_{t \geq 0}$.

At the moment of the jump, i.e at time t_n , process $(Y(t))_{t \geq 0}$ skips to a new state due to the mapping $q_\theta : Y \rightarrow Y$ and the current semiflow Π_i is displaced by Π_j . The q_θ is randomly pick out from a given set $\{q_\theta : \theta \in \Theta\}$. We assume here that $Y \times \Theta \ni (y, \theta) \mapsto q_\theta(y) \in Y$ is continuous and that the probability of choosing q_θ is related with density function $\Theta \ni \theta \mapsto p_\theta(y)$, $y \in Y$, such that $(\theta, y) \mapsto p_\theta(y)$ is continuous. The semiflows conversion is done in accordance with a matrix of continuous probabilities $\pi_{ij} : Y \rightarrow [0, 1]$, $i, j \in I$, satisfying

$$\sum_{j \in I} \pi_{ij}(y) = 1 \quad \text{for } i \in I, y \in Y.$$

The intensity of jumps is associated with Lipschitz continuous function $\lambda : Y \rightarrow (0, \infty)$ such that

$$\underline{\lambda} = \inf_{y \in Y} \lambda(y) > 0 \quad \text{and} \quad \bar{\lambda} = \sup_{y \in Y} \lambda(y) < \infty.$$

In this work we examine only the sequence of random variables given by the locations directly after the jumps, that is $(Y_n, \xi_n) := (Y(\tau_n), \xi(\tau_n))$, $n \in \mathbb{N}_0$, where τ_n is a random variable describing the jump time t_n .

We will express the above considerations for the intuitive description of how our model works in the language of random variables. Let $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ be a probability space on which we define $((Y_n, \xi_n))_{n \in \mathbb{N}_0}$. Let $(Y_0, \xi_0): \Omega \rightarrow X$ be a random variable with arbitrary and fixed distribution $\mu \in \mathcal{M}_1(X)$. Further, we introduce sequences $(\tau_n)_{n \in \mathbb{N}_0}$, $(\xi_n)_{n \in \mathbb{N}}$, $(\eta_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ of random variables which fulfill the following conditions:

- $\tau_n: \Omega \rightarrow \mathbb{R}_+$, $n \in \mathbb{N}_0$, where $\tau_0 = 0$, form a strictly increasing sequence such that $\tau_n \rightarrow \infty$ a.e., and $\Delta\tau_n = \tau_n - \tau_{n-1}$ are mutually independent and have the conditional distributions given by

$$\mathbb{P}_\mu(\Delta\tau_{n+1} \leq t | Y_n = y, \xi_n = i) = 1 - e^{-L(t, (y, i))} \quad \text{for } t \geq 0,$$

whenever $y \in Y$ and $i \in I$, where L is given by

$$(8) \quad L(t, (y, i)) = \int_0^t \lambda(\Pi_i(s, y)) ds.$$

- $\xi_n: \Omega \rightarrow I$, $n \in \mathbb{N}$, satisfy

$$\mathbb{P}_\mu(\xi_n = j | Y_n = y, \xi_{n-1} = i) = \pi_{ij}(y) \quad \text{for } i, j \in I, y \in Y.$$

- $\eta_n: \Omega \rightarrow \Theta$, $n \in \mathbb{N}_0$, is specified by

$$\mathbb{P}_\mu(\eta_{n+1} \in A | \Pi_{\xi_n}(\Delta\tau_{n+1}, Y_n) = y) = \int_A p_\theta(y) d\theta$$

for all $A \in \mathcal{B}_\Theta$ and $y \in Y$.

- $Y_n: \Omega \rightarrow Y$, $n \in \mathbb{N}$, are determined by

$$Y_{n+1} = q_{\eta_{n+1}}(\Pi_{\xi_n}(\Delta\tau_{n+1}, Y_n)) \quad \text{for } n \in \mathbb{N}_0.$$

Setting

$$U_0 = (Y_0, \xi_0), \quad U_k = (Y_0, \tau_1, \dots, \tau_k, \eta_1, \dots, \eta_k, \xi_0, \dots, \xi_k) \quad \text{for } k \in \mathbb{N},$$

we assume that, for every $k \in \mathbb{N}_0$, the random variables ξ_{k+1} and η_{k+1} are conditionally independent of U_k given $\{Y_{k+1} = y, \xi_k = i\}$ and $\{\Pi_{\xi_k}(\Delta\tau_{k+1}, Y_k) = y\}$, respectively. In addition to this, we require that ξ_{k+1} , η_{k+1} and $\Delta\tau_{k+1}$ are mutually conditionally independent given U_k , and that $\Delta\tau_{k+1}$ is independent of U_k .

These assumptions assure that $((Y_n, \xi_n))_{n \in \mathbb{N}_0}$ is a Markov chain with phase space X and stochastic kernel $\mathcal{K}: X \times \mathcal{B}_X \rightarrow [0, 1]$ given by

$$(9) \quad \mathcal{K}((y, i), A) = \sum_{j \in I} \int_{\Theta} \int_0^{\infty} \mathbb{1}_A(q_{\theta}(\Pi_i(t, y)), j) \lambda(\Pi_i(t, y)) e^{-L(t, (y, i))} \\ \times \pi_{ij}(q_{\theta}(\Pi_i(t, y))) p_{\theta}(\Pi_i(t, y)) dt d\theta,$$

where L is given by (8).

The evolution of the distributions $\mu_n(\cdot) := \mathbb{P}_{\mu}((Y_n, \xi_n) \in \cdot)$ can be described by the Markov operator $\mathcal{P}: \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ corresponding to (9), as defined in (2).

As in [4], we apply following assumptions on considered model:

(A1) There is $y_* \in Y$ such that for every $i \in I$

$$\sup_{y \in Y} \int_0^{\infty} e^{-\lambda t} \int_{\Theta} \rho(q_{\theta}(\Pi_i(t, y_*)), y_*) p_{\theta}(\Pi_i(t, y)) d\theta dt < \infty.$$

(A2) There exist $\alpha \in \mathbb{R}$, $L > 0$ and a bounded on bounded sets function $\mathcal{T}: Y \rightarrow \mathbb{R}_+$ such that for $t \geq 0$, $y_1, y_2 \in Y$, $i, j \in I$

$$\rho(\Pi_i(t, y_1), \Pi_j(t, y_2)) \leq L e^{\alpha t} \rho(y_1, y_2) + t \mathcal{T}(y_2) \phi(i, j),$$

where $\phi(i, j)$ is given by (7).

(A3) There is a constant $L_q > 0$ such that for $y_1, y_2 \in Y$

$$\int_{\Theta} \rho(q_{\theta}(y_1), q_{\theta}(y_2)) p_{\theta}(y_1) d\theta \leq L_q \rho(y_1, y_2).$$

(A4) There exists $L_{\lambda} > 0$ such that for $y_1, y_2 \in Y$

$$|\lambda(y_1) - \lambda(y_2)| \leq L_{\lambda} \rho(y_1, y_2).$$

(A5) There exists $L_{\pi} > 0$ and $L_p > 0$ such that for $y_1, y_2 \in Y$ and $i \in I$

$$\sum_{j \in I} |\pi_{ij}(y_1) - \pi_{ij}(y_2)| \leq L_{\pi} \rho(y_1, y_2),$$

$$\int_{\Theta} |p_{\theta}(y_1) - p_{\theta}(y_2)| d\theta \leq L_p \rho(y_1, y_2).$$

(A6) There exist $\delta_\pi > 0$ and $\delta_p > 0$ such that for $y_1, y_2 \in Y, i_1, i_2 \in I$

$$\sum_{j \in I} \min\{\pi_{i_1 j}(y_1), \pi_{i_2 j}(y_2)\} \geq \delta_\pi,$$

$$\int_{\Theta(y_1, y_2)} \min\{p_\theta(y_1), p_\theta(y_2)\} d\theta \geq \delta_p,$$

where $\Theta(y_1, y_2) = \{\theta \in \Theta : \rho(q_\theta(y_1), q_\theta(y_2)) \leq \rho(y_1, y_2)\}$.

The choice of c , which appears in (6), depends on constants from conditions (A1)–(A4). More details can be found in [4].

In [4] is also shown ([4, Theorem 3.1]), that if the conditions (A1)–(A6) hold and the constants $L, L_q, \alpha, \bar{\lambda}, \underline{\lambda}$ satisfy inequality

$$(10) \quad LL_q \bar{\lambda} + \alpha < \underline{\lambda},$$

then the operator \mathcal{P} corresponding to (9) is exponentially ergodic. Proof of this fact requires checking whether conditions (B0)–(B5) beyond (B1') are met.

4. The main result

Let $((Y_n, \xi_n))_{n \in \mathbb{N}_0}$ be the Markov chain described in the previous section. We extend conditions (A1) and (A3) imposed on this model to:

(A1') There exist $r \in (0, 2)$ and $\tilde{y} \in Y$ such that for $i \in I$

$$\sup_{y \in Y} \int_0^\infty e^{-\lambda t} \int_{\Theta} \left(\rho(q_\theta(\Pi_i(t, \tilde{y})), \tilde{y}) \right)^{2+r} p_\theta(\Pi_i(t, y)) d\theta dt < \infty.$$

(A3') There exists $\tilde{L}_q > 0$ such that for $y_1, y_2 \in Y$

$$\int_{\Theta} \left(\rho(q_\theta(y_1), q_\theta(y_2)) \right)^{2+r} p_\theta(y_1) d\theta \leq \tilde{L}_q \left(\rho(y_1, y_2) \right)^{2+r}.$$

It is noticeable that (A1') implies (A1) and (A3') implies (A3) with $L_q = \tilde{L}_q^{1/(2+r)}$.

Let us also define function $\mathcal{L}: X \rightarrow \mathbb{R}_+$ by

$$(11) \quad \mathcal{L}(y, i) = \rho(y, \tilde{y}) \quad \text{for } (y, i) \in X,$$

where \tilde{y} is point which appears in (A1').

THEOREM 2. *Let $((Y_n, \xi_n))_{n \in \mathbb{N}_0}$ be the Markov chain with stochastic kernel given by (9) and let (A1'), (A2), (A3'), (A4)–(A6) hold with constants satisfying*

$$(12) \quad \left(\frac{\bar{\lambda}}{\underline{\lambda}} L \right)^{2+r} \tilde{L}_q + \frac{(2+r)\alpha}{\underline{\lambda}} < 1.$$

Let $f \in \text{Lip}(X)$ and f is not constant function. If the initial distribution μ of the chain $(f(Y_n, \xi_n))_{n \in \mathbb{N}_0}$ belongs to $\mathcal{M}_{1,2+r}^{\mathcal{L}}$, where \mathcal{L} is defined by (11), then the LIL holds for $(f(Y_n, \xi_n))_{n \in \mathbb{N}_0}$.

PROOF. The following proof is based on techniques shown in proof of [3, Theorem 5.2]. First we show that (12) implies that for $L_q = \tilde{L}_q^{1/(2+r)}$ inequality (10) is fulfilled. To obtain this, let us assume (12) and in contrast to (10) that $L_q = \tilde{L}_q^{1/(2+r)}$ and

$$(13) \quad LL_q \bar{\lambda} + \alpha \geq \underline{\lambda}.$$

Since $L, \bar{\lambda}, \underline{\lambda}, \tilde{L}_q, r$ are positive numbers, it results in particular from (12) that

$$\frac{\alpha}{\underline{\lambda}} < \frac{1}{2+r} < 1$$

and so

$$1 - \frac{\alpha}{\underline{\lambda}} > 0.$$

Then, from (13)

$$\left(\frac{\bar{\lambda} LL_q}{\underline{\lambda}} \right)^{2+r} \geq \left(1 - \frac{\alpha}{\underline{\lambda}} \right)^{2+r}.$$

Further, from the Bernoulli inequality

$$\left(\frac{\bar{\lambda} LL_q}{\underline{\lambda}} \right)^{2+r} \geq \left(1 - \frac{\alpha}{\underline{\lambda}} \right)^{2+r} \geq 1 - (2+r) \frac{\alpha}{\underline{\lambda}},$$

which contradicts (12). Therefore (12) implies (10) with $L_q = \tilde{L}_q^{1/(2+r)}$.

Let us notice that $\mathcal{M}_{1,2+r}^{\mathcal{L}}(X) \subset \mathcal{M}_{1,2+r}^{\rho_c(\cdot, \tilde{x})}(X)$, where $\tilde{x} = (\tilde{y}, \tilde{i})$ for some fixed $\tilde{i} \in I$ and \tilde{y} given in (A1'). Indeed, let $\nu \in \mathcal{M}_{1,2+r}^{\mathcal{L}}(X)$. Then

$$\begin{aligned} \int_X \left(\rho_c(x, \tilde{x})\right)^{2+r} \nu(dx) &= \int_{Y \times I} \left(\rho_c((y, i), (\tilde{y}, \tilde{i}))\right)^{2+r} \nu(dy \times di) \\ &\leq \int_{Y \times I} \left(\rho(y, \tilde{y}) + c\right)^{2+r} \nu(dy \times di) < \infty. \end{aligned}$$

Considering that conditions (B0)–(B5) beyond (B1') were verified in [4] we show that (B1') also holds. In particular, using (4) and definition of stochastic kernel (9) we obtain

$$\begin{aligned} (14) \quad &\int_X \left(\rho_c(x, \tilde{x})\right)^{2+r} (P\mu)(dx) \\ &= \int_X \int_X \left(\rho_c((s, l), (\tilde{y}, \tilde{i}))\right)^{2+r} \mathcal{K}(y, i, ds \times dl) \mu(dy \times di) \\ &= \sum_{j \in I} \int_X \int_0^\infty \int_\Theta \lambda(\Pi_i(t, y)) e^{-L(t, (y, i))} \left(\rho(\Pi_i(t, y), \tilde{y}) + c\phi(j, \tilde{i})\right)^{2+r} \\ &\quad \times \pi_{ij}(q_\theta(\Pi_i(t, y))) p_\theta(\Pi_i(t, y)) d\theta dt \mu(dy \times di). \end{aligned}$$

Let us define $\nu \in \mathcal{M}_1(Z)$, where $Z = X \times \mathbb{R}_+ \times \Theta \times I$ as

$$\begin{aligned} \nu(A) &= \sum_{j \in I} \int_X \int_0^\infty \int_\Theta \lambda(\Pi_i(t, y)) e^{-L(t, (y, i))} \mathbb{1}_A(y, i, t, \theta, j) \pi_{ij}(q_\theta(\Pi_i(t, y))) \\ &\quad \times p_\theta(\Pi_i(t, y)) d\theta dt \mu(dy \times di) \quad \text{for } A \in \mathcal{B}_Z, \end{aligned}$$

and $\psi: Z \rightarrow \mathbb{R}$ by formula

$$\psi(y, i, t, \theta, j) = \rho(q_\theta(\Pi_i(t, y)), \tilde{y}) + c\phi(j, \tilde{i}).$$

From triangle inequality, for $y, i, t, \theta, j \in Z$

$$\psi(y, i, t, \theta, j) \leq \rho(q_\theta(\Pi_i(t, y)), q_\theta(\Pi_i(t, \tilde{y}))) + \rho(q_\theta(\Pi_i(t, \tilde{y})), \tilde{y}) + c\phi(j, \tilde{i}).$$

Then, using the definition of ν we can write (14) as

$$\int_X \left(\rho_c(x, \tilde{x})\right)^{2+r} (P\mu)(dx) = \int_Z \left(\psi(y, i, t, \theta, j)\right)^{2+r} \nu(dy \times di \times dt \times d\theta \times dj).$$

Applying Minkowski inequality, we obtain

$$\begin{aligned} & \left[\int_X \left(\rho_c(x, \tilde{x}) \right)^{2+r} (P\mu)(dx) \right]^{\frac{1}{2+r}} \\ &= \left[\int_Z \left(\psi(y, i, t, \theta, j) \right)^{2+r} \nu(dy \times di \times dt \times d\theta \times dj) \right]^{\frac{1}{2+r}} \\ &\leq \left[\int_Z \left(\rho(q_\theta(\Pi_i(t, y)), q_\theta(\Pi_i(t, \tilde{y}))) \right)^{2+r} \nu(dy \times di \times dt \times d\theta \times dj) \right]^{\frac{1}{2+r}} \\ &\quad + \left[\int_Z \left(\rho(q_\theta(\Pi_i(t, \tilde{y})), \tilde{y}) \right)^{2+r} \nu(dy \times di \times dt \times d\theta \times dj) \right]^{\frac{1}{2+r}} + c. \end{aligned}$$

Further, using (A3') and (A2) we obtain

$$\begin{aligned} & \int_Z \left(\rho(q_\theta(\Pi_i(t, y)), q_\theta(\Pi_i(t, \tilde{y}))) \right)^{2+r} \nu(dy \times di \times dt \times d\theta \times dj) \\ &\leq \int_X \int_0^\infty \lambda(\Pi_i(t, y)) e^{-L(t, (y, i))} \tilde{L}_q \left(\rho(\Pi_i(t, y), \Pi_i(t, \tilde{y})) \right)^{2+r} dt \mu(dy \times di) \\ &\leq \int_X \int_0^\infty \lambda(\Pi_i(t, y)) e^{-L(t, (y, i))} \tilde{L}_q L^{2+r} e^{(2+r)\alpha t} \left(\rho(y, \tilde{y}) \right)^{2+r} dt \mu(dy \times di) \\ &\leq \int_X \int_0^\infty \bar{\lambda} e^{-\lambda t} \tilde{L}_q L^{2+r} e^{(2+r)\alpha t} \left(\rho(y, \tilde{y}) \right)^{2+r} dt \mu(dy \times di) \\ &\leq \bar{\lambda} \tilde{L}_q L^{2+r} \int_0^\infty e^{-(\lambda - (2+r)\alpha)t} dt \int_X \left(\rho(y, \tilde{y}) \right)^{2+r} \mu(dy \times di) \\ &\leq \frac{\bar{\lambda} \tilde{L}_q L^{2+r}}{\lambda - (2+r)\alpha} \int_X \left(\rho_c(x, \tilde{x}) \right)^{2+r} \mu(dx). \end{aligned}$$

Let

$$a^* = \left[\frac{\bar{\lambda} \tilde{L}_q L^{2+r}}{\lambda - (2+r)\alpha} \right]^{\frac{1}{2+r}}$$

and

$$b^* = \left[\sup_{y \in Y} \left[\int_0^\infty \bar{\lambda} e^{-\lambda t} \int_\Theta \left(\rho(q_\theta(\Pi_i(t, \tilde{y})), \tilde{y}) \right)^{2+r} p_\theta(\Pi_i(t, y)) d\theta dt \right] \right]^{\frac{1}{2+r}} + c.$$

From inequality (12) it follows that $a^* \in (0, 1)$. Moreover, assumption (A1') ensures that b^* is finite and thus the (B1') is met. \square

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