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SOME EXISTENCE RESULTS FOR SYSTEMS OF IMPULSIVE STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study a class of impulsive systems of stochastic differential equations with infinite Brownian motions. Sufficient conditions for the existence and uniqueness of solutions are established by mean of some fixed point theorems in vector Banach spaces. An example is provided to illustrate the theory.

1. Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis ([29]) and then followed by a period of active research which culminated with the monograph by Halanay and Wexler ([19]). Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine and

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biology. A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra *et al.* ([2]), Graef *et al.* ([15]), Laskshmikantham *et al.* ([23]), Samoilenko and Perestyuk ([39]).

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems; see for instance the monographs of Da Prato and Zabczyk ([10]), Gard ([13]), Gikhman and Skorokhod ([14]), Sobczyk ([40]) and Tsokos and Padgett ([41]). For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett ([41]) to a closed system with a simplified heat, one organ or capillary bed, and re-circulation of blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs of Bharucha-Reid ([4]), Tsokos and Padgett ([41]), Sobczyk ([40]) and Da Prato and Zabczyk ([10]).

By using classical fixed point theory, in [9, 12, 18, 21, 24, 25, 26, 38], the authors studied the existence and asymptotic stability and exponential stability for impulsive stochastic differential equations.

In [1], the authors studied the following system of impulsive random semilinear differential equations without Brownian motion,

$$\begin{cases} x'(t,\omega) = A_1(\omega)x(t,\omega) + f_1(t, x(t,\omega), y(t,\omega), \omega), \quad t \in J = [0,b], \\ y'(t,\omega) = A_2(\omega)y(t,\omega) + f_2(t, x(t,\omega), y(t,\omega), \omega), \quad t \in J = [0,b], \\ x(t_k^+,\omega) - x(t_k^-,\omega) = I_k(x(t_k^-,\omega), y(t_k^-,\omega)), \quad k = 1, 2, \dots, m, \\ y(t_k^+,\omega) - y(t_k^-,\omega) = \overline{I}_k(x(t_k^-,\omega), y(t_k^-,\omega)), \quad k = 1, 2, \dots, m, \\ x(\omega, 0) = \varphi_1(\omega), \quad \omega \in \Omega, \\ y(\omega, 0) = \varphi_2(\omega), \quad \omega \in \Omega, \end{cases}$$

where X is a Banach space and $A_i : \Omega \times X \to X, i = 1, 2$ are random operators. They obtained the existence and uniqueness of solutions using fixed point theory in vector Banach spaces.

Recently in [6], the authors used the idea of fixed point theory in generalized Banach spaces to prove the existence of mild solutions of impulsive coupled systems of stochastic differential equations with fractional Brownian motion.

In this paper, we are interested in the questions of existence and uniqueness of solutions of the following system of problems:

$$
(1.1)\qquad \begin{cases} dx(t) = \displaystyle\sum_{l=1}^{\infty} f_l^1(t, x(t), y(t))dW^l(t) + g^1(t, x(t), y(t))dt, \ t \in J, t \neq t_k, \\ dy(t) = \displaystyle\sum_{l=1}^{\infty} f_l^2(t, x(t), y(t))dW^l(t) + g^2(t, x(t), y(t))dt, \ t \in J, t \neq t_k, \\ \Delta x(t) = I_k(x(t_k)), \quad \Delta y(t) = \overline{I}_k(y(t_k)), \quad t = t_k, \quad k = 1, 2, \dots, m, \\ x(0) = x_0 \in \mathbb{R}, \qquad y(0) = y_0 \in \mathbb{R}, \end{cases}
$$

where $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $J := [0, T]$, $f^1, f^2, g^1, g^2 \colon J \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions, W^l is an infinite sequence of independent standard Brownian motions $(l = 1, 2, \ldots)$, $I_k, \overline{I}_k \in C(\mathbb{R}, \mathbb{R})\,(k = 1, \ldots, m)$, and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim\limits_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h \to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively. Set

$$
\begin{cases} f^i(\cdot, x, y) = (f_1^i(\cdot, x, y), f_2^i(\cdot, x, y), \ldots), \\ \|f^i(\cdot, x, y)\| = \Big(\displaystyle\sum_{l=1}^{\infty} (f_l^i)^2(\cdot, x, y) \Big)^{\frac{1}{2}}, \end{cases}
$$

where $i = 1, 2$, $f^i(\cdot, x, y) \in l^2$ for all $x \in \mathbb{R}$.

In recent years, in the absence of random effect and stochastic analysis many authors studied the existence of solutions for systems of differential and difference equations with and without impulses by using the vector version of the fixed point theorem (see [5, 3, 20, 17, 22, 31, 32, 35, 30], the monograph of Graef *et al.* [15], and the references therein).

This paper is organized as follows. In Sections 2, 3, we introduce all the background material used in this paper such as stochastic calculus and some properties of generalized Banach spaces. In Section 4, we state some results for fixed point theorems in generalized Banach spaces. Finally, an application of Schaefer's and Perov fixed point theorems in generalized Banach spaces are used to prove the existence of solutions to problem (1.1).

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuity and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Assume $W(t)$ is an infinite sequence of independent standard Brownian motions, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that is, $W(t) = (W^1(t), W^2(t), \ldots)^T$. An $\mathbb{R}$-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to \mathbb{R}$ and the collection of random variables

$$S = \{x(t, \omega) : \Omega \to \mathbb{R} |\ t \in J\}$$

is called a stochastic process.

The following result is one of the elementary properties of square-integrable stochastic processes ([27]).

LEMMA 2.1 (Itô Isometry for Elementary Processes). *Let $(X_l)_{l\in\mathbb{N}}$ be a sequence of elementary processes. Assume that*

$$\int_0^T \mathbb{E}|X(s)|^2 ds < \infty, \quad |X| = \left(\sum_{l=1}^{\infty} X_l^2\right)^{\frac{1}{2}}.$$

Then

$$\mathbb{E}\left(\sum_{l=1}^{\infty} \int_0^T X_l(s) dW^l(s)\right)^2 = \mathbb{E}\left(\sum_{l=1}^{\infty} \int_0^T X_l^2(s) ds\right).$$

The next result is known as the Burkholder–Davis–Gundy inequalities. It was first proved for discrete martingales and $p > 0$ by Burkholder ([7]) in 1966. In 1968, Millar ([28]) extended the result to continuous martingales. In 1970, Davis ([11]) extended the result for discrete martingales to $p = 1$. The extension to $p > 0$ was obtained independently by Burkholder and Gundy ([8]) in 1970 and Novikov ([33]) in 1971.

THEOREM 2.1 ([36]). *For each $p > 0$ there exist constants $c_p, C_p \in (0, \infty)$, such that for any progressive process x with the property that for some $t \in [0, \infty)$, $\int_0^t X_s^2 ds < \infty$ a.s., we have*

$$c_p \mathbb{E}\left(\int_0^t X_s^2 ds\right)^{\frac{p}{2}} \leq \mathbb{E}\left(\sup_{s\in[0,t]} \int_0^t X_s^2 dW(s)\right)^p \leq C_p \mathbb{E}\left(\int_0^t X_s^2 ds\right)^{\frac{p}{2}}. \tag{2.1}$$

3. Generalized metric and Banach spaces

In this section we define vector metric spaces and generalized Banach spaces and prove some properties. If $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$. For $x \in \mathbb{R}^n$, $(x)_i = x_i$, $i = 1, \ldots, n$.

Definition 3.1. Let X be a nonempty set. By a *vector-valued metric* on X we mean a map $d\colon X \times X \to \mathbb{R}^n$ with he following properties:

(i) $d(u, v) \geq 0$ for all $u, v \in X$; $d(u, v) = 0$ if and only if $u = v$.
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$.
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Note that for any $i \in \{1, \ldots, n\}$ $(d(u, v))_i = d_i(u, v)$ is a metric space in X.

We call the pair (X, d) a generalized metric space. For $r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : \ d(x_0, x) < r\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r.

Definition 3.2. Let E be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a *vector-valued norm* on E we mean a map $\|\cdot\|\colon E \to \mathbb{R}^n_+$ with the following properties:

(i) $\|x\| \geq 0$ for all $x \in E$; if $\|x\| = 0$ then $x = 0$,
(ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in E$ and $\lambda \in \mathbb{K}$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

The pair $(E, \|\cdot\|)$ is called a *generalized normed space*. If the generalized metric generated by $\|\cdot\|$ (i.e. $d(x, y) = \|x - y\|$) is complete then the space $(E, \|\cdot\|)$ is called a *generalized Banach space*, where

$$\|x - y\| = \begin{pmatrix} \|x - y\|_1 \\ \vdots \\ \|x - y\|_n \end{pmatrix}.$$

Notice that $\|\cdot\|$ is a generalized Banach space on E if and only if $\|\cdot\|_i,\ i = 1,\dots,n$ are norms on E.

Remark 3.1. In generalized metric space in the sense of Perov's, the notations of convergence sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Definition 3.3. A square matrix of real numbers is said to be *convergent to zero* if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc.

Lemma 3.1 ([37]). *Let M be a square matrix of nonnegative numbers. The following assertions are equivalent:*

(i) *M is convergent towards zero;*

(ii) *the matrix $I - M$ is non-singular and*

$$(I-M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iii) *$\|\lambda\| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$;*

(iv) *$(I - M)$ is non-singular and $(I - M)^{-1}$ has nonnegative elements.*

In the next part, we present the versions of Banach, Schauder and Schaefer's fixed point theorems in generalized Banach spaces.

Theorem 3.1 ([34]). *Let (X, d) be a complete generalized metric space and let $N\colon X \longrightarrow X$ be such that*

$$d(N(x), N(y)) \leq Md(x, y)$$

for all $x, y \in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \longrightarrow 0$ as $k \longrightarrow \infty$, then N has a unique fixed point $x_* \in X$,

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(N(x_0), x_0)$$

for every $x_0 \in X$ and $k \geq 1$.

Theorem 3.2 ([16, 42]). *Let E be a generalized Banach space, $C \subset E$ be a nonempty closed convex subset of E and $N\colon C \to C$ be a continuous operator such that $N(C)$ is relatively compact. Then N has at least fixed point in C.*

As a consequence of Schauder fixed point theorem we present the version of Schaefer's fixed point theorem in generalized Banach space.

Theorem 3.3 ([16]). *Let $(E, \|\cdot\|)$ be a generalized Banach space and $N\colon E \to E$ be a continuous compact mapping. Moreover assume that the set*

$$\mathcal{A} = \{x \in E : x = \lambda N(x) \text{ for some } \lambda \in (0,1)\}$$

is bounded. Then N has a fixed point.

4. Existence and uniqueness results

Let $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. In order to define a solution for Problem (1.1), consider the space of piece-wise continuous functions

$$PC = \{x : \ \Omega \times J \longrightarrow \mathbb{R}, \ x \in C(J_k, \mathbb{R}), \ k = 1, \ldots, m \text{ such that}$$
$$x(t_k^+, \cdot) \text{ and } x(t_k^-, \cdot) \text{ exist with } x(t_k^-, \cdot) = x(t_k, \cdot)\}$$

endowed with the norm

$$\|x\|_{PC}^2 = \sup_{t \in J} \mathbb{E}|x(t, \cdot)|^2.$$

PC is a Banach space with norm $\|\cdot\|_{PC}$.

Definition 4.1. An $\mathbb{R}$-valued stochastic process $u = (x, y) \in PC \times PC$ is said to be a *solution of* (1.1) with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if:
1) $u(t)$ is $\mathcal{F}_t$-adapted for all $t \in J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$;
2) $u(t)$ is right continuous and has limit on the left;
3) $u(t)$ satisfies that

$$\begin{cases} x(t) = x_0 + \displaystyle\sum_{l=1}^{\infty} \int_0^t f_l^1(s, x(s), y(s)) dW^l(s) \\ \qquad + \displaystyle\int_0^t g^1(s, x(s), y(s)) ds + \sum_{0 \le t_k \le t} I_k(x(t_k)), \quad \in J, \\ y(t) = y_0 + \displaystyle\sum_{l=1}^{\infty} \int_0^t f_l^2(s, x(s), y(s)) dW^l(s) \\ \qquad + \displaystyle\int_0^t g^2(s, x(s), y(s)) ds + \sum_{0 \le t_k \le t} \overline{I}_k(y(t_k)), \quad t \in J. \end{cases}$$

Let us introduce the following hypothesis:

(H_1) There exist nonnegative numbers a_i and b_i, $i = 1, 2$ such that for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}, t \in J$ we have

$$\mathbb{E}(|f^i(t,x,y) - f^i(t,\overline{x},\overline{y})|^2) \leq a_i\mathbb{E}(|x-\overline{x}|^2) + b_i\mathbb{E}(|y-\overline{y}|^2).$$

(H_2) There exist positive constants α_i and β_i, $i = 1, 2$ such that for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}, t \in J$ we have

$$\mathbb{E}(|g^i(t,x,y) - g^i(t,\overline{x},\overline{y})|^2) \leq \alpha_i\mathbb{E}(|x-\overline{x}|)^2 + \beta_i\mathbb{E}(|y-\overline{y}|^2).$$

(H_3) There exist constants $d_k \geq 0$ and $\overline{d}_k \geq 0$, $k = 1, \dots, m$ such that for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}$

$$\mathbb{E}(|I_k(x) - I_k(\overline{x})|^2) \leq d_k\mathbb{E}(|x-\overline{x}|)^2, \quad \mathbb{E}(|\overline{I}_k(y) - \overline{I}_k(\overline{y})|^2) \leq \overline{d}_k\mathbb{E}(|y-\overline{y}|^2).$$

Our first main result in this section is based on Perov's fixed point theorem.

THEOREM 4.1. *Assume that* (H_1)–(H_3) *are satisfied and the matrix* M *is given by*

$$M = \sqrt{3}\begin{pmatrix} \sqrt{C_2a_1 + \alpha_1 T + l_1} & \sqrt{C_2b_1 + \beta_1 T} \\ \sqrt{C_2a_2 + \alpha_2 T} & \sqrt{C_2b_2 + \beta_2 T + l_2} \end{pmatrix},$$

$$l_1 = \sum_{k=1}^{m} d_k, \quad l_2 = \sum_{k=1}^{m} \overline{d}_k,$$

where $C_2 \geq 0$ *is defined in Theorem* 2.1. *If* M *converges to zero, then the problem* (1.1) *has unique solution.*

PROOF. Let $X = PC \times PC$. Consider the operator $N \colon X \to X$ defined by

$$N(x,y) = (N_1(x,y), N_2(x,y)),\ (x,y) \in PC \times PC$$

where

$$\begin{aligned} N_1(x(t),y(t)) = x_0 &+ \sum_{l=1}^{\infty}\int_0^t f_l^1(s,x(s),y(s))dW^l(s) \\ &+ \int_0^t g^1(s,x(s),y(s))ds + \sum_{0<t_k\leq t} I_k(x(t_k)), \end{aligned}$$

and

$$N_2(x(t),y(t)) = y_0 + \sum_{l=1}^{\infty}\int_0^t f_l^2(s,x(s),y(s))dW^l(s) + \int_0^t g^2(s,x(s),y(s))ds + \sum_{0<t_k\le t}\overline{I}_k(y(t_k)).$$

Fixed points of operator N are solutions of problem (1.1).

We shall use Theorem 3.1 to prove that N has a fixed point. Indeed, let $(x,y),(\overline{x},\overline{y})\in X$. Then we have for each $t\in J$

$$\begin{aligned}|N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 &\le 3\left|\sum_{l=1}^{\infty}\int_0^t (f_l^1(s,x(s),y(s)) - f_l^1(s,\overline{x}(s),\overline{y}(s)))dW^l(s)\right|^2\\ &+ 3\left|\int_0^t (g^1(s,x(s),y(s) - g^1(s,\overline{x}(s),\overline{y}(s))ds\right|^2\\ &+ 3\sum_{k=1}^{m}|I_k(x(t_k)) - I_k(\overline{x}(t_k))|^2.\end{aligned}$$

By Theorem 2.1, we get

$$\begin{aligned}\mathbb{E}|N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 &\le 3C_2\int_0^t \mathbb{E}|f^1(s,x(s),y(s)) - f^1(s,\overline{x}(s),\overline{y}(s))|^2ds\\ &+ 3t\int_0^t \mathbb{E}|g^1(s,x(s),y(s)) - g^1(s,\overline{x}(s),\overline{y}(s))|^2ds\\ &+ 3\sum_{k=1}^{m}\mathbb{E}|I_k(x(t_k)) - I_k(\overline{x}(t_k))|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{t\in J}\mathbb{E}|N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 &\le 3(C_2a_1 + \alpha_1T + l_1)\|x-\overline{x}\|_{PC}^2\\ &+ 3(C_2b_1 + \beta_1T)\|y-\overline{y}\|_{PC}^2.\end{aligned}$$

Similarly we have

$$\|N_2(x,y)-N_2(\overline{x},\overline{y})\|_{PC}^2 \le 3(C_2a_2+\alpha_2T)\|x-\overline{x}\|_{PC}^2 + (C_2b_2+\beta_2T+l_2)\|y-\overline{y}\|_{PC}^2.$$

Hence

$$\|N(x,y)-N(\overline{x},\overline{y})\|_X = \begin{pmatrix} \|N_1((x,y)-N_1(\overline{x},\overline{y})\|_{PC} \\ \|N_2(x,y)-N_2(\overline{x},\overline{y})\|_{PC} \end{pmatrix}$$

$$\le \sqrt{3}\begin{pmatrix} \sqrt{C_2a_1+\alpha_1T+l_1} & \sqrt{C_2b_1+\beta_1T} \\ \sqrt{C_2a_2+\alpha_2T} & \sqrt{C_2b_2+\beta_2T+l_2} \end{pmatrix}\begin{pmatrix} \|x-\overline{x}\|_{PC} \\ \|y-\overline{y}\|_{PC} \end{pmatrix}.$$

Therefore

$$\|N(x,y)-N(\overline{x},\overline{y})\|_X \le M\begin{pmatrix} \|x-\overline{x}\|_{PC} \\ \|y-\overline{y}\|_{PC} \end{pmatrix}, \quad \text{for all } (x,y),(\overline{x},\overline{y}) \in X.$$

From Perov's fixed point theorem, the mapping N has a unique fixed $(x,y)\in PC\times PC$ which is unique solution of problem (1.1). □

We present now the existence result under nonlinearities f^i and g^i, $i=1,2$ satisfying a Nagumo type growth conditions:

(H_4) There exist a function $p_i \in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i:[0,\infty)\to[0,\infty)$ for each $i=1,2$ such that for all $x,y\in\mathbb{R}$

$$\mathbb{E}(\|f^1(t,x,y)\|^2) \le p_1(t)\psi_1(\mathbb{E}(|x|^2+|y|^2)),$$

$$\mathbb{E}(\|f^2(t,x,y)\|)^2 \le p_2(t)\psi_2(\mathbb{E}(|x|^2+|y|^2)),$$

with

$$\int_0^T m_1(s)ds < \int_{v_1}^{\infty}\frac{ds}{\psi_1(s)+\psi_2(s)}$$

where $m_1(t)=\max\{4C_2p_1(t),\ 4Tp_2(t)\}$, $v_1 = 4\mathbb{E}|x_0|^2+4\sum_{k=1}^m c_k$.

(H_5) There exist a function $p_i \in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i:[0,\infty)\to[0,\infty)$ for each $i=3,4$ such that for all $x,y,\in\mathbb{R}$ we have

$$E(|g^i(t,x,y)|^2) \le p_i(t)\psi_3(E(|x|^2+|y|^2)),$$

with

$$\int_0^T m_2(s)ds < \int_{v_2}^{\infty}\frac{ds}{\psi_3(s)+\psi_4(s)}$$

where

$$m_2(t) = \max\{4C_2 p_3(t), 4Tp_4(t)\}, \quad v_2 = 4\mathbb{E}|y_0|^2 + \sum_{k=1}^{m} \widetilde{c}_k.$$

(H_6) There exist positive constants $c_k, \widetilde{c}_k, k = 1, \ldots, m$, such that

$$\mathbb{E}(|I_k(x)|)^2 \le c_k, \quad \mathbb{E}(|\overline{I}_k(y)|)^2 \le \widetilde{c}_k \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

THEOREM 4.2. *Assume that* (H_4)–(H_6) *hold. Then* (1.1) *has at least one solution on* J.

PROOF. Clearly, the fixed points of N are solutions to (1.1), where N is defined in Theorem 4.1. In order to apply Theorem 3.3, we first show that N is completely continuous. The proof will be given in several steps.

STEP 1. $N = (N_1, N_2)$ is continuous.

Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x, y) \in PC \times PC$ as $n \to \infty$. Then

$$\begin{aligned}
&|N_1(x_n(t), y_n(t)) - N_1(x(t), y(t))|^2 \\
&\qquad \le 3\left|\sum_{l=1}^{\infty}\int_0^t (f_l^1(s, x_n(s), y_n(s)) - f_l^1(s, x(s), y(s)))dW^l(s)\right|^2 \\
&\qquad\quad + 3\left|\int_0^t (g^1(s, x_n(s), y_n(s)) - g^1(s, x(s), y(s)))ds\right|^2 \\
&\qquad\quad + 3\sum_{k=1}^{m} \left|I_k(x_n(t_k)) - I_k(x(t_k))\right|^2.
\end{aligned}$$

From Theorem 2.1, we obtain

$$\begin{aligned}
&\mathbb{E}|N_1(x_n(t), y_n(t)) - N_1(x(t), y(t))|^2 \\
&\qquad \le 3C_2 \int_0^t \mathbb{E}|f^1(s, x_n(s), y_n(s)) - f^1(s, x(s), y(s))|^2 ds \\
&\qquad\quad + 3t\int_0^t \mathbb{E}|g^1(s, x_n(s), y_n(s)) - g^1(s, x(s), y(s))|^2 ds \\
&\qquad\quad + 3\sum_{k=1}^{m} \mathbb{E}|I_k(x_n(t_k)) - I_k(x(t_k))|^2.
\end{aligned}$$

Since f^1, g^1 are Carathéodory functions and $I_k, \overline{I}_k$ are continuous functions, by Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\sup_{t\in J}\mathbb{E}|N_1(x_n(t),y_n(t))-N_1(x(t),y(t))|^2 &\le 3C_2\mathbb{E}\|f^1(\cdot,x_n,y_n)-f^1(\cdot,x,y)\|_{L^2}^2\\ &+3T\mathbb{E}\|g^1(\cdot,x_n,y_n)-g^1(\cdot,x,y)\|_{L^2}^2\\ &+3\sum_{k=1}^m\mathbb{E}|I_k(x_n(t_k))-I_k(x(t_k))|^2\to 0 \text{ as } n\to\infty.\end{aligned}$$

Similarly

$$\begin{aligned}\sup_{t\in J}\mathbb{E}|N_2(x_n(t),y_n(t))-N_1(x(t),y(t))|^2 &\le 3C_2\mathbb{E}\|f^2(\cdot,x_n,y_n)-f^2(\cdot,x,y)\|_{L^2}^2\\ &+3T\mathbb{E}\|g^2(\cdot,x_n,y_n)-g^2(\cdot,x,y)\|_{L^2}\\ &+3\sum_{k=1}^m\mathbb{E}|\overline{I}_k(y_n(t_k))-\overline{I}_k(y(t_k))|^2\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus N is continuous.

STEP 2. N maps bounded sets into bounded sets in $PC\times PC$.

Indeed, it is enough to show that for any $q>0$ there exists a positive constant l such that for each $(x,y)\in B_q=\{(x,y)\in PC\times PC : \|x\|_{PC}\le q,\ \|y\|\le q\}$, we have

$$\|N(x,y)\|_{PC}\le l=(l_1,l_2).$$

For each $t\in J$, we get

$$\begin{aligned}|N_1(x(t),y(t))|^2 &\le 4|x_0|^2+4|\sum_{l=1}^{\infty}\int_0^t f_l^1(s,x(s),y(s))dW^l(s)|^2\\ &+4|\int_0^t g^1(s,x(s),y(s))ds|^2+4|\sum_{k=1}^m I_k(x(t_k))|^2.\end{aligned}$$

Using the inequality (2.1), we also get

$$\begin{aligned}\mathbb{E}|N_1(x(t),y(t))|^2 &\le 4\mathbb{E}|x_0|^2+4C_2\|p_1\|_{L^1}\psi_1(2q)\\ &+4T\|p_3\|_{L^1}\psi_2(2q)ds+4\sum_{k=1}^m c_k.\end{aligned}$$

Therefore

$$\|N_1(x,y)\|_{PC} \leq 4\mathbb{E}|x_0|^2 + 4C_2\|p_1\|_{L^1}\psi_1(2q) + 4\|p_2\|_{L^1}\psi_2(2q)ds + 4\sum_{k=1}^{m} c_k := l_1.$$

Similarly, we have

$$\|N_2(x,y)\|_{PC} \leq 4\mathbb{E}|x_0|^2 + 4C_2\|p_3\|_{L^1}\psi_2(q) + 4\|p_4\|_{L^1}\psi_4(q)ds + 4\sum_{k=1}^{m} \widetilde{c}_k := l_2.$$

STEP 3. N maps bounded sets into equicontinuous sets of $PC \times PC$.

Let B_q be a bounded set in $PC \times PC$ as in Step 2. Let $r_1, r_2 \in J, r_1 < r_2$ and $u \in B_q$. Then we have

$$|N_1(x(r_2), y(r_2)) - N_1(x(r_1), y(r_1))|^2 \leq 3\left|\sum_{l=1}^{\infty}\int_{r_1}^{r_2} f_l^1(s, x(s), y(s))dW^l(s)\right|^2 + 3\left|\int_{r_1}^{r_2} g^1(s, x(s), y(s))ds\right|^2 + 3\sum_{r_1 \leq t_k \leq r_2} |I_k(x(t_k))|^2.$$

Hence

$$\mathbb{E}|N_1(x(r_2), y(r_2)) - N_1(x(r_1), y(r_1))|^2 \leq 3C_2\psi_1(q)\int_{r_1}^{r_2} p_1(s)ds + T\psi_2(q)\int_{r_1}^{r_2} p_2(s)ds + 3\sum_{r_1 \leq t_k \leq t_2} c_k.$$

The right-hand term tends to zero as $|r_2 - r_1| \to 0$. As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli, we conclude that N maps B_q into a precompact set in $PC \times PC$.

STEP 4. It remains to show that

$$\mathcal{A} = \{(x,y) \in PC \times PC : \ (x,y) = \lambda N(x,y), \lambda \in (0,1)\}$$

is bounded.

Let $(x, y) \in \mathcal{A}$. Then $x = \lambda N_1(x, y)$ and $y = \lambda N_2(x, y)$ for some $0 < \lambda < 1$. Thus, for $t \in J$, we have

$$\mathbb{E}|x(t)|^2 \leq 4\mathbb{E}|x_0|^2 + 4C_2 \int_0^t p_1(s)\psi_1(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds$$

$$+ 4T \int_0^t p_2(s)\psi_2(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds + 4\sum_{k=1}^{m} c_k.$$

Hence

$$\mathbb{E}|x(t)|^2 \leq 4\mathbb{E}|x_0|^2 + 4C_2 \int_0^t p_1(s)\psi_1(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds$$

$$+ 4T \int_0^t p_2(s)\psi_2(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds + 4\sum_{k=1}^{m} c_k$$

and

$$\mathbb{E}|y(t)|^2 \leq 4\mathbb{E}|x_0|^2 + 4C_2 \int_0^t p_3(s)\psi_2(\mathbb{E}|x(s)|^2 + E|y(s)|^2)ds$$

$$+ 4T \int_0^t p_4(s)\psi_3(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds + 4\sum_{k=1}^{m} \widetilde{c}_k.$$

Therefore

$$\mathbb{E}|x(t)|^2 + \mathbb{E}|y(t)|^2 \leq \gamma + \int_0^t p(s)\phi(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds,$$

where

$$\gamma = 8\mathbb{E}|x_0|^2 + 4\sum_{k=1}^{m}(c_k + \widetilde{c}_k), \quad p(t) = m_1(t) + m(t), \quad \text{and} \quad \phi(t) = \sum_{i=1}^{m} \psi_i(t).$$

By the Gronwall inequality, we have

$$\mathbb{E}|x(t)|^2 + \mathbb{E}|y(t)|^2 \leq \Gamma^{-1}\left(\int_\gamma^T p(s)ds\right) := K, \quad \text{for each } t \in J,$$

where

$$\Gamma(z) = \int_\gamma^z \frac{du}{\phi(u)}.$$

Consequently

$$\|x\|_{PC} \leq K \quad \text{and} \quad \|y\|_{PC} \leq K.$$

This shows that $\mathcal{A}$ is bounded. As a consequence of Theorem 3.3 we deduce that N has a fixed point (x,y) which is a solution to the problem (1.1). $\square$

The goal of the second result of this section is to apply Schauder's fixed point. For the study of this problem we first introduce the following hypotheses:

(H_7) There exist nonnegative numbers $\overline{a}_i$ and $\overline{b}_i, c_i,\ i = 1,2$ such that for all $x, y \in \mathbb{R}$, we have

$$\mathbb{E}(|f_i(t,x,y)|^2) \leq \overline{a}_i \mathbb{E}(|x|)^2 + \overline{b}_i \mathbb{E}(|y|)^2 + c_1.$$

(H_8) There exist positive constants $\overline{\alpha}_i$ and $\overline{\beta}_i, \lambda_i,\ i = 1,2$ such that for all $x, y \in \mathbb{R}$, we have

$$\mathbb{E}(|g_i(t,x,y)|^2) \leq \overline{\alpha}_i \mathbb{E}(|x|)^2 + \overline{\beta}_i \mathbb{E}(|y|)^2 + \lambda_1.$$

(H_9) There exist constants $d \geq 0, \overline{d} \geq 0$ and $e_i \geq 0,\ i = 1,2$ and $k = 1, \ldots, m$ such that

$$\sum_{k=1}^{m} \mathbb{E}|I_k(x)|^2 \leq dE|x|^2 + e_1,$$

$$\sum_{k=1}^{m} \mathbb{E}|\overline{I}_k(x)|^2 \leq \overline{d}\mathbb{E}|x|^2 + e_2, \quad \text{for all } x \in \mathbb{R}.$$

Theorem 4.3. *Assume* (H_7)–(H_9) *hold and*

$$M_{a,b} = \sqrt{2} \begin{pmatrix} \sqrt{C_2\overline{a}_1 + \overline{\alpha}_1 T + d} & \sqrt{C_2\overline{b}_1 + \overline{\beta}_1 T} \\ \sqrt{C_2\overline{a}_2 + \overline{\alpha}_2 T} & \sqrt{C_2\overline{b}_2 + \overline{\beta}_2 T + \overline{d}} \end{pmatrix}$$

converges to zero. Then problem (1.1) *has at least one solution.*

Proof. Let $X = PC \times PC$. Consider the operator $N = (N_1, N_2)\colon PC \times \times PC \longrightarrow PC \times \times PC$ defined for $x, y \in PC$ by

$$N_1(x(t), y(t)) = x_0 + \sum_{l=1}^{\infty} \int_0^t f_l^1(s, x(s), y(s)) dW^l(s) + \int_0^t g^1(s, x(s), y(s)) ds + \sum_{0<t_k<t} I_k(x(t_k))$$

and

$$N_2(x(t), y(t)) = y_0 + \sum_{l=1}^{\infty} \int_0^t f_l^2(s, x(s), y(s)) dW^l(s) + \int_0^t g^2(s, x(s), y(s)) ds + \sum_{0<t_k<t} \overline{I}_k(y(t_k)).$$

Set

$$D = \{(x, y) \in PC \times PC : \|x\|_{PC} \leq R_1, \|y\|_{PC} \leq R_2\}.$$

Obviously, the set D is a bounded closed convex set in space $PC \times PC$.

It is clear that

$$|N_1(x(t), y(t))|^2 \leq 4|x_0|^2 + 4|\sum_{l=1}^{\infty} \int_0^t f_l^1(s, x(s), y(s)) dW^l(s)|^2 + 4|\int_0^t g^1(s, x(s), y(s)) ds|^2 + |4 \sum_{k=1}^{m} I_k(x(t_k))|^2.$$

From the inequality (2.1), we get

$$\begin{aligned}\mathbb{E}|N_1(x(t), y(t))|^2 \leq{}& 4\mathbb{E}|x_0|^2 + 4C_2\overline{a}_1 \int_0^t \mathbb{E}|x(s)|^2 ds \\ &+ 4\overline{b}_1 C_2 \int_0^t \mathbb{E}|y(s)|^2 ds + 4c_1 T + 4T\overline{\alpha}_1 \int_0^t \mathbb{E}|x(s)|^2 d(s) \\ &+ 4\overline{\beta}_1 T \int_0^t \mathbb{E}|y(s)|^2 ds + 4\lambda_1 T + 4d\mathbb{E}|x|^2 + 4e_1,\end{aligned}$$

thus

$$\begin{aligned}\sup_{t\in J} \mathbb{E}|N_1(x(t), y(t))|^2 \leq{}& 4(C_2\overline{a}_1 + \overline{\alpha}_1 T + d)\|x\|_{PC} + 4(C_2\overline{b}_1 + \overline{\beta}_1 T)\|y\|_{PC} \\ &+ 4\mathbb{E}|x_0|^2 + 4e_1 + 4Tc_1 + 4T\lambda_1. \end{aligned} \tag{4.1}$$

From (4.1) we obtain that

$$\|N_1(x,y))\|_{PC} \le \widetilde{a}_1\|x\|_{PC} + \widetilde{b}_1\|y\|_{PC} + \widetilde{c}_1, \tag{4.2}$$

where

$$\widetilde{a}_1 = 2\sqrt{C_2\overline{a}_1 + \overline{\alpha}_1 T + d}, \quad \widetilde{b}_1 = 2\sqrt{C_2\overline{b}_1 + \overline{\beta}_1 T},$$

$$\widetilde{c}_1 = 2\sqrt{\mathbb{E}|x_0|^2 + e_1 + Tc_1 + T\lambda_1}.$$

Similarly we have

$$\|N_2(x,y)\|_{PC} \le \widetilde{a}_2\|x\|_{PC} + \widetilde{b}_2\|y\|_{PC} + \widetilde{c}_2, \tag{4.3}$$

where

$$\widetilde{a}_2 = 2\sqrt{C_2\overline{a}_2 + 4\overline{\alpha}_2 T}, \quad \widetilde{b}_2 = 2\sqrt{C_2\overline{b}_2 + \overline{\beta}_2 T + \overline{d}},$$

$$\text{and} \quad \widetilde{c}_2 = 2\sqrt{\mathbb{E}|y_0|^2 + 4e_2 + Tc_2 + T\lambda_2}.$$

Now (4.2), (4.3) can be put together as

$$\begin{aligned}\|N(x,y)\|_X &= \begin{pmatrix}\|N_1(x,y)\|_{PC}\\ \|N_2(x,y)\|_{PC}\end{pmatrix}\\ &\le 2\begin{pmatrix}\sqrt{C_2\overline{a}_1 + \overline{\alpha}_1 T + d} & \sqrt{C_2\overline{b}_1 + \overline{\beta}_1 T}\\ \sqrt{C_2\overline{a}_2 + \overline{\alpha}_2 T} & \sqrt{C_2\overline{b}_2 + \overline{\beta}_2 T + \overline{d}}\end{pmatrix}\begin{pmatrix}\|x\|_{PC}\\ \|y\|_{PC}\end{pmatrix} + \begin{pmatrix}\widetilde{c}_1\\ \widetilde{c}_2\end{pmatrix}.\end{aligned}$$

Therefore

$$\|N(x,y)\|_X \le M_{a,b}\begin{pmatrix}\|x\|_{PC}\\ \|y\|_{PC}\end{pmatrix} + \begin{pmatrix}\widetilde{c}_1\\ \widetilde{c}_2\end{pmatrix}.$$

Since $M_{a,b} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, $N(x,y)$ converges to zero. Next, we look for two positive numbers R_1, R_2 such that if $\|x\|_{PC} \le R_1$, $\|y\|_{PC} \le R_2$, then $\|N_1(x,y)\|_{PC} \le R_1$, $\|N_2(x,y)\|_{PC} \le R_1$. To this end it is sufficient that

$$\begin{pmatrix}R_1\\ R_2\end{pmatrix} \le M_{a,b}\begin{pmatrix}R_1\\ R_2\end{pmatrix} + \begin{pmatrix}\widetilde{c}_1\\ \widetilde{c}_2\end{pmatrix}$$

whence

$$(I - M_{a,b})\begin{pmatrix}R_1\\ R_2\end{pmatrix} \le \begin{pmatrix}\widetilde{c}_1\\ \widetilde{c}_2\end{pmatrix}$$

that is

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \leq (I - M_{a,b})^{-1} \begin{pmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \end{pmatrix}.$$

Thus, it is clear that there exist $R_1, R_2 > 0$ such that

$$N(D) \subseteq D,$$

where

$$D = \{(x, y) \in PC \times PC : \|x\|_{PC} \leq R_1, \|y\|_{PC} \leq R_2\}.$$

Hence, by Theorem 3.1, the operator N has at least one fixed point which is solution of (1.1). $\square$

5. An example

In this section we consider the following example of stochastic differential equation:

$$(5.1) \qquad \begin{cases} dx(t) = \displaystyle\sum_{l=1}^{\infty} (a_{2l+1} \sin k^2 x + a_{2l} \cos l^2 y) dW^l(t) \\ \qquad\qquad + d_1(t + x(t) + y(t))dt, \quad t \in [0,1], t \neq \dfrac{1}{2} \\ dy(t) = \displaystyle\sum_{l=1}^{\infty} (b_{2l+1} \sin k^2 x + b_{2l} \cos l^2 y) dW^l(t) \\ \qquad\qquad + d_2(t + x(t) + y(t))dt, \quad t \in [0,1], t \neq \dfrac{1}{2} \\ \Delta x(t) = c_1 \dfrac{x(t)}{1 + |x(t)|}, \quad \Delta y(t) = c_1 \dfrac{y(t)}{1 + |y(t)|}, \quad t = \dfrac{1}{2} \\ x(0) = x_0, \quad y(0) = y_0, \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$, $(a_l)_{l\in\mathbb{N}}, (b_l)_{l\in\mathbb{N}} \in l^2$, $f_1, f_2 \colon [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are defined by

$$f_1(t,x,y) = \sum_{k=1}^{\infty}(a_{2k+1}\sin k^2 x + a_{2k}\cos k^2 y),$$

$$f_2(t,x,y) = \sum_{k=1}^{\infty}(b_{2k+1}\sin k^2 x + b_{2k}\cos k^2 y).$$

We deduce that

$$\|f_1(t,x,y)\|^2 \le 4\sum_{k=1}^{\infty} a_k^2 < \infty, \quad \|f_2(t,x,y)\|^2 \le 4\sum_{k=1}^{\infty} b_k^2 < \infty.$$

Hence

$$\mathbb{E}|f_1(t,x,y)|^2 \le 4\sum_{k=1}^{\infty} a_k^2 + \mathbb{E}(|x|^2+|y|^2),$$

$$\mathbb{E}|f_2(t,x,y)|^2 \le 4\sum_{k=1}^{\infty} b_k^2 + \mathbb{E}(|x|^2+|y|^2) \quad \text{for all } x,y\in\mathbb{R}.$$

Also we have

$$I_1(x) = c_1\frac{x(t)}{1+|x(t)|}, \quad I_2(y) = c_2\frac{y}{1+|y|} \;\Rightarrow\; \mathbb{E}|I_1(x)|^2 \le c_1, \quad \mathbb{E}|I_2(x)|^2 \le c_2,$$

and

$$g^1(t,x,y) = d_1(t+x+y), \quad g^2(t,x,y) = d_2(t+x+y), \quad x,y\in\mathbb{R},\ t\in[0,1].$$

Hence

$$\mathbb{E}|g^1(t,x,y)|^2 \le 3d_1^2(1+\mathbb{E}|x|^2+\mathbb{E}|y|^2), \quad \mathbb{E}|g^2(t,x,y)|^2 \le 3d_2^2(1+\mathbb{E}|x|^2+\mathbb{E}|y|^2).$$

Thus all the conditions of Theorem 4.2 hold, and then Problem (5.1) has at least one solution.

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