sciendo

Annales Mathematicae Silesianae **35** (2021), no. 2, 289–301
DOI: 10.2478/amsil-2021-0003

SOME FIXED POINT THEOREMS VIA COMBINATION OF WEAK CONTRACTION AND CARISTI CONTRACTIVE MAPPING

Kushal Roy, Sayantan Panja, Mantu Saha, Zoran D. Mitrović

Abstract. In this paper we introduce some new types of contractive mappings by combining Caristi contraction, Ćirić-quasi contraction and weak contraction in the framework of a metric space. We prove some fixed point theorems for such type of mappings over complete metric spaces with the help of φ-diminishing property. Some examples are given in strengthening the hypothesis of our established theorems.

1. Introduction and preliminaries

In recent years, there appeared a considerable interest in the fixed point theory. Currently fixed point theory has various applications in different branches of mathematics such as boundary value problems, nonlinear differential and integral equations, nonlinear matrix equations, homotopy theory etc. The main purpose of fixed point theory is to deal with several mappings either of contractive type or non-expansive type in nature over various generalized metric spaces beyond our usual metric spaces and to investigate the existence of their fixed points therein.

Received: 14.06.2020. Accepted: 11.03.2021. Published online: 13.04.2021.

(2020) Mathematics Subject Classification: 47H10, 54H25.

Key words and phrases: fixed point, metric space, orbital Banach–Caristi contractive map, weak Banach–Caristi contractive map, weak Ćirić–Caristi contractive map.

First and second authors acknowledge financial support awarded by the Council of Scientific and Industrial Research, New Delhi, India, through research fellowship for carrying out research work leading to the preparation of this manuscript.

In the year 1922, S. Banach ([1]) introduced the most celebrated theorem in fixed point theory known as *Banach Contraction Principle.*

Theorem 1.1 ([1]). *Let (M, d) be a complete metric space. If a map $T\colon M \to M$ satisfies*

$$d(T\xi, T\eta) \leq ad(\xi, \eta) \quad \text{for all } \xi, \eta \in M \text{ and for some } a \in [0, 1), \tag{1.1}$$

then T has a unique fixed point in M.

In 1976, Caristi ([3]) had established another famous fixed point theorem using a suitably defined lower semi-continuous function.

Theorem 1.2 ([3, 9]). *If (M, d) is a complete metric space and $\varphi\colon M \to [0, \infty)$ is a lower semi-continuous function, then a mapping $T\colon M \to M$ satisfying*

$$d(\xi, T\xi) \leq \varphi(\xi) - \varphi(T\xi) \quad \text{for each } \xi \in M, \tag{1.2}$$

has a fixed point in M.

Subsequently Ćirić ([4, 5]) had introduced a new contractive condition and proved a prominent fixed point theorem for a quasi contraction mapping. This mapping might be discontinuous in nature.

Theorem 1.3 ([4, 5]). *Let (M, d) be a complete metric space and T be a self mapping on M. If T satisfies*

$$d(T\xi, T\eta) \leq kR(\xi, \eta) \quad \text{for all } \xi, \eta \in M \text{ and for some } k \in [0, 1) \tag{1.3}$$

where $R(\xi, \eta) = \max\{d(\xi, \eta), d(\xi, T\xi), d(\eta, T\eta), d(\xi, T\eta), d(T\xi, \eta)\}$, then T has a unique fixed point in M.

In the year 2004, V. Berinde ([2]) had introduced a generalized contractive mapping, namely weak contraction mapping, which is so strong that it generalizes not only just Banach contraction maps but also Kannan maps, Chatterjea maps, Zamfirescu maps and also partially Ćirić quasi contraction maps and proved a fixed point theorem on it, which is given below.

Theorem 1.4 ([2, 10]). *Let (M, d) be a complete metric space. If a mapping $T\colon M \to M$ satisfies*

$$d(T\xi, T\eta) \leq \delta d(\xi, \eta) + Ld(\eta, T\xi) \quad \text{for all } \xi, \eta \in M,$$

for some $\delta \in [0, 1)$ and for some $L \geq 0$ then T has at least one fixed point in M.

Though the approaches of the renowned results of Caristi ([3]) and Banach ([1]) are quite different and the corresponding proofs vary, Karapınar et al. ([7]) have proposed a new contractive type condition which is the combination of the two contractive conditions (1.1) and (1.2), and proved a fixed point theorem in a complete metric space, stated below.

Theorem 1.5 ([7]). *Let (M, d) be a complete metric space and $T\colon M \to M$ be a mapping. If there exists a function $\varphi\colon M \to [0, \infty)$ such that*

$$d(T\xi, T\eta) \le (\varphi(\xi) - \varphi(T\xi))d(\xi, \eta) \tag{1.4}$$

for all $\xi, \eta \in M$ with $\xi \ne T\xi$, then T has a fixed point in M.

The contractive condition (1.4) has been further generalized by combining the contractive conditions (1.3) and (1.2) and the following fixed point theorem has been proved by Karapınar et al. (see [8]).

Theorem 1.6 ([8]). *Let (M, d) be a complete metric space and $T\colon M \to M$ be a mapping. If there exists a function $\varphi\colon M \to [0, \infty)$ such that*

$$d(T\xi, T\eta) \le (\varphi(\xi) - \varphi(T\xi))R(\xi, \eta) \tag{1.5}$$

for all $\xi, \eta \in M$ with $\xi \ne T\xi$, where $R(\xi, \eta) = \max\{d(\xi, \eta), d(\xi, T\xi), d(\eta, T\eta), d(\xi, T\eta), d(T\xi, \eta)\}$, then T has at least one fixed point in M.

There are some mappings which are not Ćirić-quasi contraction mappings but satisfy the contractive condition (1.5). See the following example.

Example 1.7. Let $M = [0, 1]$ be the metric space equipped with the usual metric d. Let $T\colon M \to M$ be defined by

$$T(\xi) = \begin{cases} 0, & \text{if } 0 \le \xi < \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \le \xi \le 1. \end{cases}$$

Then T satisfies the contractive condition (1.5) for $\varphi\colon M \to [0, \infty)$, which is defined by

$$\varphi(\xi) = \begin{cases} 1, & \text{if } \xi = 0, \\ 4, & \text{if } 0 < \xi < \frac{1}{2}, \\ 6, & \text{if } \frac{1}{2} \le \xi < 1, \\ 3, & \text{if } \xi = 1. \end{cases}$$

If T satisfies the contractive condition (1.3) then there exists $0 \leq q < 1$ such that

$$d(T\xi, T\eta) \leq q \ \max\{d(\xi, \eta), d(\xi, T\xi), d(\eta, T\eta), d(\xi, T\eta), d(T\xi, \eta)\},$$

for all $\xi, \eta \in M$. Then for $\xi = \frac{1}{2}$ and $\eta < \frac{1}{2}$ we have

$$\begin{aligned} 1 = d(T\xi, T\eta) &\leq q \max\{d(\xi, \eta), d(\xi, T\xi), d(\eta, T\eta), d(\xi, T\eta), d(T\xi, \eta)\} \\ &= q \max\left\{\frac{1}{2} - \eta, \frac{1}{2}, \eta, \frac{1}{2}, 1 - \eta\right\}. \end{aligned} \tag{1.6}$$

Taking $\eta \to (1/2)^-$ in (1.6) we see that $1 \leq q/2 < 1/2$, a contradiction. Therefore T is not a Ćirić-quasi contraction map.

Any mapping satisfying (1.4) also satisfies the contractive condition (1.5) but the converse is not true in general.

EXAMPLE 1.8. Let $M = [0, 3]$ be the metric space endowed with the usual metric d. Let $T \colon M \to M$ be defined by

$$T(\xi) = \begin{cases} 1, & \text{if } 0 \leq \xi < 1, \\ 2, & \text{if } 1 \leq \xi < 2, \\ 3, & \text{if } 2 \leq \xi \leq 3. \end{cases}$$

Then T satisfies the contractive condition (1.5) for $\varphi \colon M \to [0, \infty)$, which is defined by

$$\varphi(\xi) = \begin{cases} 5, & \text{if } 0 \leq \xi < 1, \\ 4, & \text{if } 1 \leq \xi < 2, \\ 3, & \text{if } 2 \leq \xi < 3, \\ 2, & \text{if } \xi = 3. \end{cases}$$

But T does not satisfy the contractive condition (1.4). If T satisfies (1.4) for some $\varphi \colon M \to [0, \infty)$ then for $\xi = 1$ and $\eta < 1$ we have

$$\begin{aligned} 1 = d(T\xi, T\eta) &\leq (\varphi(\xi) - \varphi(T\xi))d(\xi, \eta) \\ &= (\varphi(1) - \varphi(2))(1 - \eta). \end{aligned}$$

By taking $\eta \to 1^-$ we arrive at a contradiction.

In this article, we define some new contractive conditions which generalize the contractive conditions (1.5) and (1.6) and prove some fixed point theorems in a complete metric space which generalize the results of Karapınar et al. (see [7], [8]).

2. Main results

In this section we correlate the weak contraction mapping with Caristi contractive mapping in the framework of metric spaces. First we define some new contractive mappings.

Definition 2.1. Let (M, d) be a metric space and $T\colon M \to M$ be a self map. Then T is said to be

(i) *orbital Banach–Caristi contractive* mapping if there exists $\varphi\colon M \to [0, \infty)$ such that for all $\xi \in M$,

$$d(T\xi, T^2\xi) \le (\varphi(\xi) - \varphi(T\xi))d(\xi, T\xi). \tag{2.1}$$

(ii) *weak Banach–Caristi contractive* mapping if there exist $\varphi\colon M \to [0, \infty)$ and $L \ge 0$ such that for all $\xi, \eta \in M$ with $d(\xi, T\xi) > 0$ we have

$$d(T\xi, T\eta) \le (\varphi(\xi) - \varphi(T\xi))d(\xi, \eta) + Ld(\eta, T\xi). \tag{2.2}$$

(iii) *weak Ćirić–Caristi contractive* mapping if there exist $\varphi\colon M \to [0, \infty)$ and $L \ge 0$ such that for all $\xi, \eta \in M$ with $d(\xi, T\xi) > 0$ we have

$$d(T\xi, T\eta) \le (\varphi(\xi) - \varphi(T\xi))R(\xi, \eta) + Ld(\eta, T\xi), \tag{2.3}$$

where $R(\xi, \eta) = \max\{d(\xi, \eta), d(\xi, T\xi), d(\eta, T\eta), d(\xi, T\eta), d(\eta, T\xi)\}$.

Remark 2.2. Condition (2.1) generalizes condition (1.1) in the paper [6].

Definition 2.3 (φ-diminishing property). Let (M, d) be a metric space and $T\colon M \to M$ be a mapping. Also let $\varphi\colon M \to [0, \infty)$ be a given mapping. Then T is said to have *φ-diminishing property* in M if for any $\epsilon > 0$ there exists $\xi \in M$ with $d(\xi, T\xi) > 0$ such that $0 \le \varphi(\xi) - \varphi(T\xi) < \epsilon$.

Lemma 2.4. *If in a metric space (M, d), T does not have φ-diminishing property for some $\varphi\colon M \to [0, \infty)$ then T has a fixed point in M.*

PROOF. Since T does not have φ-diminishing property for $\varphi\colon M\to[0,\infty)$, there exists $\epsilon>0$ such that $\varphi(\xi)-\varphi(T\xi)\geq\epsilon$ for all $\xi\in M$ with $d(\xi,T\xi)>0$. Let $\xi_0\in M$ be chosen as arbitrary and let us construct the Picard iterating sequence $\{\xi_n\}$, where $\xi_n=T^n\xi_0$ for all $n\geq 1$. If possible let $\xi_n\neq\xi_{n+1}$ for all $n\geq 1$. Then we have $\{\varphi(\xi_n)\}_{n\in\mathbb{N}}$ is monotone decreasing sequence of real numbers which is bounded below. So it is convergent that is a Cauchy sequence. But since $\epsilon\leq\varphi(\xi_n)-\varphi(\xi_{n+1})$ for all $n\geq 1$ then $\{\varphi(\xi_n)\}_{n\in\mathbb{N}}$ can not be Cauchy, arrives at a contradiction. Therefore there must exists some $m\in\mathbb{N}$ such that $\xi_m=\xi_{m+1}$ and hence T has a fixed point in M. □

Now we recall the definition of orbital continuity of a mapping (see [5]): For a self mapping T over a metric space (M,d) the set $O(\xi,T):=\{T^n\xi : n=0,1,2,\cdots\}$, $\xi\in M$, is called an orbit of the mapping T and T is said to be orbitally continuous at a point $p\in M$ if for any sequence $\{\xi_n\}\subset O(\xi,T)$ (for some $\xi\in M$) $\xi_n\to p$ implies $T\xi_n\to Tp$ as $n\to\infty$. Moreover, if T is orbitally continuous at each point of M then T is said to be orbitally continuous on M.

THEOREM 2.5. *Let (M,d) be a complete metric space and $T\colon M\to M$ be an orbital Banach–Caristi contractive mapping. Then T has a fixed point in M, provided T is orbitally continuous on M.*

PROOF. If T does not have φ-diminishing property for the function φ appearing in the definition of T then T has a fixed point in M by Lemma 2.4. So we assume that T has φ-diminishing property in M.

Let us choose $a_0\in M$ and we construct the Picard iterating sequence $\{a_n\}$, where $a_n=T^na_0$ for all $n\geq 1$. If $a_n=a_{n+1}$ then T has a fixed point in M. So without loss of generality we assume that $d(a_{n-1},Ta_{n-1})>0$ for all $n\in\mathbb{N}$. Then due to condition (2.1) we have

$$\begin{aligned} d(a_n,a_{n+1}) &= d(Ta_{n-1},T^2a_{n-1}) \\ &\leq (\varphi(a_{n-1})-\varphi(a_n))d(a_{n-1},Ta_{n-1}) \\ &= (\varphi(a_{n-1})-\varphi(a_n))d(a_{n-1},a_n) \quad \text{for all } n\geq 1. \end{aligned} \tag{2.4}$$

Let us denote $d(a_{n-1},Ta_{n-1})=s_n$ for all $n\in\mathbb{N}$. Then from (2.4) we get

$$0<\frac{s_{n+1}}{s_n}\leq\varphi(a_{n-1})-\varphi(a_n) \quad \text{for all } n\geq 1. \tag{2.5}$$

Thus for any $k\in\mathbb{N}$ we have

$$\sum_{i=1}^{k}\frac{s_{i+1}}{s_i}\leq\sum_{i=1}^{k}\{\varphi(a_{i-1})-\varphi(a_i)\}=\varphi(a_0)-\varphi(a_k). \tag{2.6}$$

Now from (2.5) it follows that $\{\varphi(a_n)\}_{n\in\mathbb{N}}$ is a monotone decreasing sequence of real numbers which is bounded below. So it is convergent and let it converge to some $r(\geq 0)$. Thus (2.6) implies that

$$\sum_{i=1}^{\infty} \frac{s_{i+1}}{s_i} \leq \varphi(a_0) - r < \infty.$$

Therefore $\lim_{i\to\infty} \frac{s_{i+1}}{s_i} = 0$ and for some fixed $\rho \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $s_{i+1} \leq \rho s_i$ for all $i \geq n_0$. Now for $n_0 \leq p \leq q$ we have

$$d(a_p, a_q) \leq \sum_{i=p}^{q-1} s_{i+1} \leq \Big(\sum_{i=p}^{q-1} \rho^i\Big) s_1 \leq \frac{\rho^p}{1-\rho} s_1 \to 0 \quad \text{as } p \to \infty.$$

Therefore, $\{a_n\}$ is a Cauchy sequence and since M is complete, there exists $e \in M$ such that $\{a_n\}$ converges to e. Since T is orbitally continuous in M it follows that Ta_n converges to Te as $n \to \infty$. Hence $Te = e$ and T has a fixed point. □

Remark 2.6. The class of mappings satisfying the contractive condition (2.1) is a larger class than the class of mappings satisfying the contractive condition (1.4). If we take $\eta = T\xi \in M$ in the contractive condition (1.4) then clearly it reduces to the contractive condition (2.1), but the converse is not true. The following example proves our assertion.

Example 2.7. Let $M = \big[0, \frac{3}{2}\big]$ be the metric space endowed with the usual metric d and $T\colon M \to M$ be defined by

$$T(\xi) = \begin{cases} 0, & \text{if } \ \xi \in [0,1), \\ \frac{\xi}{2}, & \text{if } \ 1 \leq \xi \leq \frac{3}{2}. \end{cases}$$

Also let $\varphi\colon M \to [0,\infty)$ be defined as $\varphi(\xi) = 2\xi$ for all $\xi \in M$. Then T is an orbital Banach–Caristi contractive mapping (2.1). Also T is orbitally continuous and hence by Theorem 2.5, T has a unique fixed point $0 \in M$. But T does not satisfy the contractive condition (1.4). If T satisfies (1.4) for some $\varphi\colon M \to [0,\infty)$ then for $\xi = 1$ and $\eta < 1$ we have

$$\begin{aligned} \frac{1}{2} = d(T\xi, T\eta) &\leq (\varphi(\xi) - \varphi(T\xi))d(\xi,\eta) \\ &= \Big(\varphi(1) - \varphi\Big(\frac{1}{2}\Big)\Big)(1-\eta). \end{aligned}$$

By taking $\eta \to 1^-$, we arrived at a contradiction.

Example 2.8. Let $M = [0, \infty)$ be the metric space endowed with the usual metric and $T\colon M \to M$ be defined by

$$T(\xi) = \begin{cases} 1, & \text{if } \xi \in [0, 2), \\ \xi, & \text{if } \xi \geq 2. \end{cases}$$

Also let $\varphi\colon M \to [0, \infty)$ be defined as $\varphi(1) = 0$ and $\varphi(\xi) = \xi$ for all $\xi \neq 1$. Then T is clearly an orbital Banach–Caristi contractive mapping and also orbitally continuous in M. We see that T has uncountably many fixed points in M.

Example 2.9. Consider $M = c_0$, the space of all real sequences convergent to zero, equipped with its usual metric $d_\infty(\xi, \eta) = \sup_n |\xi_n - \eta_n|$ for all $\xi = (\xi_n)_n$, $\eta = (\eta_n)_n \in M$. Then (M, d_∞) forms a complete metric space. Define $T\colon M \to M$ by

$$T(\xi) = T(\{\xi_n\}) = \begin{cases} \frac{\xi}{2}, & \text{if there is at least one } \xi_n \text{ with } |\xi_n| \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Take $\varphi\colon M \to [0, \infty)$ defined by $\varphi(\xi) = \sup\limits_n |\xi_n|$ for all $\xi = \{\xi_n\} \in M$. Then it can be easily checked that T is orbital Banach–Caristi contractive mapping and also T is orbitally continuous. Clearly, the zero sequence $(0, 0, \cdots, 0, \cdots) \in M$ is a fixed point of T.

Theorem 2.10. *Let (M, d) be a complete metric space and $T\colon M \to M$ be a weak Banach–Caristi contractive mapping. Then T has a fixed point in M.*

Proof. If T does not possess φ-diminishing property for the function φ appearing in (2.2) then T has a fixed point in M, by Lemma 2.4. So we assume that T has φ-diminishing property in M.

Let us choose $a_0 \in M$ and we construct the Picard iterating sequence $\{a_n\}$, where $a_n = T^n a_0$ for all $n \geq 1$. If $a_n = a_{n+1}$ then T has a fixed point in M. So without loss of generality we assume that $d(a_{n-1}, Ta_{n-1}) > 0$ for all $n \in \mathbb{N}$. Then due to condition (2.2) we have

$$\begin{aligned} d(a_n, a_{n+1}) &= d(Ta_{n-1}, Ta_n) \\ &\leq (\varphi(a_{n-1}) - \varphi(a_n))d(a_{n-1}, a_n) + Ld(a_n, Ta_{n-1}) \\ &= (\varphi(a_{n-1}) - \varphi(a_n))d(a_{n-1}, a_n) \quad \text{for all } n \geq 1. \end{aligned} \tag{2.7}$$

Let us denote $d(a_{n-1}, Ta_{n-1}) = t_n$ for all $n \in \mathbb{N}$. Then from (2.7) we get

$$0 < \frac{t_{n+1}}{t_n} \leq \varphi(a_{n-1}) - \varphi(a_n) \quad \text{for all } n \geq 1.$$

Arguing in a similar manner as in Theorem 2.5 we see that $\{\varphi(a_n)\}_{n\in\mathbb{N}}$ converges to some $r \geq 0$ and $\{a_n\}$ is a convergent sequence in M. Let it be convergent to $\omega \in M$. Then from the contractive condition (2.2) we get

$$\begin{aligned} d(a_{n+1}, T\omega) &= d(Ta_n, T\omega) \\ &\leq (\varphi(a_n) - \varphi(a_{n+1}))d(a_n, \omega) + Ld(\omega, a_{n+1}) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore $T\omega = \omega$ and ω is a fixed point of T in M. □

From Theorem 2.10 we get the following remark.

REMARK 2.11. If we take $L = 0$ in the contractive condition (2.2) then it reduces to the contractive condition (1.4). More precisely, Theorem 1.5 directly follows from Theorem 2.10 by choosing $L = 0$.

Therefore any mapping satisfying contractive condition (1.4) is a weak Banach–Caristi contractive mapping but the converse is not true in general. The following example proves our assertion.

EXAMPLE 2.12. Let $M = \mathbb{R}$, equipped with the usual metric d, and $T\colon M \to M$ be given by

$$T(\xi) = \begin{cases} 1, & \text{if } \xi \geq 0, \\ -1, & \text{if } \xi < 0. \end{cases}$$

Then it can be easily checked that T is a weak Banach–Caristi contractive mapping, i.e. satisfies (2.2) with suitable choice of φ and $L \geq 2$ but it does not satisfy the contractive condition (1.4). For if, T satisfy (1.4) then for $\xi = 0$ and $\eta < 0$ we have

$$\begin{aligned} 2 = d(T0, T\eta) &\leq (\varphi(0) - \varphi(1))d(0, \eta) \\ &= (\varphi(0) - \varphi(1))|\eta| \to 0 \quad \text{as } \eta \to 0^-, \text{ a contradiction.} \end{aligned}$$

Here all the conditions of Theorem 2.10 are satisfied and T has two fixed points -1 and 1.

Theorem 2.13. *In a complete metric space (M,d) a weak Ćirić–Caristi contractive mapping T has at least one fixed point.*

Proof. It is immediate that T has a fixed point in M without having φ-diminishing property of T. So we suppose that T has φ-diminishing property in M.

Let us choose $a_0 \in M$ and we construct the Picard iterating sequence $\{a_n\}$, where $a_n = T^n a_0$ for all $n \geq 1$. If $a_n = a_{n+1}$ then T has a fixed point in M. So without loss of generality we assume that $d(a_{n-1}, Ta_{n-1}) > 0$ for all $n \in \mathbb{N}$. Then due to condition (2.3) we have

$$\begin{aligned} d(a_n, a_{n+1}) &= d(Ta_{n-1}, Ta_n) \\ &\leq (\varphi(a_{n-1}) - \varphi(a_n))R(a_{n-1}, a_n) + Ld(a_n, Ta_{n-1}) \\ &= (\varphi(a_{n-1}) - \varphi(a_n)) \max\{d(a_{n-1}, a_n), d(a_n, a_{n+1}), d(a_{n-1}, a_{n+1})\} \\ &\leq (\varphi(a_{n-1}) - \varphi(a_n))[d(a_{n-1}, a_n) + d(a_n, a_{n+1})] \quad \text{for all } n \geq 1. \end{aligned} \tag{2.8}$$

Let us denote $d(a_{n-1}, Ta_{n-1}) = u_n$ for all $n \in \mathbb{N}$. Then from (2.8) we get

$$0 < \frac{u_{n+1}}{u_n + u_{n+1}} \leq \varphi(a_{n-1}) - \varphi(a_n) \quad \text{for all } n \geq 1. \tag{2.9}$$

Thus for any $k \in \mathbb{N}$ we see that

$$\sum_{i=1}^{k} \frac{u_{i+1}}{u_i + u_{i+1}} \leq \sum_{i=1}^{k} \{\varphi(a_{i-1}) - \varphi(a_i)\} = \varphi(a_0) - \varphi(a_k). \tag{2.10}$$

Now from (2.9) it follows that $\{\varphi(a_n)\}_{n\in\mathbb{N}}$ is a real monotonically decreasing sequence which is bounded below. So it is convergent and let it converge to some $r(\geq 0)$. Thus (2.10) implies that

$$\sum_{i=1}^{\infty} \frac{u_{i+1}}{u_i + u_{i+1}} \leq \varphi(a_0) - r < \infty.$$

Therefore $\lim_{i\to\infty} \frac{u_{i+1}}{u_i+u_{i+1}} = 0$ and for some fixed $\sigma \in (0, \frac{1}{2})$ there exists $N \in \mathbb{N}$ such that $\frac{u_{i+1}}{u_i+u_{i+1}} \leq \sigma$ i.e. $u_{i+1} \leq \mu u_i$ for all $i \geq N$, where $\mu = \frac{\sigma}{1-\sigma}$. Now for $N \leq p \leq q$ we have

$$d(a_p, a_q) \leq \sum_{i=p}^{q-1} u_{i+1} \leq \Big(\sum_{i=p}^{q-1} \mu^i\Big) u_1 \leq \frac{\mu^p}{1-\mu} u_1 \to 0 \quad \text{as } p \to \infty.$$

Therefore, $\{a_n\}$ is a Cauchy sequence and, since M is complete, there exists $\upsilon \in M$ such that $\{a_n\}$ converges to υ. Now,

$$\begin{aligned} d(\upsilon, T\upsilon) &\le d(\upsilon, a_n) + d(Ta_{n-1}, T\upsilon) \\ &\le d(\upsilon, a_n) + (\varphi(a_{n-1}) - \varphi(a_n))R(a_{n-1}, \upsilon) + Ld(\upsilon, Ta_{n-1}) \\ &= d(\upsilon, a_n) + (\varphi(a_{n-1}) - \varphi(a_n)) \max\{d(a_{n-1}, \upsilon), d(a_{n-1}, a_n), \\ &\qquad d(\upsilon, T\upsilon), d(\upsilon, a_n), d(a_{n-1}, T\upsilon)\} + Ld(\upsilon, a_n) \quad \text{for any } n \in \mathbb{N}. \end{aligned}$$

Taking $n \to \infty$, by continuity of d we get $d(\upsilon, T\upsilon) = 0$. Hence T has a fixed point in M. □

Example 2.14. Let $M = \{1, 2, 3, 4\}$, endowed with the usual metric, and $T\colon M \to M$ be defined by $T1 = 2$, $T2 = 3$, $T3 = 4$ and $T4 = 4$. Also let $\varphi\colon M \to [0, \infty)$ be given by $\varphi(1) = 10$, $\varphi(2) = 9$, $\varphi(3) = 8$ and $\varphi(4) = 7$. Then clearly T is a Ćirić–Caristi contractive mapping and all the conditions of Theorem 2.13 are satisfied. Here T has a unique fixed point in M.

Remark 2.15. Theorem 1.6 directly follows from Theorem 2.13, by choosing $L = 0$.

We can use one of the following contractive conditions instead of the contractive condition (2.3).

Corollary 2.16.

(i) *For all $\xi, \eta \in M$ with $d(\xi, T\xi) > 0$, T satisfies*

$$d(T\xi, T\eta) \le (\varphi(\xi) - \varphi(T\xi))[\alpha d(\xi, T\xi) + \beta d(\eta, T\eta)] + Ld(\eta, T\xi),$$

where $0 \le \alpha + \beta \le 1$ and $L \ge 0$.

(ii) *For all $\xi, \eta \in M$ with $d(\xi, T\xi) > 0$, T satisfies*

$$d(T\xi, T\eta) \le (\varphi(\xi) - \varphi(T\xi))[\gamma d(\xi, T\eta) + \delta d(\eta, T\xi)] + Ld(\eta, T\xi),$$

where $0 \le \gamma + \delta \le 1$ and $L \ge 0$.

(iii) *For all $\xi, \eta \in M$ with $d(\xi, T\xi) > 0$, T satisfies*

$$\begin{aligned} d(T\xi, T\eta) \le (\varphi(\xi) - \varphi(T\xi))[\lambda d(\xi, \eta) + \mu d(\xi, T\xi) + \nu d(\eta, T\eta)] \\ + Ld(\eta, T\xi), \end{aligned}$$

where $0 \le \lambda + \mu + \nu \le 1$ and $L \ge 0$.

(iv) *For all* $\xi, \eta \in M$ *with* $d(\xi, T\xi) > 0$, T *satisfies*

$$d(T\xi, T\eta) \leq (\varphi(\xi) - \varphi(T\xi))[\beta d(\xi, \eta) + \gamma d(\xi, T\eta) + \delta d(\eta, T\xi)] \\ + Ld(\eta, T\xi),$$

where $0 \leq \beta + \gamma + \delta \leq 1$ *and* $L \geq 0$.

(v) *For all* $\xi, \eta \in M$ *with* $d(\xi, T\xi) > 0$, T *satisfies*

$$d(T\xi, T\eta) \leq (\varphi(\xi) - \varphi(T\xi))[\alpha d(\xi, \eta) + \beta d(\xi, T\xi) + \gamma d(\eta, T\eta) \\ + \delta d(\xi, T\eta) + \zeta d(\eta, T\xi)] + Ld(\eta, T\xi),$$

where $0 \leq \alpha + \beta + \gamma + \delta + \zeta \leq 1$ *and* $L \geq 0$.

Acknowledgments. The authors are thankful to the learned referee for suggesting some improvements in the presentation of the paper.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[2] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum **9** (2004), no. 1, 43–53.

[3] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251.

[4] L.B. Ćirić, *Generalized contractions and fixed-point theorems*, Publ. Inst. Math. (Beograd) (N.S.) **12(26)** (1971), 19–26.

[5] L.B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.

[6] G.S. Jeong and B.E. Rhoades, *Maps for which* $F(T) = F(T^n)$, in: Y.J. Cho et al. (eds.), *Fixed Point Theory and Applications*, 6, Nova Sci. Publ., New York, 2007, pp. 71–104.

[7] E. Karapınar, F. Khojasteh, and Z.D. Mitrović, *A proposal for revisiting Banach and Caristi type theorems in b-metric spaces*, Mathematics **7** (2019), no. 4, 308, 4 pp. DOI:10.3390/math7040308.

[8] E. Karapınar, F. Khojasteh, and W. Shatanawi, *Revisiting Ćirić-type contraction with Caristi's approach*, Symmetry **11** (2019), no. 6, 726, 7 pp. DOI:10.3390/sym11060726.

[9] K. Roy and M. Saha, *Fixed point theorems for generalized contractive and expansive type mappings over a* C^**-algebra valued metric space*, Sci. Stud. Res. Ser. Math. Inform. **28** (2018), no. 1, 115–129.

[10] K. Roy and M. Saha, *Fixed point theorems for a pair of generalized contractive mappings over a metric space with an application to homotopy*, Acta Univ. Apulensis Math. Inform. **60** (2019), 1–17.

Kushal Roy, Sayantan Panja, Mantu Saha
Department of Mathematics
The University of Burdwan
Purba Bardhaman-713104, West Bengal
India
e-mail: kushal.roy93@gmail.com
e-mail: spanja1729@gmail.com
e-mail: mantusaha.bu@gmail.com

Zoran D. Mitrović
University of Banja Luka
Faculty of Electrical Engineering
patre 5, 78000 Banja Luka
Bosnia and Herzegovina
e-mail: zoran.mitrovic@etf.unibl.org