Annales Mathematicae Silesianae **35** (2021), no. 2, 250–259
DOI: 10.2478/amsil-2021-0006

SANDWICH TYPE RESULTS FOR m-CONVEX REAL FUNCTIONS

Teodoro Lara, Edgar Rosales

Abstract. We establish necessary and sufficient conditions allowing separation of pair of real functions by an m-convex and by an m-affine function. Some examples and a geometric interpretation of m-convexity of a function is exhibited, as well as a Jensen's inequality for this kind of function.

1. Introduction and preliminaries

Since apparition of sandwich type theorems of separation for real convex functions in 1994 ([1, Theorem 1]), a quite number of researchers have obtained similar results for different kinds of convexity around. It is well-known that, basically, the idea consists of establishing necessary and sufficient conditions for a couple of given functions, under which the existence of a third function, between them, with the kind of convexity considered. Nowadays, we have at our disposal results in this context, strong convexity ([13, Theorem 2]); h-convexity ([16, Theorem 3]); in the case of convexity for set-valued functions ([18, Theorem 1]); and more recent, in the context of harmonically convex functions, and reciprocally strongly convex functions ([2, Theorem 2.4, Theorem 3.1]), as well as versions for the case of convexity and strong convexity of functions defined on time scales ([8, Theorem 2.10], [9, Theorem 13]).

Received: 04.10.2020. Accepted: 14.04.2021. Published online: 26.05.2021.
(2020) Mathematics Subject Classification: 26A51, 26B25, 39B62.
Key words and phrases: m-convex function, m-affine function, Jensen type inequality, sandwich theorem.

For the case of m-convex functions for functions defined on the positive real line a separation result has been established in [12]; for set-valued functions a separation result is given in [7, Theorem 2.28]; and in [4, Theorem 4.2] there is a result of a sandwich type as well, involving m-convexity of sets and convexity of functions. In our present research we establish necessary and sufficient conditions to separate a pair of real functions by a real m-convex function defined on a convex subset D of a linear space X; at the same time we set and prove a similar result for the case of m-affine functions. We begin by recalling some known concepts and results; next we set our results, illustrate with examples, and perform a geometric interpretation of m-convexity of real functions.

Definition 1.1 ([5, 19, 20]). Let X be a real linear space and $m \in [0,1]$. A nonempty set $D \subseteq X$ is called *m-convex* if for any $x, y \in D$ and $t \in [0,1]$ the point $tx + m(1-t)y \in D$.

In [5], the incoming results are established and used afterward to prove several statements.

Lemma 1.2 ([5, Lemma 3.3]). *A set $D \subseteq X$ is m-convex if and only if D coincides with the set of all m-convex combinations of elements of D (denoted by $D^\star_m$); these combinations are of the form $\sum_{i=1}^n m^{1-\delta_{i1}} t_i x_i$, $m \in (0,1)$, n any natural number, δ_{ij} is the known Delta of Kronecker function, and the real numbers t_i are nonnegative ($i = 1, \ldots, n$) with $0 < \sum_{i=1}^n t_i \leq 1$.*

Remark 1.3 ([5, Remark 3.5]). The m-convex hull of a set $D \subseteq X$, denoted by $Conv_m(D)$, satisfies among others, the following statements:
(1) $D \subseteq Conv_m(D)$.
(2) $Conv_m(D)$ is an m-convex set of X.

Theorem 1.4 ([5, Theorem 3.6]). *If $\emptyset \neq D \subseteq X$, then $Conv_m(D) = D^\star_m$.*

Theorem 1.5 ([5, Theorem 3.11]). *Let X be a linear space of dimension n and D be any nonempty set of X. For all $x \in Conv_m(D)$ there exists a set $D_x \subseteq D$ such that $\#(D_x) \leq n+1$ and $x \in Conv_m(D_x)$.*

Definition 1.6 ([4, 6, 11]). Let D be an m-convex subset of a real linear space X, and $m \in [0,1]$. A function $f \colon D \to \mathbb{R}$ is called *m-convex* (respectively *m-concave*, *m-affine*) if for any $x, y \in D$ and $t \in [0,1]$, it verifies

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y),$$

(respectively, if converse inequality or equality holds).

Remark 1.7. It is not difficult to check that if f is an m-convex function ($m \neq 1$), then $f(0) \leq 0$ ([11, Remark 3]), and this fact in turn implies $f(tx) \leq tf(x)$ for all $x \in D$ and $t \in [0,1]$.

At this point we are going to give geometric interpretation of m-convexity of a real function (on the real plane). The geometric meaning of convex functions is well-known. For a similar geometric interpretation of m-convexity of a real function (on the real plane) we consider the m-convex function $f\colon I \to \mathbb{R}$, where I is a real interval containing 0 (and therefore an m-convex set of $\mathbb{R}$ ([10, Theorem 2.6])). Let $x_1, x_2 \in I$ such that $x_1 \neq mx_2$. The straight line through the points $(x_1, f(x_1))$ and $(mx_2, mf(x_2))$ is given by equation

$$r(x) = \frac{mf(x_2) - f(x_1)}{mx_2 - x_1}(x - x_1) + f(x_1).$$

For any point p in the interval $[\min\{x_1, mx_2\}, \max\{x_1, mx_2\}]$, there exists $t \in [0,1]$ such that $p = tx_1 + m(1-t)x_2$. So,

$$r(p) = \frac{mf(x_2) - f(x_1)}{mx_2 - x_1}(tx_1 + m(1-t)x_2 - x_1) + f(x_1)$$

hence,

$$r(p) = tf(x_1) + m(1-t)f(x_2).$$

By m-convexity of f, $f(p) \leq r(p)$; that is, geometrically, m-convexity of f means that the points on the graph of f, are under the chord (or on the chord) joining the endpoints $(x_1, f(x_1))$, $(mx_2, mf(x_2))$ on $[\min\{x_1, mx_2\}, \max\{x_1, mx_2\}]$.

Example 1.8. In [4, Example 3.5] authors show a function which is m-convex but not convex. Another example of such a type of function is as follows.

Let $f\colon \left[0, \frac{1}{2}\right] \to \mathbb{R}$ given by $f(x) = -x^2 - 1$. So, for all $x, y \in \left[0, \frac{1}{2}\right]$, $t \in [0,1]$ and $m = \frac{1}{2}$,

$$tf(x) + m(1-t)f(y) - f(tx + m(1-t)y) = \frac{(1-t)(2 - y^2 - t(2x-y)^2)}{4} \geq 0$$

if and only if

$$2 - y^2 \geq t(2x - y)^2. \tag{1.1}$$

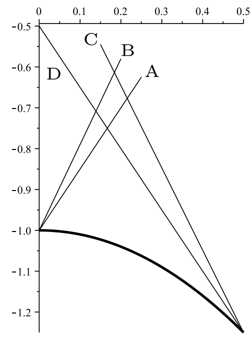

Figure 1. The graph of a function that is m-convex but not convex

Since $0 \leq x, y \leq \frac{1}{2}$, it follows $2 - y^2 \geq \frac{7}{4}$ and $2x - y \leq 1 \leq \frac{\sqrt{7}}{2}$. Moreover, if $2x \geq y$, and since $0 \leq t \leq 1$, (1.1) holds.

On the other hand, if $2x < y$, then $-\frac{\sqrt{7}}{2} < -\frac{1}{2} \leq 2x - y < 0$, and again (1.1) holds. So, f is $\frac{1}{2}$-convex. Clearly, f is not a convex function. The graph of f together with some of the above-mentioned chords are showed in Figure 1. Specifically, the chord A joining the endpoints $(0, -1)$ and $\left(\frac{1}{2}, -\frac{5}{4}\right)$; the chord B joining the endpoints $(0, -1)$ and $\left(\frac{2}{5}, -\frac{29}{25}\right)$; the chord C joining the endpoints $\left(\frac{1}{2}, -\frac{5}{4}\right)$ and $\left(\frac{3}{10}, -\frac{109}{100}\right)$; and the chord D joining the endpoints $\left(\frac{1}{2}, -\frac{5}{4}\right)$ and $(0, -1)$.

In [4, Theorem 4.2] it was proved that, for $0 < m < 1$, if $f\colon [0, +\infty) \to \mathbb{R}$ is an m-convex function, then there exists a convex function $h\colon [0, +\infty) \to \mathbb{R}$ such that $f(x) \leq h(x) \leq mf\left(\dfrac{x}{m}\right)$. This fact is a sandwich type theorem. Our main results refer to this kind of properties.

2. Main results

We start up this section with the following inequality involving an m-convex function, which is a Jensen's type inequality ([3, 14, 17]) for this kind of convexity of real functions.

Theorem 2.1. *Let $m \in [0, 1]$, and let X be a linear space and $D \subseteq X$ a nonempty m-convex set. If $f\colon D \to \mathbb{R}$ is an m-convex function, then for* *all* $t_1, \dots, t_n \geq 0$ *with* $\sum_{i=1}^{n} t_i \in (0,1]$*, and for all* $x_1, \dots, x_n \in D$ *(*n *is any natural number), we have*

$$f\left(\sum_{i=1}^{n} m^{1-\delta_{i1}} t_i x_i\right) \leq \sum_{i=1}^{n} m^{1-\delta_{i1}} t_i f(x_i).$$

Proof. From the m-convexity of f, $f(t_1x_1) \leq t_1 f(x_1)$ for all $x_1 \in D$ and $t_1 \in [0,1]$ (Remark 1.7). So, the result holds for $m = 0$. For $m \in (0,1)$, we apply to $-f$ (which is an m-concave function) the Jensen type inequality for m-concave functions ([6, Theorem 3.1]). Thus, for all $t_1, \dots, t_n \geq 0$ with $\sum_{i=1}^{n} t_i \in (0,1]$, and for all $x_1, \dots, x_n \in D$ ($n \geq 2$),

$$(-f)\left(\sum_{i=1}^{n} m^{1-\delta_{i1}} t_i x_i\right) \geq \sum_{i=1}^{n} m^{1-\delta_{i1}} t_i (-f)(x_i),$$

and conclusion follows. □

Now we establish necessary and sufficient conditions under which two real functions can be separated by an m-convex function.

Theorem 2.2. *Let* $m \in [0,1]$*, and let* X *be a real linear space of dimension* n*,* $D \neq \emptyset$ *an* m*-convex subset of* X*, and* $f, g \colon D \to \mathbb{R}$*. Then, there exists an* m*-convex function* $h \colon D \to \mathbb{R}$ *such that* $f \leq h \leq g$ *if and only if*

$$f\left(\sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i x_i\right) \leq \sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i g(x_i) \tag{2.1}$$

for all $x_1, \dots, x_{n+2} \in D$ *and all* $t_1, \dots, t_{n+2} \geq 0$ *with* $\sum_{i=1}^{n+2} t_i \in (0,1]$*.*

Proof. First, we assume that f, g satisfy (2.1), and let A be the m-convex hull of the epigraph of g; that is,

$$A = Conv_m\{(x,y) \in D \times \mathbb{R} \colon g(x) \leq y\}.$$

Let $(x,y) \in A$, by Theorem 1.5, there exist at most $n+2$ points in epigraph of g, say $(x_1,y_1), \dots, (x_{n+2}, y_{n+2})$, such that

$$(x,y) \in Conv_m\{(x_1,y_1), \dots, (x_{n+2}, y_{n+2})\}.$$

Now, by Theorem 1.4, $(x,y) \in \{(x_1,y_1),\dots,(x_{n+2},y_{n+2})\}^\star_m$; consequently, and accordance with Lemma 1.2, there exist $t_1,\dots,t_{n+2} \geq 0$, $\sum_{i=1}^{n+2} t_i \in (0,1]$ such that

$$(x,y) = \sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i (x_i, y_i).$$

In other words,

$$y = \sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i y_i \geq \sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i g(x_i) \geq f\left(\sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i x_i\right) = f(x).$$

Then, the set $\{y \in \mathbb{R} : (x,y) \in A\}$ is bounded from below, and we are able to define a function $h\colon D \to \mathbb{R}$ as

$$h(x) = \inf\{y \in \mathbb{R} : (x,y) \in A\};$$

hence $f(x) \leq h(x)$ for all $x \in D$. Furthermore, because $(x, g(x)) \in A$ (Remark 1.3 (1)), $h(x) \leq g(x)$ for all $x \in D$. To show that h is an m-convex function, we let $x_1, x_2 \in D$ and $t \in [0,1]$. So, for any couple of real numbers y_1, y_2 with $(x_1,y_1),(x_2,y_2) \in A$ (which is m-convex (Remark 1.3 (2)), $t(x_1,y_1) + m(1-t)(x_2,y_2) \in A$, or further, $(tx_1 + m(1-t)x_2, ty_1 + m(1-t)y_2) \in A$ and therefore,

$$h(tx_1 + m(1-t)x_2) \leq ty_1 + m(1-t)y_2.$$

Passing now to infimum we obtain $h(tx_1+m(1-t)x_2) \leq th(x_1)+m(1-t)h(x_2)$.

For the converse, we set $x_1,\dots,x_{n+2} \in D$ and $t_1,\dots,t_{n+2} \geq 0$ with $\sum_{i=1}^{n+2} t_i \in (0,1]$. Then, by Theorem 2.1 (applied to h),

$$f\left(\sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i x_i\right) \leq h\left(\sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i x_i\right) \leq \sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i h(x_i).$$

Consequently,

$$f\left(\sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i x_i\right) \leq \sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i g(x_i). \qquad \square$$

The above result implies a Hyers–Ulam stability result, actually we have the following

Corollary 2.3. *Let $f\colon D \to \mathbb{R}$ be a function satisfying*

$$f\left(\sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i x_i\right) \leq \sum_{i=1}^{n+2} m^{1-\delta_{i1}} t_i f(x_i) + \epsilon$$

with t_i, x_i as in Theorem 9. Then there exists an m-convex function $h\colon D \to \mathbb{R}$ such that

$$f(x) \leq h(x) \leq f(x) + \epsilon$$

for all $x \in D$.

This is readily obtained by considering $g = f + \epsilon$ in (2.1). This foregoing inequality also can be rewritten as

$$|f(x) - h(x)| < \epsilon$$

since the function $h - \frac{1}{2}$ is also m-convex.

Remark 2.4. If $X = \mathbb{R}$ and D is a real interval containing 0, then condition (2.1) becomes

$$f(t_1x_1 + mt_2x_2 + mt_3x_3) \leq t_1g(x_1) + mt_2g(x_2) + mt_3g(x_3) \tag{2.2}$$

for all $x_1, x_2, x_3 \in D$ and all $t_1, t_2, t_3 \geq 0$, with $0 < t_1 + t_2 + t_3 \leq 1$.

Example 2.5. If $f, g\colon \left[0, \frac{1}{2}\right] \to \mathbb{R}$ are given by $f(x) = -2$ and $g(x) = x$ respectively, it is clear that (2.2) holds for all $x_1, x_2, x_3 \in \left[0, \frac{1}{2}\right]$ and arbitrary $t_1, t_2, t_3 \geq 0$. But then, there exists a real m-convex function (defined in $\left[0, \frac{1}{2}\right]$) between f and g. Note that the given function in Example 1.8 is one of such functions.

Example 2.6 ([4, Example 4.3]). For the functions $f, g\colon [0, +\infty) \to \mathbb{R}$ defined as $f(x) = x + 1$ and $g(x) = x + 2$ respectively, there is no $\frac{1}{2}$-convex function between them; althoug they satisfy the following condition.

$$f\left(tx + \frac{1}{2}(1-t)y\right) \leq tg(x) + \frac{1}{2}(1-t)g(y)$$

for all $x, y \in [0, +\infty)$ and $t \in [0, 1]$. This fact tells us that an analogue (regarding the conditions) of the sandwich theorem for convex functions ([1]) is not true in the class of m-convex functions. Note that (for $m = \frac{1}{2}$), (2.2) is not true when, for instance, $t_1 = t_2 = \frac{1}{4}$ and $t_3 = \frac{1}{5}$.

For the proof of forthcoming result we need the following result, which is a part of [15, Theorem 1], and also a consequence of Helly's theorem ([21]).

Proposition 2.7 ([15]). *Let f, g be real functions defined on $[0, +\infty)$. If the inequalities*

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$$

and

$$g(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$$

hold for all $x, y \in [0, +\infty)$ and $t \in [0, 1]$, then there exists an affine function $h\colon [0, +\infty) \to \mathbb{R}$ such that $f \leq h \leq g$ on $[0, +\infty)$.

The next result provides conditions to be able to separate a pair of real functions by an m-affine one, and it is inspired on ideas of [15].

Theorem 2.8. *Let $0 < m < 1$ and $f, g\colon [0, +\infty) \to \mathbb{R}$ be two functions such that*

$$f(mx) = mf(x) \quad \text{and} \quad g(mx) = mg(x) \tag{2.3}$$

for all $x \in [0, +\infty)$. Then, the following conditions are equivalent.

(1) *There exists an m-affine function $h\colon [0, \infty) \to \mathbb{R}$ such that $f \leq h \leq g$ on $[0, +\infty)$.*
(2) *There exist an m-convex function $h_1\colon [0, +\infty) \to \mathbb{R}$ and an m-concave function $h_2\colon [0, +\infty) \to \mathbb{R}$ such that $f \leq h_1 \leq g$ and $f \leq h_2 \leq g$ on $[0, +\infty)$.*
(3) *The following inequalities hold*

$$f(tx + m(1-t)y) \leq tg(x) + m(1-t)g(y)$$

and

$$g(tx + m(1-t)y) \geq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, +\infty)$ and $t \in [0, 1]$.

Proof. (1) $\Rightarrow$ (2) is a consequence of the fact that any m-affine function is both m-convex and m-concave.

(2) $\Rightarrow$ (3) follows from the m-convexity of h_1 and the m-concavity of h_2, respectively.

(3) $\Rightarrow$ (1): First of all, notice that (because $m \neq 1$) the condition (2.3) implies that $f(0) = g(0) = 0$.

Let $x, y \in [0, +\infty)$, and $t \in [0, 1]$. It is clear that there exists $\bar{y} \in [0, +\infty)$ such that $y = m\bar{y}$. So,

$$\begin{aligned} f(tx + (1-t)y) &= f(tx + m(1-t)\bar{y}) \\ &\leq tg(x) + m(1-t)g(\bar{y}) \quad \text{(by assuming (3))} \\ &= tg(x) + (1-t)g(y) \text{ (by (2.3))}. \end{aligned}$$

In a similar way, we obtain $g(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$. Therefore, by applying Proposition 2.7, there exists an affine function $h\colon [0, +\infty) \to \mathbb{R}$ such that $f \leq h \leq g$ on $[0, +\infty)$. Thus, $h(x) = ax + b$ for some $a, b \in \mathbb{R}$. The fact $f(0) = g(0) = 0$ forces to $h(0) = 0$; and hence, $h(x) = ax$. □

3. Conclusion

We have presented some result of separation of two functions by means of an m-convex one, defined on m-convex subset of a real linear space X. Examples were given. More can be done in this direction, for example separation by means of strongly m-convex functions.

References

[1] K. Baron, J. Matkowski, and K. Nikodem, *A sandwich with convexity*, Math. Pannon. **5** (1994), no. 1, 139–144.

[2] M. Bracamonte, J. Giménez, and J. Medina, *Sandwich theorem for reciprocally strongly convex functions*, Rev. Colombiana Mat. **52** (2018), no. 2, 171–184.

[3] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Second edition, Birkhäuser Verlag, Basel, 2009.

[4] T. Lara, J. Matkowski, N. Merentes, R. Quintero, and M. Wróbel, *A generalization of m-convexity and a sandwich theorem*, Ann. Math. Sil. **31** (2017), 107–126.

[5] T. Lara, N. Merentes, Z. Páles, R. Quintero, and E. Rosales, *On m-convexity on real linear spaces*, UPI Journal of Mathematics and Biostatistics **1** (2018), no. 2, JMB8, 16 pp.

[6] T. Lara, N. Merentes, R. Quintero, and E. Rosales, *On m-concave functions on real linear spaces*, Bol. Asoc. Mat. Venez. **23** (2016), no. 2, 131–137.

[7] T. Lara, N. Merentes, R. Quintero, and E. Rosales, *On m-convexity of set-valued functions*, Adv. Oper. Theory **4** (2019), no. 4, 767–783.

[8] T. Lara, N. Merentes, E. Rosales, and A. Tineo, *Properties and characterizations of convex functions on time scales*, Ann. Math. Sil. **32** (2018), 237–245.
[9] T. Lara and E. Rosales, *Strongly convex functions on time scales*, UPI Journal of Mathematics and Biostatistics **1** (2018), no. 2, JMB9, 10 pp.
[10] T. Lara and E. Rosales, *Log m-convex functions*, Moroc. J. of Pure and Appl. Anal. (MJPAA) **5** (2019), no. 2, 117–124.
[11] T. Lara, E. Rosales, and J.L. Sánchez, *New properties of m-convex functions*, Int. J. Math. Anal. (Ruse) **9** (2015), no. 15, 735–742.
[12] J. Matkowski and M. Wróbel, *Sandwich theorem for m-convex functions*, J. Math. Anal. Appl. **451** (2017), no. 2, 924–930.
[13] N. Merentes and K. Nikodem, *Remarks on strongly convex functions*, Aequationes. Math. **80** (2010), no. 1-2, 193–199.
[14] C.P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics, 23, Springer, New York, 2006.
[15] K. Nikodem and S. Wąsowicz, *A sandwich theorem and Hyers-Ulam stability of affine functions*, Aequationes Math. **49** (1995), no. 1-2, 160–164.
[16] A. Olbryś, *On separation by h-convex functions*, Tatra Mt. Math. Publ. **62** (2015), 105–111.
[17] A.W. Roberts and D.E. Varberg, *Convex Functions*, Pure and Applied Mathematics, 57, Academic Press, New York, 1973.
[18] E. Sadowska, *A sandwich with convexity for set-valued functions*, Math. Pannon. **7** (1996), no. 1, 163–169.
[19] G. Toader, *Some generalizations of the convexity*, in: I. Muruşciac and W.W. Breckner (eds.), *Proceedings of the Colloquium on Approximation and Optimization*, Univ. Cluj-Napoca, Cluj-Napoca, 1985, pp. 329–338.
[20] G. Toader, *On a generalization of the convexity*, Mathematica (Cluj) **30(53)** (1988), no. 1, 83–87.
[21] F.A. Valentine, *Convex Sets*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York, 1964.

Departamento de Física y Matemáticas
Universidad de los Andes
Núcleo "Rafael Rangel"
Trujillo
Venezuela
e-mail: tlara@ula.ve
e-mail: edgarr@ula.ve